Interdependence among material objects and voids

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Abstract. Material-spatial interdependence (mat-dep) is a type of dependence in which the physical extents of two entities are necessarily and mutually contingent, e.g. an object and its matter, or a hole and its host. Such dependence is commonly found amongst arrangements of physical entities, particularly in models of the natural environment. In this paper, we analyze and formally characterize mat-dep, and show how it augments the physical characterization of the containment, constitution, and hosting relations, primarily for development of a hydro ontology.

Keywords. formal ontology, first-order logic, spatial ontology, physical space, hydro ontology, ontological dependence, interdependence, matter, constitution, mereotopology, void, hole

1. Introduction

Two physical entities are in a physical dependence relation when they are intrinsically linked physically: they share matter, they are bound by gravitational forces, they are interlocked, or for some other similar reason. Physical dependence in this sense helps explain the material, spatial, and temporal arrangement of physical entities. Such arrangements are at the heart of representations of the natural environment as found in scientific models or associated data standards. Ontologies that support use of these representations, e.g. for various e-Science activities, should then account for physical dependence.

Of particular concern here are physical arrangements in hydrology utilizing the containment, constitution, and void-hosting relations, which are often inter-related. For example, while it is accurate to say that a subsurface water body is contained within a rock body, scientific requirements often demand a finer expression, one that specifies that the water body is contained in the spaces hosted by the rock body’s constituting matter, or hosted by the rock grains that itself constitute this matter. There are clearly physical dependencies here: between the rock body and its matter and the matter’s constituents, and between the matter and the spaces it hosts (assuming, of course, separate distinct identities for the rock body and its matter). Likewise, physical independence is also evident between the water body and both the rock body and its matter. A principled physical account of such relations, one that incorporates physical (in)dependence, is required to disentangle these nuances. However, such physical characteristics are largely absent
from existing ontologies, as the relations are typically expressed in metaphysical terms related to existence or essence, or via purely spatial relations such as mereotopological. Therefore, to enhance their physical characterization we propose a much narrower version of physical dependence: (1) only material endurants and immaterial voids (i.e. holes and gaps) can participate, and (2) the participants must share matter or voids (i.e. be made of, or host, the same (im)material stuff at the same time and place) such that if the shared stuff is different in one then it must be different in the other, unavoidably. We call this notion of physical dependence \textit{material-spatial interdependence, or mat-dep}. To our knowledge, this version of dependence, including its association with other foundational relations, is not yet characterized in a formal ontology despite its importance to virtually all physical settings.

In previous work, we outline mat-dep as a minor component of full physical containment and the direct constitution of an object by its matter [9, 10]. In this paper we significantly extend mat-dep, and express its interaction with the foundational relations of physical containment, generic constitution, and hosting a void, thus enhancing their physical characterization. The paper makes the following original contributions: (1) it identifies and describes the mat-dep relation; (2) it orients participants of this relation within the DOLCE ontology, and (3) it provides a first-order logical theory for mat-dep.

The paper is organized as follows: Section 2 describes related work; Section 3 introduces background concepts; Section 4 describes mat-dep spatially, while Section 5 describes it materially; and Section 6 concludes with a brief summary and some implications.

2. Related Work

Mat-dep is one of a handful of tightly interwoven relations that are foundational to the representation of physical entities. Included among these are physical containment, constitution, hosting, ontological dependence, and general physical dependence. However, prior ontological engineering of these relations either does not capture the physical dependence between participants, as they are defined metaphysically or mereotopologically, or the representation is not fully developed or too broad for mat-dep.

\textit{Containment} is the enclosure of one physical entity by another, and it can be fundamentally delineated into dependent and independent types according to whether the physical arrangement is necessary or accidental [10], e.g. a rock is accidentally enclosed by a lake, but the lake's hollow in the ground is necessarily enclosed by the ground surface. Dependent containment between two material entities requires them to made of the same matter at the same time in the same place, hence removing or adding such shared matter will physically affect both. In dependent containment between a void and its material host, growth or reduction in one requires a reciprocal alteration in the other. In dependent containment between two overlapping voids with a common host, both voids would be mutually affected by a change to their overlapping part, e.g. a canyon and the lake hollow at its bottom. As this shows, necessary physical dependence is vital to the characterization of dependent containment, but previous ontological engineering of physical containment provides an outline that is incomplete and imprecise [10]: for example, the permissible types and spatial dimensions of the participants are too broad and the physical dependence is indistinct.
Constitution is the relation of a physical object to its constituting matter, such as a statue to its clay matter. When these are seen as two distinct entities with separate identity, as in [15,21], the relation between them is special case of dependent containment in which they share the same material and spatial extents, and are thus necessarily materially dependent. Other constitution-like scenarios, which also exhibit dependent containment and necessary material dependence, involve relations in which the material extents of an entity and its matter do not fully coincide due to a shift in granular level, e.g. a statue and its clay minerals, or the statue’s clay matter and its clay minerals. The shift in granularity, from statue or clay matter to clay minerals, reveals tiny spaces between the mineral particles that are not voids in the statue nor its matter, hence accounting for the variation in (im)material extents across granular levels. This scenario is vital to proper handing of material granularity issues, for example, to ensure a water-bearing rock (or sponge) is physically independent from its contained water—otherwise drainage of the water would erroneously entail a loss of matter in the rock (or sponge). However, current formal ontologies for either type of constitution do not capture the notion of necessary physical dependence: same-extents constitution characterizes the dependence metaphysically and spatially [15], typically in terms of existence with full spatial co-location, and different-extents constitution is usually characterized using parthood without physical dependence of any kind [13, 16].

