Changes in Topological Relations when Splitting and Merging Regions

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Abstract

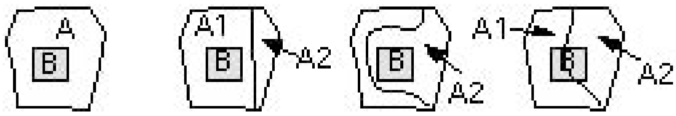
This paper addresses changes in topological relations as they occur when splitting a region into two. It derives systematically what qualitative inferences can be made about binary topological relations when one region is cut into two pieces. The new insights about the possible topological relations obtained after splitting regions form a foundation for high-level spatio-temporal reasoning without explicit geometric information about each object's shapes, as well as for transactions in spatio-temporal databases that want to enforce consistency constraints.

1 Introduction

Efforts in spatio-temporal modeling have significantly enhanced the computational capabilities of otherwise static models of geographic space. In recent years the primary focus has been on moving objects (Wolfson *et al.* 1998), emphasizing point-like representations of objects and their trajectories. These investigations have led to a plethora of methods for querying and indexing of space-time samples as they are stored in and retrieved from spatio-temporal databases (Güting and Schneider 2005; Pfoser and Jensen 2003). Methods for making higher-level inferences about changes to spatial configurations, however, have been confined to objects that retain their identity over time, considering such changes as movement, rotation, expansion, and shrinking (Egenhofer and Al-Taha 1992).

More complex changes have been addressed at the level of the identity of objects (Hornsby and Egenhofer 1998), covering the splitting of objects into several autonomous pieces, the spawning off of parts from a continuing entity, the merging of several items into a unified object, or an item joining a collection. When such identity changes occur with respect to spatial objects these changes imply not only modifications at the level of the individuals' identities, but also involve spatial changes. Few considerations, however, have been given to the spatial ramifications of such spatio-temporal change operations, for instance topological changes when merging regions (Clementini *et al.*, 1995; Tryfona and Egenhofer 1997) or by introducing holes into regions (Egenhofer *et al.*, 1994).

This paper addresses changes in topological relations as they occur when splitting an object into two pieces. For example, when subdividing a land parcel with a building on it into two pieces, there are several possibilities for the building to be located with respect to the two newly created land parcels (Fig. 1). Unless the exact location of the newly introduced boundary is known, the actual situation is one among several choices. Such inferences without graphical or detailed geometric information typically occur when analyzing and reasoning with verbal descriptions.



(a) (b) (c) (d)

Fig. 1. Three scenarios of subdividing land parcel A into two, A1 and A2, such that building B has a different topological relation with respect to the two subdivisions, A1 and A2: (a) A1 *contains* B and A2 is *disjoint* from B; (b) A1 is *disjoint* from B and A2 *contains* B; and (c) A1 *overlaps* B and A2 *overlaps* B.

A comprehensive understanding of all possible topological configurations would provide a basis for making temporal inferences about spatial relations, which may yield interesting, high-level information without the need of information about the actual geometric representations and, therefore, supports qualitative spatio-temporal reasoning. The inferences about the changes in topological relations are also critical in transactions so that one can assess whether a particular change was performed consistently with the operation's semantics.

The remainder of this paper is organized as follows: Section 2 summarizes the model used for describing binary topological relations as well as the inference mechanisms available for dealing with spatial objects that do not change their identities. Section 3 defines splitting and introduces the process used for deriving the set of topological relations that holds after splitting a region into two regions. Sections 4 and 5 determine potential and feasible relations, respectively, the results of which are integrated into achievable splitting configurations (Section 6). Section 7 draws conclusions and stimulates future work.

