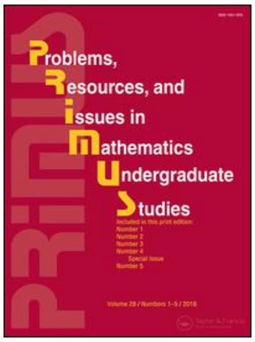

PRIMUS
Problems, Resources, and Issues in Mathematics Undergraduate Studies

ISSN: 1051-1970 (Print) 1935-4053 (Online) Journal homepage: https://www.tandfonline.com/loi/upri20

Calculus Students' Deductive Reasoning and Strategies when Working with Abstract Propositions and Calculus Theorems

Joshua Case & Natasha Speer

To cite this article: Joshua Case & Natasha Speer (2019): Calculus Students' Deductive Reasoning and Strategies when Working with Abstract Propositions and Calculus Theorems, PRIMUS, DOI: 10.1080/10511970.2019.1660931

To link to this article: https://doi.org/10.1080/10511970.2019.1660931

Accepted author version posted online: 30 Aug 2019.
Published online: 23 Sep 2019.

PRIMUS, 0(0): 1–20, 2019

ISSN: 1051-1970 print / 1935-4053 online
DOI: 10.1080/10511970.2019.1660931

Calculus Students' Deductive Reasoning and Strategies when Working with Abstract Propositions and Calculus Theorems

Joshua Case and Natasha Speer

Abstract: In undergraduate mathematics, deductive reasoning plays important roles in teaching and learning various ideas, and is primarily characterized by the concept of logical implication. This comes up whenever conditional statements are applied, i.e., one checks if a statement's hypotheses are satisfied and then makes inferences. In calculus, students must learn to work with such statements; however, most have not studied propositional logic. How do these students comprehend the abstract notion of logical implication, and how do they reason conditionally with calculus theorems? Study results indicate that students struggle with logical implication in abstract contexts, but perform better when working in calculus contexts. Findings indicate that some students use "example generating" strategies to successfully determine the validity of calculus implications. We discuss ways instructors might support students' use of such strategies, as well as further avenues of inquiry.

Keywords: Student thinking, logic, implication, calculus, theorems, conditionals

1. INTRODUCTION

In a typical calculus course, students are often presented with definitions, lemmas, propositions, and theorems. Often, these statements are conditional, that is, they are sentences of the *if-then* form. For example, the *differentiability implies continuity* theorem is the following conditional statement:

If f is differentiable at $x = c$, then it is continuous at $x = c$.

Address all correspondence to Natasha Speer, University of Maine, 234 Neville Hall Orono, ME 04469, USA. E-mail: speer@math.umaine.edu.
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/upri.

Calculus students must be able to make proper inferences from these statements in order to build their calculus knowledge, and they must then know when it is or is not appropriate to apply this theorem, such as when they are given a function that is differentiable or when they are given a function that is continuous. This deductive process, characterized by logical implication, is a hallmark of mathematical thinking. To apply a theorem correctly, a student must comprehend logical implication. This requires understanding conditional statements and their standard conditional variations: *inverse*, *contrapositive* and *converse*. Applying this reasoning can enable a student to know, for example, that a function being continuous at a point does not necessarily imply that it is differentiable at that point. As calculus is a major aspect of science, technology, engineering and mathematics (STEM) undergraduate programs, the goal of improving STEM enrollment and retention rates [4] can be well-served by improving the teaching and learning of calculus. One component of this effort can involve investigating how students think about calculus theorems and about logical implication itself to inform instructional materials and approaches. In particular, in the study reported on here, we examined calculus students' understanding of logical implication and the extent to which that understanding played a role in their abilities to make sense of calculus theorems. Knowing more about this kind of student thinking allows instructors to be able to serve the needs of their calculus students and thus to contribute to efforts to address STEM enrollment and retention rates.

2. STUDENT UNDERSTANDING OF LOGICAL IMPLICATION AND CALCULUS

It is well-established that both children and adults may struggle with certain logical reasoning tasks [6, 14, 15]. For example, participants in Wason's study [14] were asked to carry out the following abstract reasoning task (popularly known as the "Wason Selection Task"):

> Given four cards (one with a vowel shown face up, one with a consonant shown face up, one with an even number shown face up, and one with an odd number shown face up), the assumption that each card has a letter on one side and a number on the other, and the rule "if there is a vowel on one side of a card, then there is an even number on the other side" ([15, p. 273], turn over only the cards needed to find out if the rule has actually been followed.

The solution: Turn over the card with a vowel shown face up and the card with an odd number shown face up. The vast majority of participants were unable to carry out the task successfully. Mistakes often involved turning over the card with an even number shown face up (assuming the converse is true) and not turning over the card with an odd number shown face up (not recognizing the truth of the contrapositive).