Hosting is the relation of a physical object to an intrinsic feature. The hosted features may be voids or physical parts (e.g. a bump on a rock), which satisfy the criteria for dependent containment and thus also mat-dep. Other hosted features, such as shadows and boundaries, might also exhibit physical dependence (though possibly not dependent containment), but these are out of scope for our purposes; only material hosts and features that are physical parts or voids are considered here. This form of hosting, like constitution, is characterized metaphysically [15] or mereotopologically [2] in related work, without any explicit physical dependency.

Ontological dependence is the relation in which an entity relies intrinsically on the other for metaphysical characteristics such as its existence or essence [5, 14]. Prototypical examples include same-extents constitution, such as a statue and its clay, and void-hosting, such a hole and its donut. In both cases the former could not exist, nor be the way the way they are, without the latter. The statue could not exist without the clay, nor without the clay grossly having a specific shape; and the hole could not exist without its host (e.g. a donut), nor without its host also broadly having the shape it does. Importantly, these examples of ontological dependence are also examples of mat-dep, but the two types of dependence only partially overlap: not all examples of one are examples of the other. For instance, a material object is not necessarily ontologically dependent on a material part, but they are necessarily materially dependent if they share matter: the statue can exist without its arm (under commonly held identity assumptions for such objects and even for some collections e.g. [13, 16, 18]), but the removal of a finger from the arm affects the material extents of both. Conversely, when a physical quality is held to be ontologically dependent on its bearer (as assumed in DOLCE [15]), such as the colour of a rock on the rock, then they are not necessarily materially dependent, because addition or removal of matter from the rock would not necessarily alter its colour (but might accidentally, e.g. with a coating of paint). Consequently, while mat-dep and ontological dependence overlap considerably, they are not identical, at least not under the assumptions made above. It follows that existing ontology engineering of ontological
dependence [12, 15, 19] cannot be directly reused for mat-dep here. But, in many (all?) physical cases mat-dep might complement core relations defined in terms of ontological dependence, such as same-extents constitution or void-hosting, providing an augmented physical characterization.

Physical dependence is a general dependence between physical entities. Its ontology engineering is manifest primarily in the representation of physical laws and processes [1, 4], but these representations are not directly transferable to mat-dep, because they either do not represent appropriate entities, e.g. most physical laws relate physical qualities rather than their bearers, or they involve causal, relational or functional necessity. For example, transfer of mass from an iceberg to the surrounding ocean causes material and spatial change in both due to physical causality, but they are not in mat-dep because the iceberg and ocean do not share matter at any given time. More relevantly, ideas similar to mat-dep are used to partially define a unifying criteria for objects in BFO [18], one reliant on physical causality, however the physical dependence is not explicitly recognized nor formalized, and the consideration of voids is minimal.

3. Axiomatic foundation

This work utilizes a fundamental distinction between physical endurants, the named objects in a physical world that we are ultimately interested in, and abstract regions of space, which are extended regions of space denoting the location of physical endurants. For the former we specialize the DOLCE foundational ontology [15], and for the latter we reuse the first-order spatial ontology from [8,11], which is a multidimensional generalization of the first-order mereotopology RCC (Region Connection Calculus) [3]. This section reviews the formalization of abstract space and the categories of material and void endurants from [8, 9] with some extensions in [10]. We also show how the material and void endurants fit into DOLCE’s category of physical entities, but without their time-indexing—we assume a static ontology throughout. We maintain notation and axiom number from [10], and refer the reader to the appropriate work [8–10] for the detailed axiomatizations. All free variables in our logical sentences are assumed to be implicitly universally quantified.

3.1. Theory of abstract space

Spatial regions (in short: regions; denoted by the class \(S\)) are topologically closed regions of abstract space obtainable by gluing together finite sets of manifolds with boundaries [8]. Regions are mathematical-geometrical abstractions that might not correspond to the location of any particular object. They can have arbitrary dimensions and are compared dimensionally using predicates \(<_{\text{dim}}, =_{\text{dim}}, \text{ and } \prec_{\text{dim}}\) to denote lower, equal, and next lower dimension, respectively. For mathematical simplicity, we assume that an empty region and a unique greatest region of maximal dimension, \(S_u\), exist (see [8,9,11] for detailed explanations). Then the regions of maximal dimension are mereologically closed: the intersection \(x \cdot y\), sum \(x + y\), and difference \(x - y\) of any two regions is again a region and every region has a complement \(x' = S_u - x\).