2 Binary Topological Relations between Regions

A *region* is a non-empty proper subset of a connected topological space such that the region's interior is connected and the region is identical to the closure of the region's interior. Each region is closed, bounded, homogeneously two-dimensional, and homeomorphic to a 2-disk. For pairs of such regions embedded in $\mathbb{R}^2$ a set of eight binary topological relations has been identified whose elements are mutually exclusive and provide a complete coverage between any two regions, that is, there holds exactly one of the eight topological relations (Egenhofer and Franzosa 1991). Their semantics are captured by the 4-intersections (Equations 1a-1i) among the two regions' interiors (A° and B°) and boundaries (∂A and ∂B). The regions' exteriors (denoted by A^- and B^-) capture their regions' complements (i.e., $\mathbb{R}^2 \setminus (A^\circ \cup \partial A)$ and $\mathbb{R}^2 \setminus (B^\circ \cup \partial B)$, respectively).

$$A\ \textit{disjoint}\ B\text{:}\ A^\circ \cap B^\circ = \varnothing \ \wedge\ \partial A \cap \partial B = \varnothing \tag{1a}$$

$$A\ \textit{meet}\ B\text{:}\ A^\circ \cap B^\circ = \varnothing \ \wedge\ \partial A \cap \partial B = \neg\varnothing \tag{1b}$$

$$A\ \textit{equal}\ B\text{:}\ A^\circ \cap B^\circ = \neg\varnothing \ \wedge\ \partial A \cap \partial B = \neg\varnothing \ \wedge \\ A^\circ \cap \partial B = \varnothing \ \wedge\ \partial A \cap B^\circ = \varnothing \tag{1c}$$

$$A\ \textit{overlap}\ B\text{:}\ A^\circ \cap B^\circ = \neg\varnothing \ \wedge\ \partial A \cap \partial B = \neg\varnothing \ \wedge \\ A^\circ \cap \partial B = \neg\varnothing \ \wedge\ \partial A \cap B^\circ = \neg\varnothing \tag{1d}$$

$$A\ \textit{inside}\ B\text{:}\ A^\circ \cap B^\circ = \neg\varnothing \ \wedge\ \partial A \cap \partial B = \varnothing \ \wedge \\ A^\circ \cap \partial B = \varnothing \ \wedge\ \partial A \cap B^\circ = \neg\varnothing \tag{1e}$$

$$A\ \textit{contains}\ B\text{:}\ A^\circ \cap B^\circ = \neg\varnothing \ \wedge\ \partial A \cap \partial B = \varnothing \ \wedge \\ A^\circ \cap \partial B = \neg\varnothing \ \wedge\ \partial A \cap B^\circ = \varnothing \tag{1f}$$

$$A\ \textit{covers}\ B\text{:}\ A^\circ \cap B^\circ = \neg\varnothing \ \wedge\ \partial A \cap \partial B = \neg\varnothing \ \wedge \\ A^\circ \cap \partial B = \neg\varnothing \ \wedge\ \partial A \cap B^\circ = \varnothing \tag{1g}$$

$$A\ \textit{coveredBy}\ B\text{:}\ A^\circ \cap B^\circ = \neg\varnothing \ \wedge\ \partial A \cap \partial B = \neg\varnothing \ \wedge \\ A^\circ \cap \partial B = \varnothing \ \wedge\ \partial A \cap B^\circ = \neg\varnothing \tag{1h}$$

U is the universal relation {*disjoint, meet, overlap, inside, covers, contains, coveredBy, equal*} and $topRel \in U$. If several topological relations are referred to, they are distinguished by indices $topRel_i$, $topRel_j$, etc. The set of eight topological region-region relations also enables qualitative spatial reasoning in the form of the *composition* of relations, that is, given a pair of topological relations $A\ topRel_i\ B$ and $B\ topRel_j\ C$ the composition derives candidates for the topological relation $topRel_k$ between *A* and *C*

(Egenhofer 1994). With the composition table—the complete set of all possible compositions among the eight topological relations—one can make topological inferences among the set of regions of a spatial configuration using constraint propagation techniques (Egenhofer and Sharma 1993; Smith and Park 1992).