Although individuals seem to struggle with abstract logical tasks, it also appears that people are more successful when the tasks are posed in a familiar context. For example, in [13], the authors found that education majors reasoned about contraposition better in a verbal, syllogistic environment than in a purely mathematical environment with abstract symbols and sentences. Wason and Shapiro [16] also found that participants performed better on the Wason Selection Task when given a rule with familiar content, as opposed to an abstract rule. For example, a rule such as "every time I go to Manchester, I travel by car" [16, p. 68] would be considered "familiar" whereas the rule "every card which has a D on one side has a 3 on the other side" [16, p. 68] would be considered "abstract." It is important to note that although these studies do seem to suggest that familiar contexts can help students to reason correctly, Stylianides et al. [13, pp. 155–156] state that

> research, however, provides a weak basis on which to formulate hypotheses about the relation between students' performance in tasks with non-meaningful words and symbolic tasks that investigate the same logical principles

and that research tends to favor logical reasoning in "meaningful verbal contexts." For additional work regarding the teaching and learning of logical implication, see Yopp [17] for a study related to eighth grade learning of the contrapositive and Attridge, et al. [1] for undergraduate understanding of conditionals given previous logic experience. Also, see [7] and [10] for recent work related to the Wason Selection Task.

Findings from decades of research have provided insights into student thinking and the challenges that students encounter with calculus ideas such as limit, differentiation, and integration. For a history of this work, see [8], and for more detailed reviews of the literature and findings specific to sub-topics in calculus, see [2] and [3], and SIGMAA on RUME conference proceedings (http://sigmaa.maa.org/rume/). The instruction students receive about key calculus ideas often includes theorem or theorem-like statements and students are expected to reason logically from them. However, the vast majority of research into student understanding of this kind of logical reasoning has occurred in the context of introduction to proof or other proof-focused courses [11] and has not focused on students in introductory calculus.

Although much work has been done separately on both the issues of logical implication and calculus learning, we know little about how students understand and work with ideas of logical implication that appear in theorems and theorem-like statements in a calculus context. Researchers have examined related ideas through studies of student thinking about sequences and series (see, for example, [9]) and some work (e.g., [11]) has examined calculus students' meanings for quantifiers found in calculus theorems. However, the focus in that work was specifically on quantifiers appearing in complex theorems. To date, beginning calculus student understanding of conditionals in the form of if-then statements that occur in introductory calculus has not been closely examined. To explore this, we were interested in whether calculus students had the same kinds of difficulties with calculus-based conditional statement tasks as they did with the purely abstract tasks. In other words, we wondered if calculus theorems provided enough of a "context" to support students' productive reasoning or whether those tasks were treated in the same way as the classic, abstract tasks.

To examine this, we focused on particular types of tasks set in abstract and calculus contexts that are characteristic of one kind of reasoning expected of students. In particular, in calculus, students are told to take for granted the truth of a particular theorem and then asked to draw conclusions given a true or false antecedent or consequent. It is important to note that we did not ask students to consider the truth or falseness of an entire conditional statement. Below are the tasks, in the abstract, that students were asked to consider:

- *Inverse Task*: Suppose that $p \Rightarrow q$ is true and you know that p is false. Is q true, false, or is it not possible to tell? Explain.
- *Converse Task*: Suppose that $p \Rightarrow q$ is true and you know that q is true. Is p true, false, or is it not possible to tell? Explain.
- *Contrapositive Task*: Suppose $p \Rightarrow q$ is true and you know that q is false. Is p true, false, or is it not possible to tell? Explain.
- *Modus-ponens Task*: Suppose that $p \Rightarrow q$ is true and you know that p is true. Is q true, false, or is it not possible to tell? Explain.

In abstract terms, they are given that $p \Rightarrow q$ for a particular p and q as well as a situation in which p (or q) is either true or false. An inference may then be made by reasoning with these two pieces of information.

As an example of such a task, students may be told to assume that the following theorem is true:

> *For all functions f, if f is differentiable at a point $x = c$, then f is also continuous at the point $x = c$.*

Then, given a particular function and point such as $f(x) = x^2$ at $x = 0$, the student is then expected to investigate whether $f(x) = x^2$ at the point $x = 0$ is differentiable. If the student determines that the antecedent is met, the student may then use the theorem to infer that $f(x) = x^2$ is continuous at the point $x = 0$. Given the importance of this kind of reasoning, we focus on student reasoning involving the inference of q (or p) given a true implication statement $p \Rightarrow q$ and true or false p (or q).

Note that we are not concerned with students' abilities to validate the truth or falseness of an *entire* conditional statement given an antecedent and consequent. Rather, we are interested in how students infer the status of an antecedent or consequent given a true conditional statement and a true or false antecedent or consequent. This connects to the use of truth tables, such as the one in Table 1 that characterizes logical implication. In this table, conditions

Table 1. Truth table for a conditional statement

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

are placed on the antecedent p and the consequent q that allow for a true conditional statement or a false conditional statement. In the case of our study, we are interested in students' abilities to evaluate the truth status of a particular p or q.

We designed our study to examine how calculus students engage with logic tasks. In particular, we sought answers to the following questions:

- How successful are calculus students with logical implication tasks set in calculus and abstract contexts?
- What, if any, relationship exists between success on one type of task and success on the other?
- What strategies do students use when engaged in calculus theorem tasks involving logical implications?

Our answers to these questions provide insights into how students make sense of problems that can be used to inform instructional design aimed at improving student understanding of theorems and definitions in calculus.

3. RESEARCH DESIGN

Similar to much of the prior work on student thinking about calculus, this study was performed from a cognitive theoretical perspective, and thus, students' written and spoken statements were used as data regarding their thinking and understanding of the ideas. Surveys were given in a first semester differential Calculus I class at a university in New England near the end of the fall semester. In total, there were 52 participants. The surveys consisted of two parts. Part I consisted of a well-known calculus theorem and four tasks that were modeled after the theorem and its standard conditional variations. In Part II, the same four tasks were given, but presented in an abstract manner. Many of these tasks resembled syllogisms (e.g., All men are mortal. Socrates is a man. Therefore, Socrates is mortal), but were stated in a formal context using letters and symbols to represent statements. See Figure 1 for sample tasks.

Theorem: ***For all functions f, if f is differentiable at a point x = c, then f is also continuous at the point x = c.***

2) Suppose h is a function that is continuous at $x = 7$. Then
a. h is differentiable at $x = 7$.
b. h is not differentiable at $x = 7$.
c. not enough information to decide whether or not h is differentiable at $x = 7$.

Explain the reason for your answer:

Proposition: ***For integers a and b, if a§b then aba§bab.***

8) Suppose (7)(4)(7)§(4)(7)(4) is true.
Then 7§4 is
a. True.
b. False.
c. Not enough information to decide if True or False.

Explain the reason for your answer:

Figure 1. (Left) A sample calculus task (*converse reasoning*) from Part I. (Right) A sample abstract task (*converse reasoning*) from Part II.

We note that students are not determining whether a full conditional statement is true or false given a premise and a conclusion. Rather, given a conditional statement and an antecedent (or consequent), students are asked to determine if the corresponding consequent (or antecedent) is true, false, or if there is not enough information to decide. Although other researchers have established the difficulties students have with these kinds of abstract tasks (see beginning of previous section), we sought to examine the extent to which these difficulties were apparent in the (relatively) less abstract context of calculus theorems.

As our focus was on whether (and how) students reason about the various conclusions drawn from a calculus theorem and a premise, our data collection occurred after the calculus theorems used on our instruments had been introduced in the course. As a result, students were likely familiar with those theorems and had used them in various ways, but were unlikely to have been asked to reason in the manner presented on the survey. Although some students may have had instruction in formal logic in a high school course, that topic is not addressed explicitly in the calculus course from which study participants were recruited. This design enables us to examine student thinking as it is apt to occur post-instruction in a calculus course. Investigating the extent to which that thinking may have changed from pre-instruction to post-instruction is beyond the scope of the present study.

Two versions of the survey were distributed, each consisting of a different, well-known calculus theorem, as well as a different abstract conditional statement (see Figure 1). Each student received one version of the survey. There was no statistically significant difference in student performance between the two versions.

To learn about student strategies for solving the survey tasks, 10 students were interviewed. During these clinical interviews [5], participants were asked to work through a version of the survey and prompted to explain their reasoning. Various follow-up and probe questions were used to generate further data on student thinking and solution strategies. Interviews were recorded using LiveScribe technology to capture both their written work and spoken answers. Interviews lasted approximately 20–40 minutes each.

4. DATA ANALYSIS

Survey responses were coded as "correct" or "incorrect." For example, consider the calculus task given in Figure 1. If a student believed that the correct choice was "h is differentiable at $x = 7$" or "h is not differentiable at $x = 7$," then the response was coded as "incorrect" since, according to the truth table given in Table 1, the conclusion might be either true or false. Thus, the correct answer for this task would be "not enough information." To compare performance on the abstract and calculus versions of the tasks and to examine potential relationships between performances on the two types of tasks, we carried out several kinds of statistical tests (described below in conjunction with the presentation of the results). For the interview data, in addition to coding interviewees' responses as correct or incorrect, analysis centered on the manner in which interviewees explained their answers. In particular, the focus was on the kinds of strategies participants used when working through the problems. This phase of the analysis was informed, in part, by prior research (e.g., [6]) on student thinking about implication and by the use of Grounded Theory ([12]) to further characterize student strategies beyond the general characterizations previously documented in the literature. Categories and sub-categories were developed to describe these strategies in detail.

5. STUDENT PERFORMANCE ON TASKS

Consistent with prior research [13, 16] and as Figure 2 shows, students were more successful on the contextual tasks (in this case, the calculus tasks) than on the abstract tasks. On the calculus tasks, 63% of the 52 survey participants answered at least three of the four tasks correctly and 33% answered all four correctly. In contrast, only 8% of students produced correct answers for at least three of the abstract tasks, and none had all four correct. These differences between the calculus and abstract consistency percentages were statistically significant, suggesting that the calculus context prompts students to engage differently with the calculus tasks than with the abstract tasks.

We were also interested in potential relationships between success on one type of task and success on the other. For example, given that a student identified the correct answer to an abstract task, what is the conditional probability that they also answered the calculus version of that same task correctly? Given that a student did not correctly answer an abstract task, how likely are they to answer the calculus version of that same task correctly? The results show that, for the *modus ponens*, *converse*, and *inverse* reasoning tasks, using a 2-proportion z-test, there was no statistically significant advantage when answering the calculus version of a task given a correct answer on the abstract version (see Table 2). However, for the *contrapositive* task, students who answered the abstract version correctly did have an advantage

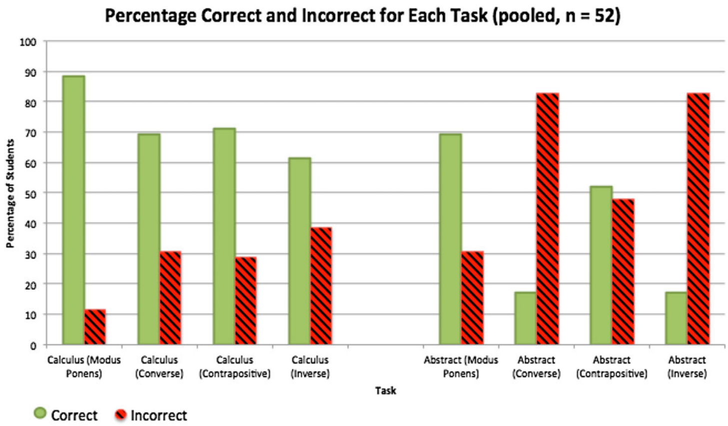

Figure 2. Student performance on calculus and abstract tasks from survey data.