3 Splitting a Region into Two Regions

Splitting a region into two regions is defined in terms of the outcome of a geometric operation. A region *A* is *split* into two parts such that each part is a region as well and that the two parts *meet* (Figs. 2a-d). Such splitting may be achieved by cutting *A* into two pieces with a non-self-intersecting simple line starting at a point in *A*'s boundary and extending through *A*'s interior back to a different point in *A*'s boundary than the starting point. This type of splitting excludes related operations, such as creating a hole in a region by cutting out an island (Fig. 2e), or partitioning the region into more than two parts (Fig. 2f). Regions with holes are known to fall into a different setting beyond simply connected spatial regions and their eight basic topological relations (Egenhofer *et al.*, 1994). Likewise, splitting excludes a separation of the two parts by inserting a non-linear object, as it might be introduced when a flooded river ploughs through some terrain, carving out another extended spatial object. Subsequently, let A_1 and A_2—the parts of *A*—be two regions such that A_1 *meets* A_2 and the union of A_1 and A_2 is *equal* to *A*.

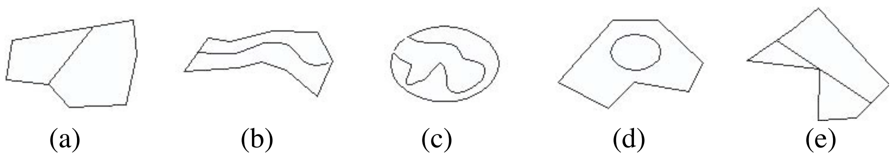


Fig. 2. Scenarios with (a-c) legally split regions and illegally split regions (d) due to the insertion of a hole and (e) due to splitting the region into more than two parts.

The topological relations after splitting a region into two regions are derived through three successive steps:

- Identifying *potential splitting configurations* that are based on the constraint that the two split regions must *meet* (Section 4). This first step is performed as a consistency check using the composition property of binary topological relations.

- Deriving systematically the set of *feasible splitting configurations* based on the propagation of empty and non-empty intersections from the to-be-split region to its parts (Section 5). This second step requires a detailed elimination process based on constraints of the split parts' interior, boundary, and exterior intersections with the to-be-split object.
- Integrating the results of potential and feasible splitting configurations into *achievable splitting configurations* (Section 6).

4 Potential Splitting Configurations

When splitting region A into two parts, A_1 and A_2, the topology with respect to region B (i.e., A $topRel_i$ B) is captured by two binary topological relations, A_1 $topRel_j$ B and A_2 $topRel_k$ B. The domain of these topological relations is the set of eight topological region-region relations; therefore, $8^3 = 512$ different combinations would be possible, among the 64 combinations of $topRel_j$ x $topRel_k$. When considering all of these combinations, however, one does not take into account any constraints imposed by the splitting requirement that the two parts, A_1 and A_2, must *meet* and that both A_1 and A_2 must be *coveredBy* A; therefore, the set of possible post-splitting configurations is smaller. For instance, A_1 *contains* B and A_1 *contains* B cannot be realized, after splitting A into A_1 and A_2, because this conjunction is inconsistent with the constraint that A_1 *meets* A_2. On the other hand, A_1 *inside* B and A_2 *inside* B would be consistent with A_1 *meets* A_2.

We define *potential* relations as those that can be obtained by applying systematically a constraint satisfaction algorithm over the network of all binary topological relations among the regions A, A_1, A_2, and B (Egenhofer and Sharma 1993). Such constraint satisfaction enforces converse relations (through the arc consistency constraint) and, along paths in the network, ensures that inconsistencies based on the relations' compositions are eliminated. This approach implies that the set of binary topological relations that hold between each pair of each region is *equal* to itself; A *covers* A_1 and A_2; A_1 *meets* A_2; the unknown relations with B are the universal relation U, and converse relations are used consistently (Fig. 3).

By replacing iteratively the universal relations U from A to B, from A_1 to B, and from A_2 to B with one concrete relation out of the set of eight topological relations, such that B $topRel_l$ A is converse to A $topRel_i$ B, B $topRel_m$ A_1 is converse to A_1 $topRel_j$ B, and A_2 $topRel_n$ B is converse to B $topRel_k$ A_2 (in order to satisfy the arc consistency constraint), one can perform a consistency check for all possible configurations, eliminating inconsistent and, therefore, impossible configurations.

	A	A_1	A_2	B
A	*equal*	*covers*	*covers*	U
A_1	*coveredBy*	*equal*	*meet*	U
A_2	*coveredBy*	*meet*	*equal*	U
B	U	U	U	*equal*

Fig. 3. The sixteen topological relations between region A, its split parts A_1 and A_2 and another region B.