Table 2. Conditional probabilities of answering calculus tasks correctly (* indicates statistical significance with $\alpha = 0.05$)

	Probability of correct calculus answer given a correct abstract answer	Probability of correct calculus answer given an incorrect abstract answer	p-value
Modus Ponens	0.89	0.88	$p > 0.05$
Converse	0.89	0.65	$p > 0.05$
Contrapositive	0.85	0.56	$0.01 < p < 0.05$*
Inverse	0.56	0.63	$p > 0.05$

when answering the calculus version (more will be discussed about this in Section 6 of this article). Thus, overall, we found that students who were not able to answer the abstract tasks correctly were still able to make sense of calculus theorems and definitions.

6. STUDENTS' REASONING STRATEGIES

Although analyses of the survey data provided some insights (e.g., the calculus context seems to make some of the reasoning easier for students, the abstractly stated tasks are generally much more difficult for students, etc.), we wanted to understand more about student thinking concerning the inferences, to gain further insight into the findings from the survey data analyses. From analysis of the interview data, we identified several different ways in which students approached the tasks. As displayed in Figure 3, there were three main ways of thinking (plus "other'), some of which had sub-categories that characterized the thinking at even finer levels of detail. The "other" category was used for

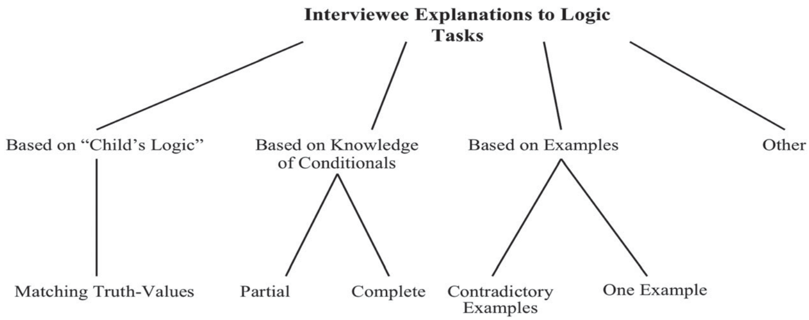

Figure 3. Types of reasoning exhibited by interviewees.

responses that were difficult to categorize and/or did not seem to fit the main categories. Characteristics of the three main categories are discussed here.

6.1. Child's Logic and Knowledge of Conditonals

We first consider the strategies located on the two left-most branches in Figure 3. Interviewees who responded with "Child's Logic" (a common logical misunderstanding) tended to match truth-values (that is, they responded with "True" when given a true premise and responded with "False" when given a false premise). This strategy, when applied with complete consistency, generates correct answers to two of the four tasks. While interviewees sometimes exhibited "Child's Logic" [6] on the calculus portion of the interview, they appeared more apt to consistently apply this kind of thinking on the abstract section. For example, 5 of the 10 interviewees responded to all four abstract tasks with Child's Logic whereas only one responded to all four calculus tasks with Child's Logic.

Interview responses based on some knowledge of conditionals were also given a category. Here, participants explained their work by following some rule or rules that they appear to have already internalized prior to their response. For example, at least one student explained that if you are given a conditional statement, only the *modus ponens* and *contrapositive* inferences could be made conclusively. This approach is correct, however, it appeared to be based primarily on knowledge students had about conditionals and not on any reasoning actions that they performed during the interview. Some believed that, given a conditional statement, it was only possible to make the *modus ponens* inference because the given conditional does not allow for any other possibilities. Thus, these students believe, incorrectly, that a conclusive deduction cannot be made regarding the task requiring *contrapositive* reasoning. This kind of partially correct thinking does provide correct responses for the inverse and converse tasks, since a conclusive inference cannot be made for them. However, this reasoning does not represent a complete understanding of how rules of logic apply to conditionals. In contrast with responses described below, these interviewees seemed to be recalling a rule to apply to the situation and were not engaged in extensive reasoning about the situations themselves. This strategy seemed to be the least prominent of the strategies used by the interviewees.

6.2. Reasoning with Examples

Many interviewees engaged with the tasks in a different way by generating an example or examples (via a graph or a verbalized scenario) to illustrate their thinking on at least one of the eight tasks. Although this occurred mostly with the calculus tasks, one student also used the example generating strategy when working on the abstract versions of the tasks. Two forms of this strategy were evident in the data: one utilized a single example to provide a justification and the other involved multiple examples. Interviewees used both forms of this strategy in their explanations of correct answers; however, they did not appear to be equally useful for reaching a correct conclusion.

6.2.1. Reasoning with a Single Example

In the single example approach, students generate an example graph or verbalize a relationship to explain the thinking behind their answer. Some students used this approach when providing explanations for why the *contrapositive* reasoning task is true. For example, Jordan was working with the theorem

> *For all functions f, if f has a local maximum value at $x = c$, then c is a critical point of f*

and was asked to explain his answer to the associated contrapositive reasoning task. Jordan drew a graph and tried to explain that if point *c* on the graph was not a critical point, then the function would not have a local maximum at that point. Although Jordan referred to various features of his graph as he tried to explain the thinking behind his correct response, he was unable to provide clear and convincing justification. After being asked to explain his answer, Jordan's response included quite a bit of hesitation and did not provide a clear chain of reasoning. He said:

> **Jordan:** *Um, well when you're um . . . when you see a graph . . . and you're trying . . . and you take the* [pauses for a moment]. *Well, let me think about* [pauses again for a moment]. *Oh, oh okay, well from a graph the two critical points are your local max and min and if its not a critical point than it can't be* [inaudible] *max or min.*

Then the interviewer asked: "*is there a way you could illustrate maybe what you're saying . . .* ?"

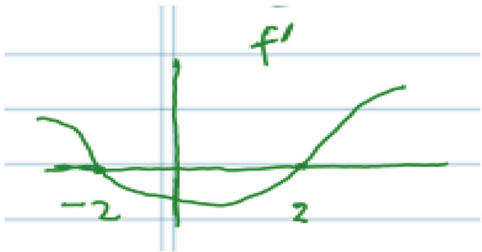

Figure 4. Jordan's example for the calculus *contrapositive* reasoning task.

This prompted Jordan to generate the graph he used to illustrate his reasoning (see Figure 4). During the discussion, Jordan was unable to provide a more compelling answer and eventually suggested the use of an equation to illustrate what was going on. A similar type of conversation also occurred when another student tried to use a single example as part of his explanation for his answer to the calculus contrapositive task.

When taking the single example approach to answering the calculus contrapositive task, students' struggles to explain their reasoning are not surprising. Visualizing one example in this situation is not going to provide the kind of solid evidence needed to obtain the appropriate conclusion for contrapositive reasoning. The two-example strategy described below was effective for students as they reasoned to obtain the correct answer for the *converse* and *inverse reasoning* tasks. This appears to be effective because it involves generating two examples and then noting that they provide contradictory information about the truth status of the conclusion. This generates evidence that there is not enough information to decide whether the conclusion is true or false. We refer to this as the "contradictory examples" approach. Generating single examples to explore and explain the truth of the contrapositive variations to the theorems may have resulted in correct answers. However, they were not productive approaches in the sense that students did not appear able to provide a complete explanation for their response. In large part, this is due to the nature of the contrapositive tasks. Here, contradictory examples cannot be obtained to properly infer the correct solution.

In contrast with the example-generating approaches used by interviewees on the *converse* and *inverse reasoning* tasks, interviewees who gave correct answers to the abstract *contrapositive* reasoning tasks sometimes justified their answers by appealing to logic or by deriving the contrapositive rule itself via a contradiction argument. Findings from the survey data analysis support the notion that the pure, logical understanding exhibited by some of these students when answering the abstract *contrapositive* reasoning task may be useful even as they consider the calculus version of the same task. More specifically, for the surveys, performance on the contrapositive calculus task was the only task variation that *was* correlated with performance on the abstractly presented version (see Table 2). In other words, having some formal (or abstract) understanding of contrapositives makes it more likely that students would answer the calculus contrapositive task correctly. These findings suggest that, for the calculus *contrapositive* task, the example generating strategy may not be as effective compared to knowing and applying rules of logic.

6.2.2. Reasoning with Two Contradictory Examples

As mentioned above, the power of the *two*-example generating strategy was evident when interviewees worked with tasks where there was not enough information to decide whether the antecedent or consequent was true or false (and thus showing that both the *converse* and *inverse* are invalid). In this approach, some interviewees utilized two, contradictory examples in order to logically deduce the correct answer. These discussions generated rich data on student thinking and potentially useful instructional implications.

We now present a transcript that illustrates this kind thinking. Here, the student (Jack) provides an explanation for his answer to the calculus *inverse* task that involved the following theorem:

> *For all functions f, if f is differentiable at a point $x = c$, then f is also continuous at the point $x = c$.*

While drawing the graphs shown in Figure 5, he says the following:

> **Jack:** *So, it's just like* [pauses to draw axes and says something inaudible] *and something goes like ... this* [draws a continuous function with a sharp corner]. *And I mean you could define it as maybe two different line segments and try to do it that way, but the function itself isn't continuous* [we suspect, from the context, that he meant "differentiable"] *because at that point there's no specific, um, rate of change. However, for "b", um ... a function ... very well could be not continuous and not differentiable. Say the function just* [draws a linear function with a hole] ... *so you have some function that just has a hole in it. It's not continuous and it's not differentiable.*

Jack produced two function graphs that invalidate two of the multiple-choice options ("*f* is continuous at the point" and "*f* is not continuous at the point") in order to infer the correct answer (not enough information to decide). This strategy allowed Jack (and others who used this strategy) to take advantage of the calculus ideas presented in the problem and create a scenario so that the correct answer became clear. Five interviewees used examples at some point during the calculus portion of the interview. Four out of these five interviewees used contradictory examples.