Whenever the path consistency constraint generates an empty relation, the configuration is *impossible*; however, the converse inference of possible configurations from a consistent network of binary topological relations does not always hold true (Papadimitriou *et al*. 1999); therefore a non-empty relation as the result of the path consistency constraint confirms that a particular configuration is a *potential* topological relation after splitting A into A_1 and A_2 (Fig. 4).

A_1 *topRel$_j$* B	potential topological relations for A_2 *topRel$_k$* B
disjoint	*disjoint* ∨ *meet* ∨ *overlap* ∨ *covers* ∨ *contains*
meet	*disjoint* ∨ *meet* ∨ *overlap* ∨ *covers* ∨ *coveredBy* ∨ *equal*
overlap	*inside* ∨ *coveredBy* ∨ *overlap* ∨ *meet* ∨ *disjoint*
coveredBy	*inside* ∨ *coveredBy* ∨ *overlap* ∨ *meet*
inside	*disjoint* ∨ *meet*
covers	*disjoint* ∨ *meet*
contains	*disjoint*
equal	*meet*

Fig. 4. Potential topological relations for the parts A_1 and A_2 with respect to B.

5 Feasible Splitting Configurations

Splitting a region into two parts requires the introduction of a new line, which extends from the boundary of the region, through its interior, to a point in the boundary. This line implies that some properties of the topological relations of the split regions can be derived from the topological properties before splitting. These properties rely primarily on the intersections of the interiors and boundaries of the to-be-split region and, therefore, trigger propagations of empty and non-empty interior, boundary, and exterior intersections from the to-be-split region to the parts (Sections 5.1-5.3). Since the newly introduced boundary runs through the to-be-split region's interior, corrective measures must be taken to account for the introduction of the corresponding boundary intersections (Section 5.4).

5.1 Interior Propagations

A's interior, $A°$, has three relations with respect to *B* and its parts:

R1: $A°$ is a subset of $B°$ ($A° \subseteq B°$).

R2: $A°$ is a true subset of *B*'s exterior ($A° \subset B^-$).

R3: $A°$ has non-empty intersections with all three parts of *B* ($A° \cap B° \neq \emptyset \wedge A° \cap \partial B \neq \emptyset \wedge A° \cap B^- \neq \emptyset$).

These three relations cover all possible cases and no other scenarios need to be considered. For instance, because a region's boundary has no extent it cannot contain the non-empty interior or non-empty exterior of another region ($A° \not\subset \partial B$). Since the regions are embedded in $\mathbb{R}^2$, the interior of a region cannot coincide with the exterior of another region ($A° \neq B^-$). Finally, if the interior of a region *A* contains another region's interior *B*, this implies that *A*'s interior has non-empty intersections with all parts of *B* ($A° \supset B° \Rightarrow A° \cap B° \neq \emptyset \wedge A° \cap \partial B \neq \emptyset \wedge A° \cap B^- \neq \emptyset$), therefore, this last scenario is covered by R3. The three relations with respect to *A*'s interior give rise to Theorems 1-3.

Theorem 1: $A° \subseteq B° \Rightarrow A_1° \subset B° \wedge A_2° \subset B°$

Proof: This follows from the definition of a subset (i.e., all parts of a contained connected set are also subsets of the containing set). Since $A_1° \subseteq A°$ and $A_2° \subseteq A°$, $A_1°$ and $A_2°$ are transitively contained in everything in which $A°$ is contained. □

Theorem 2: $A° \subset B^- \Rightarrow A_1° \subset B^- \wedge A_2° \subset B^-$

Proof: In analogy to the proof of Theorem 1, substituting $B°$ with B^-. □

Theorem 3: $A° \cap B° \neq \emptyset \wedge A° \cap \partial B \neq \emptyset \wedge A° \cap B^- \neq \emptyset \Rightarrow$

$$(A_1° \subset B° \wedge A_2° \subset B^-) \vee$$
$$(A_1° \subset B° \wedge A_2° \cap B° \neq \emptyset \wedge A_2° \cap \partial B \neq \emptyset \wedge A_2° \cap B^- \neq \emptyset) \vee$$
$$(A_1° \subset B^- \wedge A_2° \subset B°) \vee$$
$$(A_1° \subset B^- \wedge A_2° \cap B° \neq \emptyset \wedge A_2° \cap \partial B \neq \emptyset \wedge A_2° \cap B^- \neq \emptyset) \vee$$
$$(A_1° \cap B° \neq \emptyset \wedge A_1° \cap \partial B \neq \emptyset \wedge A_1° \cap B^- \neq \emptyset \wedge$$
$$A_2° \cap B° \neq \emptyset \wedge A_2° \cap \partial B \neq \emptyset \wedge A_2° \cap B^- \neq \emptyset)$$

Proof: When *A*'s interior has a non-empty intersection with all three parts of *B*, then five constellations for the split interiors ($A_1°$ and $A_2°$) are possible: (1) $A_1°$ is completely contained in $B°$ and $A_2°$ is completely contained in the other extended part of *B* (i.e., B^-) such that

$A^\circ \setminus A_1{}^\circ \setminus A_2{}^\circ = \partial B \cap A^\circ$, which is non-empty; $A_1{}^\circ$ is completely contained in B° and $A_2{}^\circ$ has non-empty intersections with all three parts of *B*; (3) reversing in (1) $A_1{}^\circ$ and $A_2{}^\circ$; (4) reversing in (2) $A_1{}^\circ$ and $A_2{}^\circ$; and (5) $A_1{}^\circ$ and $A_2{}^\circ$ both extend through all three parts of *B*. □

5.2 Boundary Propagations

Similar to the propagation of non-empty interior intersections, non-empty boundary intersections between the to-be-split region and the related region are also propagated to the split regions' parts. Relevant for this propagation from *A*'s boundary to region *B* is that *A*'s boundary ∂A has six relations with respect to the parts of *B*:

R4: ∂A is a true subset of B° ($\partial A \subset B^\circ$).
R5: ∂A is a true subset of *B*'s exterior ($\partial A \subset B^-$).
R6: ∂A is a subset of *B*'s boundary ($\partial A \subseteq \partial B$).
R7: ∂A has non-empty intersections with *B*'s interior and *B*'s boundary ($\partial A \cap B^\circ \neq \emptyset \;\wedge\; \partial A \cap \partial B \neq \emptyset$), but no intersection with *B*'s exterior ($\partial A \cap B^- = \emptyset$).
R8: ∂A has non-empty intersections with *B*'s exterior and *B*'s boundary ($\partial A \cap B^- \neq \emptyset \;\wedge\; \partial A \cap \partial B \neq \emptyset$), but no intersection with *B*'s interior ($\partial A \cap B^\circ = \emptyset$).
R9: ∂A has non-empty intersections with all three parts of *B* ($\partial A \cap B^\circ \neq \emptyset$ $\wedge\; \partial A \cap \partial B \neq \emptyset \;\wedge\; \partial A \cap B^- \neq \emptyset$).

Other set-theoretic combinations of ∂A and *B*'s parts are not meaningful or would not yield further insights when splitting *A*. For instance, considering only the non-empty intersections of ∂A with *B*'s interior and *B*'s exterior ($\partial A \cap B^\circ \neq \emptyset \;\wedge\; \partial A \cap B^- \neq \emptyset$), while assuming that $\partial A \cap \partial B = \emptyset$ is impossible, because of the role of a region's boundary as a Jordan curve, the non-empty intersections of $\partial A \cap B^\circ \neq \emptyset$ and $\partial A \cap B^- \neq \emptyset$ imply that $\partial A \cap \partial B \neq \emptyset$ as well. These six relations with respect to *A*'s boundary give rise to Theorems 4-9.