To illustrate this productive strategy further, we provide another transcript excerpt involving contradictory examples. Here Ryan used two contradictory examples to deduce the correct answer ("not enough information to decide")

Figure 5. Jack's contradictory examples to the calculus *inverse* task.

Figure 6. Ryan's contradictory example for the calculus *converse* task.

for one of the calculus *converse* tasks. In particular, Ryan worked with the theorem:

> *For all functions f, if f has a local maximum value at $x = c$, then c is a critical point of f.*

> **Interviewer:** *So if you could explain to me again how do you go from c is a critical point to deducing that ... we don't know if it's a ... if there's a maximum there.*

> **Ryan:** *So, the, the c is a critical point ... of the function k, but we don't know if c is a local maximum or local minimum ... or is undefined. But, the answer* "a" *it's a local maximum ... we don't know much about it and c does not have a local minimum value at $x = c$ we also don't know it. So it should be* "c" *not enough information.*

In the above excerpt, Ryan explained that just because we are told that a critical point exists, that does not imply that it is a maximum. After all, the point could be a minimum or have a vertical tangent line. Therefore, the correct answer is "not enough information to decide." After some more discussion, Ryan illustrated his explanation with a graph (Figure 6).

As we have seen, contradictory examples seem to provide a logical foothold for calculus students thinking on calculus *converse* and *inverse* reasoning tasks. Students' use of this method allowed them to answer these tasks correctly with sound logical reasoning. On the abstract portion, however, only one student tried to answer a task with a generated example. As discussed above, survey participants did not perform as well on the abstract tasks. This may be in part because they were unable to create scenarios based on the task with which they could work and reason about. Note also that although the single example approach may have led students to correct answers, it did not mean that the answer explanation was always satisfactory or valid.

7. DISCUSSION AND IMPLICATIONS

Consistent with other researchers' findings (e.g., [14]), participants in this study encountered difficulties when asked to complete tasks based on statements presented in abstract ways. Other researchers have found that success is higher when people are asked to engage in the same kinds of reasoning but with familiar language-based examples (e.g., [13]). Findings from the present study suggest that whatever cognitive supports that familiar English-language statements provide also seems to be provided by mathematical statements (i.e., lemmas and theorems) involving calculus. In other words, overall, students reason more successfully with conditional statements about calculus ideas than they do with purely abstract statements. However, patterns in the data suggest that this is not the case for tasks involving contrapositive statements. That is, students who possess a purely logical understanding of the contrapositive reasoning appear to have an advantage when approaching tasks involving the same conditional variation in the calculus context. It was also found that students could engage in example-generating strategies to successfully explore and explain the validity of calculus statements and to demonstrate an understanding of the inverse and converse reasoning tasks in calculus contexts.

These findings suggest a variety of learning opportunities instructors can provide to strengthen understanding of and fluency with logical implication. Here we provide four specific instructional suggestions using the theorem:

If a function is differentiable at point x, then it is continuous at point x.

Analogous examples can, of course, be generated based on other statements that have if–then structures (e.g., extrema-critical point definitions, Mean Value Theorem, Intermediate Value Theorem, etc.). We envision students doing these kinds of tasks as part of their first encounter with a calculus theorem, lemma or definition and then again throughout the course as they are exposed to more and more conditional statements.

Instructional strategy #1: Generate versions of a statement. The first type of instructional task is designed to give students opportunities to generate versions of statements based on the standard conditional variations. The goals here are to introduce (or reinforce) the structure of a conditional and its variations and, as a result, increase students' abilities to recognize the variations of statements they may encounter or generate themselves. For example:

A) Using the theorem:

If a function is differentiable at point x, then it is continuous at point x,

as a starting point, write the converse, inverse and contrapositive variations of this statement. Recall that if you start with a statement of the form A implies B, the converse is B implies A, the inverse is not A implies not B, and the contrapositive is not B implies not A.

Instructional strategy #2: Use the contrapositive. Our findings suggest that in drawing conclusions related to the contrapositive of a statement, students are successful when they understand this variation in abstract, formal ways. Recall that students who correctly answered the abstractly presented contrapositive tasks were more likely to be successful in their reasoning about the contrapositive task of the given calculus theorem. This suggests that there is value in providing students with opportunities to develop (or strengthen) their understanding of why the contrapositive of a true statement is also true. We suggest that students be given opportunities to see and work with the proof for the contrapositive. For example:

B) Theorem:

For all functions f, if f is differentiable at $x = c$*, then it is continuous at* $x = c$*.*

What we know: g is not continuous at $x = c$.

Show that g must not be differentiable at $x = c$ by first assuming that g is differentiable at $x = c$ and deriving a contradiction.

By assuming that g is differentiable at $x = c$, the given theorem would allow the student to find that g is continuous at $x = c$. However, this contradicts the given fact that g is not continuous at $x = c$. Therefore, g is not differentiable at $x = c$.

These next two strategies are extensions of the previous activity in the sense that we are having students evaluate an entire conditional statement, as opposed to simply the antecedent or consequent. Here the focus is on developing students' abilities to draw conclusions based on examples and in this case, we have students make judgements about a complete statement of the sort they encounter in their work in calculus.

Instructional strategy #3: Generate examples. Students in the study who were successful in determining the conclusions of statements given a particular premise often did so by generating examples. This allowed them to examine whether there were instances when a consequent of a statement could be true and other instances when it could be false. If contradictory observations were made, they were able to determine that the truth or falsity of the statement's conclusion could not be decided.

We suggest that instructors raise students' awareness about the value of this approach so that all students are armed with this productive strategy. One way to provide these learning opportunities is to have students work on tasks that scaffold the process (extensively at first and then less so) used successfully by students in our study. Two versions of this approach are described here. In the first version, students are presented with a true statement and a variation of it based on either the inverse or converse of the statement. For scaffolding purposes, they are told that the variant of the statement is NOT true in general and instructed to generate two examples to illustrate this. For example:

C) Consider this theorem:

If a function is differentiable at point x, then it is continuous at point x.

The following statement is NOT true in general:

If a function is continuous at point x, then it is differentiable at point x.

You are now going to produce a pair of examples to show that this statement may be true in certain instances, but not true in general. Create an example of a function that is continuous at point x and is also differentiable at point x. Now create another example of a function that is continuous at point x but is NOT differential at point x.

Another version of this type of task entails presenting students with an initial true statement as well as a variant of the statement, prompting them to generate examples, and having them determine what those examples mean for the general truth of the variant statement. For example:

D) Consider this theorem:

If a function is differentiable at $x = a$*, then it is continuous at* $x = a$*.*

Here is a variation of this statement:

If a function is NOT differentiable at $x = a$*, then it is NOT continuous at* $x = a$*.*

Create an example of a function that is NOT differentiable at $x = a$ but IS continuous at $x = a$. Then create a second example function that is NOT differential at $x = a$ and is NOT continuous at $x = a$. What can you conclude about the statement:

If a function is NOT differentiable at $x = a$*, then it is NOT continuous at* $x = a$

based on your two examples?

Instructional strategy #4: Draw conclusions based on examples. A final type of task instructors can use to support student reasoning also follows from our finding about the prevalence of example generating as a strategy students used productively.

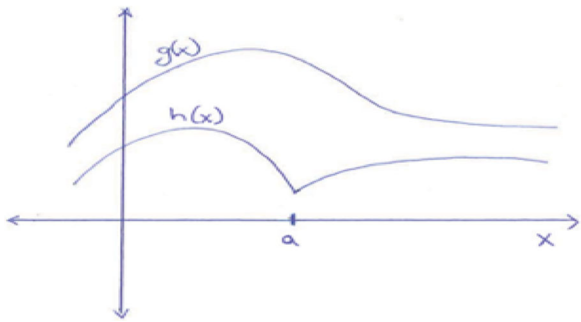

Figure 7. Contradictory examples for the converse task involving the differentiability implies continuity theorem

In these tasks, students are given a statement and a set of examples, and asked to determine whether the examples provide evidence that the statement is true in general or false. For example:

E) This problem makes use of this theorem:

If a function is differentiable at point x, then it is continuous at point x.

A friend of yours says the following is also true:

If a function is continuous at point x, then it is differentiable at point x.

Examine the graphs of function g and function h. Taken together, do the graphs in Figure 7 provide evidence that your friend is correct or incorrect? In other words, what can you conclude based on the graphs of g and h?

Findings from the present investigation provided insights into student performance and thinking about conditional statements, but examining additional research questions could enhance our understanding of these issues further. Here we provide a few suggestions for future research.

The fact that students who were successful with calculus-based reasoning tasks often used an "example-generating" strategy gives us hope that providing all students with opportunities to learn about these strategies will be valuable. However, determining the extent to which this is effective would be a valuable next step. In particular, do students' theorem-reasoning abilities improve if they engage in the type of tasks listed above?

Conditional reasoning situations are pervasive throughout mathematics but it remains to be seen if what we found in terms of performance and reasoning strategies with calculus students would also be apparent in other student populations. For example, would we find similar performance and reasoning strategies if we presented abstract and context-specific tasks to students in number theory, differential equations, abstract algebra or other courses? At what point during the sequence of mathematics major courses do students become able to reason equally well with abstract and context-specific statements and what supports that learning?

As noted earlier, findings from the contrapositive-structured tasks were different from the others. Although we hypothesize that this is because the nature of the contrapositive structure makes the example-generating strategy unproductive, investigations to reveal more about student thinking in these situations would be useful to further inform design of instructional tasks to help strengthen student understanding of contrapositive reasoning.

We also wonder what relationships, if any, exist among students' abilities to reason when the task is set in a non-calculus (but "real life") context, in a calculus context, and in the abstract. In this study, we examined relationships between student thinking in calculus contexts and the abstract. Other researchers have examined relationships between the "real life" and abstract contexts, but further insights might be gained from a study designed to examine all three simultaneously and the relationships among them.

A fruitful follow-up study might involve investigating how the teaching of logical reasoning in calculus affects students' ability to reason in abstract and calculus contexts. Is it true that students are better at reasoning in either the abstract or the calculus tasks (or both) if they are given explicit instruction about logical reasoning with theorems in class? Carrying out such a study might reveal ways in which the teaching of logic in relation to theorems may or may not affect students' ability to reason with the tasks given in the present research.