Theorem 4: $\partial A \subset B^\circ \Rightarrow \partial A_1 \subset B^\circ \wedge \partial A_2 \subset B^\circ$
Proof: If the boundary of the to-be-split region *A* is fully contained in the interior of another region *B*, then the boundary of each split part (∂A_1 and ∂A_2) must be located in that region's interior (B°) as well. The newly introduced part of the boundary between A_1 and A_2 must be a subset of B°, because it falls into A°, which is a subset of B° at the same time as ∂A is a subset of B°. □

Theorem 5: $\partial A \subset B^- \Rightarrow \partial A_1 \subset B^- \wedge \partial A_2 \subset B^-$

Proof: In analogy to the proof of Theorem 4, substituting B° with B^-. □

Theorem 6: $\partial A \subseteq \partial B \Rightarrow$

$$\partial A_1 \cap \partial B \neq \emptyset \wedge \partial A_1 \cap B^\circ \neq \emptyset \wedge \partial A_2 \cap \partial B \neq \emptyset \wedge \partial A_2 \cap B^\circ \neq \emptyset$$

Proof: For region objects, $\partial A \subseteq \partial B$ implies $\partial A = \partial B$, that is, when splitting A into A_1 and A_2, the boundaries ∂A_1 and ∂A_2 will both have non-empty intersections with ∂B. In addition, a newly introduced boundary part, which belongs to both A_1 and A_2 such that it separates $A_1{}^\circ$ from $A_2{}^\circ$, will need to extend through B°, yielding non-empty intersections of B° with respect to ∂A_1 and ∂A_2. □

Theorem 7: $\partial A \cap B^\circ \neq \emptyset \wedge \partial A \cap \partial B \neq \emptyset \wedge \partial A \cap B^- = \emptyset \Rightarrow$

$$(\partial A_1 \subset B^\circ \wedge \partial A_2 \cap B^\circ \neq \emptyset \wedge \partial A_2 \cap \partial B \neq \emptyset \wedge \partial A_2 \cap B^- = \emptyset) \vee$$

$$(\partial A_1 \cap B^\circ \neq \emptyset \wedge \partial A_1 \cap \partial B \neq \emptyset \wedge \partial A_1 \cap B^- = \emptyset \wedge$$

$$\partial A_2 \cap B^\circ \neq \emptyset \wedge \partial A_2 \cap \partial B \neq \emptyset \wedge \partial A_2 \cap B^- = \emptyset)$$

Proof: If—after splitting A into A_1 and A_2— ∂A_1 is completely contained in B°, then, since ($\partial A_2 \subseteq \partial A \setminus \partial A_1$) ∂A_2 must have non-empty intersections with B's interior and B's boundary (i.e., $\partial A_2 \cap B^\circ \neq \emptyset$ and $\partial A_2 \cap \partial B \neq \emptyset$) no intersection with B's exterior ($\partial A_2 \cap B^- = \emptyset$). Conversely, if ∂A_1 is not contained in B° then both ∂A_1 and ∂A_2 must extend through B's interior and boundary, but not through B's exterior. □

Theorem 8: $\partial A \cap B^- \neq \emptyset \wedge \partial A \cap \partial B \neq \emptyset \wedge \partial A \cap B^\circ = \emptyset \Rightarrow$

$$(\partial A_1 \subset B^- \wedge \partial A_2 \cap B^- \neq \emptyset \wedge \partial A_2 \cap \partial B \neq \emptyset \wedge \partial A_2 \cap B^\circ = \emptyset) \vee$$

$$(\partial A_1 \cap B^- \neq \emptyset \wedge \partial A_1 \cap \partial B \neq \emptyset \wedge \partial A_1 \cap B^\circ = \emptyset \wedge$$

$$\partial A_2 \cap B^- \neq \emptyset \wedge \partial A_2 \cap \partial B \neq \emptyset \wedge \partial A_2 \cap B^\circ = \emptyset)$$

Proof: In analogy to the proof of Theorem 7, exchanging B° and B^-. □

Theorem 9: $\partial A \cap B^\circ \neq \emptyset \wedge \partial A \cap \partial B \neq \emptyset \wedge \partial A \cap B^- \neq \emptyset \Rightarrow$

$$(\partial A_1 \cap B^\circ \neq \emptyset \vee \partial A_2 \cap B^\circ \neq \emptyset) \wedge$$

$$(\partial A_1 \cap \partial B \neq \emptyset \vee \partial A_2 \cap \partial B \neq \emptyset) \wedge$$

$$(\partial A_1 \cap B^- \neq \emptyset \vee \partial A_2 \cap B^- \neq \emptyset)$$

Proof: If $\partial A \cap B^\circ \neq \emptyset$ then it is impossible that $\partial A_1 \cap B^\circ = \emptyset$ and $\partial A_2 \cap B^\circ = \emptyset$, which is equivalent to $\partial A_1 \cap B^\circ \neq \emptyset \vee \partial A_2 \cap B^\circ \neq \emptyset$. The other three implications can be found accordingly by replacing B° with ∂B and B^-, respectively. □

5.3 Exterior Propagations

A's exterior has four relevant relations R10–R13 to A's and B's parts. R13 is a stronger statement than R11, but yields additional inferences. Likewise, R10 and R12 may coincide with R13, but there are configurations in which only R10 and R12 hold, but not R13. These four relations with A's exterior yield Theorems 10–13.

R10: A^- has a non-empty intersection with B°.
R11: A^- has a non-empty intersection with B^-.
R12: A^- has a non-empty intersection with ∂B.
R13: A^- is a superset of B^- ($A^- \supseteq B^-$).

Theorem 10: $A^- \cap B^\circ \neq \varnothing \Rightarrow A_1^- \cap B^\circ \neq \varnothing \wedge A_2^- \cap B^\circ \neq \varnothing$

Proof: Splitting A into A_1 and A_2 implies that $A_1^- \supset A_1$ and $A_2^- \supset A_2$. Also $A_1^- \supset A^-$ and $A_2^- \supset A^-$. Therefore, $B^\circ \cap A^- \neq \varnothing$ and $A^- \subset A_1^-$ implies $B^\circ \cap A_1^- \neq \varnothing$. Likewise, $A^- \subset A_2^-$ implies $B^\circ \cap A_2^- \neq \varnothing$. □

Theorem 11: $A^- \cap \partial B \neq \varnothing \Rightarrow A_1^- \cap \partial B \neq \varnothing \wedge A_2^- \cap \partial B \neq \varnothing$

Proof: In analogy to the proof of Theorem 10, substituting B° with ∂B. □

Theorem 12: $A^- \cap B^- \neq \varnothing \Rightarrow A_1^- \cap B^- \neq \varnothing \wedge A_2^- \cap B^- \neq \varnothing$

Proof: In analogy to the proof of Theorem 10, substituting B° with B^-. □

Theorem 13: $A^- \supseteq B^- \Rightarrow A_1^- \supset B^- \wedge A_2^- \supset B^-$

Proof: Splitting A into A_1 and A_2 implies that $A_1^- \supset A_1$. By transitivity $A_1^- \supset A_1 \supseteq B^- \Rightarrow A_1^- \supset B^-$. Substituting A_1^- with A_2^- it follows $A_2^- \supset B^-$. □

5.4 Boundary Overwrite

When splitting a region into two regions a new piece of boundary is introduced that must be connected to the to-be-split region's boundary and must run through its interior until it reaches the boundary again. Therefore, non-empty intersections of the to-be-split object's interior may overwrite empty boundary intersections of the copied boundary intersections.

Theorem 14: $A^\circ \subseteq B^\circ \Rightarrow \partial A_1 \cap B^\circ \neq \varnothing \wedge \partial A_2 \cap B^\circ \neq \varnothing$

Proof: The added boundary is part of A's interior. If A's interior is completely contained in some other component, then that component must intersect with the newly added boundary, which belongs to both A_1 and A_2; therefore, their boundaries must intersect with that component. □

Theorem 15: $A^\circ \subseteq B^- \Rightarrow \partial A_1 \cap B^- \neq \emptyset \wedge \partial A_2 \cap B^- \neq \emptyset$

Proof: In analogy to the proof of Theorem 14, substituting B° with B^-. □

6 Achievable Splitting Configurations

The feasible splitting configurations yield a set of pairs of candidate relations that might hold between the two split objects, depending on the relation that the to-be-split region held prior to splitting. Among these candidate sets, only those relations are *achievable* that lead to *potential* (Section 4) ***and*** *feasible* (Section 5) splitting configurations. We derived systematically those patterns of topological relations that fulfill Theorems 1–15. The stepwise elimination process leads to 21 achievable relations, each of which was confirmed by generating an example drawing (Figs. 5–12). The stepwise elimination also enabled us to confirm that all sixteen theorems were necessary and no combination of a subset of these theorems would yield the same result as one of the sixteen theorems.