8. CONCLUSIONS

Logical reasoning is foundational to all of mathematics, and calculus is essential for students pursuing STEM majors. Unfortunately, not all students have had rich opportunities to study logical reasoning in secondary school, and thus may still be developing their understanding when they arrive at their study of calculus and the theorems they will eventually encounter. This also highlights the potential challenges for those students who go on to take further courses related to proof. Thus, the study of conditional reasoning in calculus may potentially help lay the groundwork for students who wish to move beyond calculation-based mathematics. Although these issues are difficult, from the findings of the research presented here, it appears that calculus students who struggle with formal, traditional inference tasks may still be quite able to draw appropriate logical conclusions when the statements are calculus-oriented. This finding provides encouragement to those who strive to create learning opportunities that build student understanding of calculus and expose students to the ideas of logical reasoning. If we aim for students to obtain a full, logical understanding of the results found in calculus, then finding ways to refine their abilities to comprehend and make sense of calculus theorems and their variants could help them reach this goal.

FUNDING

This work was supported in part by the Maine Academic Prominence Initiative through a grant to the Maine Center for Research in STEM Education at the University of Maine.

REFERENCES

1. Attridge, N., A. Aberdein, and M. Inglis. 2016. Does studying logic improve logical reasoning? In C. Csikos, A. Rausch, and J. Szitányi (Eds), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*. 2: 27–34. Szeged, Hungary: PME.
2. Bressoud, D. M., I. Ghedamsi, V. Martiniez-Luaces, and G. Törner (Eds). 2016. Teaching and learning of calculus. In G. Kaiser (Ed.), *Teaching and Learning of Calculus*. ICME-13 Topical Surveys. Cham, Switzerland: Springer.
3. Carlson, M. and C. Rasmussen. 2008. *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*. Washington, DC: Mathematical Association of America.
4. Holdren, J. P. and Lander, E. S. 2012. Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering, and Mathematics. Washington, DC: Executive Office of the President.
5. Hunting, R. 1997. Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior*. 16(2): 145–165.
6. O'Brien, T. C., B. J. Shapiro, and N. C. Reali. 1971. Logical thinking - language and context. *Educational Studies in Mathematics*. 4(2): 201–219.
7. Ragni, M., I. Kola, and P. N. Johnson-Laird. 2017. The Wason selection task: A meta analysis. In G. Gunzelmann, A. Howes, T. Tenbrink, and E. J. Davelaar (Eds), *Proceedings of the 32nd Annual Conference of the Cognitive Science Society*, pp. 980–985. Austin, TX: Cognitive Science Society.
8. Rasmussen, C., K. Marrongelle, and M. C. Borba. 2014. Research on calculus: What do we know and where do we need to go? *ZDM: Zentralblatt Für Didaktik Der Mathematik*. 46(4): 507–515.
9. Roh, K. H. 2008. Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*. 69(3): 217–233.
10. Rossi, S., M. Cassotti, S. Moutier, N. Delcroix, and O. Houdé. 2015. Helping reasoners succeed in the Wason selection task: When executive learning discourages heuristic response but does not necessarily encourage logic. *PLoS ONE*. 10(4): e0123024. doi:10.1371/journal.pone.0123024.
11. Sellers, M. E., K. H. Roh, and E. J. David. 2017. A comparison of calculus, transition-to-proof, and advanced calculus student quantifications in complex mathematical statements. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds), *Proceedings of the 20th Annual Conference on*

Research in Undergraduate Mathematics Education, pp. 285–297. San Diego, CA: Mathematical Association of America.

12. Strauss, A. and J. Corbin. 1990. *Basics of Qualitative Research: Grounded Theory Procedures and Techniques*. Newbury Park, CA: Sage.
13. Stylianides, A. J., G. J. Stylianides, and G. N. Philippou. 2004. Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics*. 55: 133–162.
14. Wason, P. C. 1966. Reasoning. In B. Foss (Ed.), *New Horizons in Psychology*, pp. 135–151. Harmondsworth, UK: Penguin Books.
15. Wason, P. C. 1968. Reasoning about a rule. *Quarterly Journal of Experimental Psychology*. 20(3): 273–281.
16. Wason, P. C. and D. Shapiro. 1971. Natural and contrived experience in a reasoning problem. *Quarterly Journal of Experimental Psychology*. 23(1): 63–71.
17. Yopp, D. A. 2017. Eliminating counterexamples: A Grade 8 students' learning trajectory for countrapositive proving. *The Journal of Mathematical Behavior*. 45: 150–166.

BIOGRAPHICAL SKETCHES

Joshua Case is a doctoral student at West Virginia University pursuing study in undergraduate mathematics education. As an undergraduate at the University of Maine Farmington, he studied both music and mathematics and pursued the study of musical composition. He also holds a Master of Arts in Mathematics and a Master of Science in Teaching, both from the University of Maine. His main research interest is related to undergraduate understanding of logic and proof. Josh is also interested in music theory and its relationship to algebra and combinatorics. In his spare time, he enjoys contemporary classical music and world cinema.

Natasha Speer is an associate professor in the Department of Mathematics & Statistics at the University of Maine. She has an undergraduate degree in mathematics from Cornell University and a Ph.D. in mathematics education from the University of California at Berkeley. The focus of her research is on the knowledge teachers need, build, and use to teach college mathematics and on professional development for novice college mathematics instructors. She spends her summers in Eastport, Maine where she can see Canada from her porch.