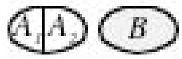
A_1 *disjoint B*	A_2 *disjoint B*	

Fig. 5. Achievable splitting relations for *A disjoint B*.

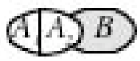
A_1 *disjoint B*	A_2 *meet B*	
A_1 *meet B*	A_2 *meet B*	

Fig. 6. Achievable splitting relations for *A meet B*.

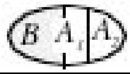
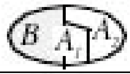
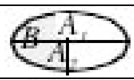
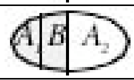
A_1 *covers B*	A_2 *disjoint B*	
A_1 *covers B*	A_2 *meet B*	
A_1 *equal B*	A_2 *meet B*	
A_1 *overlap B*	A_2 *overlap B*	
A_1 *coveredBy B*	A_2 *overlap B*	

Fig. 7. Achievable splitting relations for *A covers B*.

A_1 *disjoint B*	A_2 *overlap B*	
A_1 *meet B*	A_2 *overlap B*	
A_1 *meet B*	A_2 *coveredBy B*	
A_1 *overlap B*	A_2 *overlap B*	
A_1 *overlap B*	A_2 *coveredBy B*	
A_1 *overlap B*	A_2 *inside B*	

Fig. 8. Achievable splitting relations for *A overlap B*.

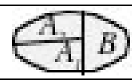
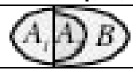
A_1 *coveredBy B*	A_2 *coveredBy B*	
A_1 *coveredBy B*	A_2 *inside B*	

Fig. 9. Achievable splitting relations for *A coveredBy B*.

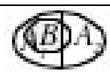
A_1 *contains B*	A_2 *disjoint B*	
A_1 *covers B*	A_2 *meet B*	
A_1 *overlap B*	A_2 *overlap B*	

Fig. 10. Achievable splitting relations for *A contains B*.

A_1 *coveredBy B*	A_2 *coveredBy B*	

Fig. 11. Achievable splitting relations for *A equal B*.

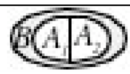
A_1 *inside B*	A_2 *inside B*	

Fig. 12. Achievable splitting relations for *A inside B*.

This set of 21 splitting configurations enables a new sort of qualitative spatial reasoning about change from successive snapshots. For instance, with the knowledge that at some time *t1* three regions *X*, *Y*, and *Z* have the topological relations *X contains Z* and *Y disjoint Z*, then *X* and *Y* could have resulted from splitting region *W* into *X* and *Y* (Fig. 10) and at an earlier time *t0*, prior to splitting, *W* would have *contained Z*. The cumulative inferences from Figs. 5–12 show that such inferences about the pre-splitting relation of *X* to *Y* are typically unique, except for four ambiguous cases: (1) *X overlaps Z* and *Y* overlaps *Z* leads to *W overlaps*, *contains*, or *covers Z*; (2) *X covers Z* and *Y meets Z* leads to *W covers* or *contains Z*; (3) *X coveredBy Z* and *Y coveredBy Z* leads to *W equal* or *coveredBy Z*; and (4) *X coveredBy Z* and *Y overlaps Z* leads to *W overlaps* or *covers Z.*

7 Conclusions

We have derived the set of binary topological relations that may hold for each part if one splits a region into two region parts. Constraint satisfaction, establishing an arc-consistent and path-consistent network of topological relations, lead to a set of potential relations. An elimination process then propagated interior, exterior, and boundary properties from the to-be-split region to its parts, yielding feasible relations. The combination of potential and feasible relations led to 21 configurations that may occur for such a region-splitting process, which enables qualitative spatio-temporal reasoning from sequences of snapshots.

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