Regions can be in various spatial relationships, all defined in terms of the primitive spatial relation \(\text{Cont}(x, y)\), which is reflexive, antisymmetric, and transitive and denotes ‘\(x\) is a subregion (of equal or lower dimension) of \(y\)’. Additionally, we have a primitive relation \(\leq_{\text{dim}}\) that dimensionally compares regions [8,11]. Among the defined relations,
we have contact $C$ (sharing a region of any dimension), parthood $P$ (a subregion of equal dimension), proper parthood $PP$, overlap $PO$ (sharing a subregion of equal dimension), strong contact $C_S$ (sharing a subregion of the next lower dimension), and the class of internally connected (“one-piece”) regions $ICon$. All spatial relations only apply to regions. Two regions that overlap the same set of regions are considered equivalent. The reader is referred to [10] for the axioms and detailed explanations.

3.2. Material endurants and voids

Physical endurants are entities that exist in physical reality. They are either physical objects $POB$ (e.g. rock bodies), amounts of matter $M$ (e.g. rock matter or organic matter), or features $F$, the latter further delineated into relevant parts $RPF$ (e.g. a bump or a surface) and dependent places $DPF$. All physical endurants except for dependent places are material, denoted by $mat(x)$. Dependent places are inherently immaterial and not constituted by any matter. Voids $V$ are the only kind of dependent places we include here; their full characterization is found in [9]. Fig. 1 illustrates these central notions.

Following [6, 8, 9], we assign each physical endurant a location in abstract space using the function $r(x)$. The class of physical endurants $PED$ and of regions $S$ are disjoint, the former denoting real objects of interest and the latter denoting mathematical abstractions. The regions of both material endurants and voids are extended regions of codimension 0. For example, in a three-dimensional space all objects and voids are three-dimensional. Additionally, we use a primitive function $ch(x)$ to denote the convex hull of regions and physical endurants, see [3, 8] for details.

A void, $V(x)$, always depends on at least one non-void endurant as its host (denoted by the $hosts-v(y, x)$ relation), and is located within the convex hull of all its hosts. While voids may be hosted by other dependent places, e.g. shadows, we are only interested in voids hosted by material endurants in this work. Identifying voids is nontrivial and somewhat arbitrary, i.e., the difference $ch(x) - r(x)$ between a host’s convex hull and its own region may not be completely covered by voids. For example, the space between the base and the bulb of a wine glass is typically not called a void. Thus, $hosts-v$ is a primitive, i.e., undefined relation.

While macroscopic voids are hosted by a material object, microscopic voids are hosted by a constituent of the object at a lower level of granularity [9, 10]. Often, we are interested in the region of all the voids hosted by some constituent at same level of granularity within a physical endurant. We call this the entire void space of a physical endurant $x$, written as $voidspace_{all}(x)$ [10]. It is important to be aware of the fact that a material endurant and its entire void space may overlap, because the entire void space could include tiny voids in the endurant’s matter not apparent at the object level.
4. Analysis of material-spatial interdependence

Significantly, mat-dep is not accidental but necessary: valid examples include a rock body and its cavity (Fig. 2(d)), a rock body and the granite that constitutes it (Fig. 2(d)), as well as a canyon and the lake hollow at its bottom (Fig. 2(a)). Examples that are invalid include an object wedged into a hole (due to caprice), a person and their parents (causal dependence), lungs and some air (functional dependence), an addict and their addicted substance (physiological dependence) [17], and an iceberg and the ocean into which it melts, or a planet and its moon (relational dependence in physical laws). The valid examples illustrate that such dependence implies a material or spatial contingency between physical entities: they are made of, or host, some same matter or associated void at the same time and place, such that if one entity were to be physically different in this matter or void then so must be the other, unavoidably. For example, if the rock body did not host the cavity or was a different shape, then the cavity would not exist or would be a different shape; if the granite were not present or had a different shape, then so would the rock body, and likewise for the canyon and lake hollow.

4.1. Physical, material, and immaterial extent of physical endurants

In order to characterize mat-dep, the spatial extent of a physical endurant is defined as the sum of its material and immaterial regions. More precisely, it is the sum of the endurant’s region together with its entire void space (Ex-A1). The entire void space [9, 10] of an endurant encompasses the region of all the voids it hosts as well as all voids hosted by its entire matter (at any level of granularity). The spatial extent of a void is its region (Ex-T1), while the spatial extent of a material endurant is the subregion of its convex hull that includes its region (the material portion) and its void space (Ex-T2). For uniformity, we use the term material extent $\text{me}(x)$ to denote an endurant’s region occupied by its matter, which is equivalent to a material endurant’s region but is empty for voids (Ex-A2), and the term immaterial extent $\text{ve}(x)$ to denote an endurant’s region occupied by voids, which is equivalent to the entire void space of a material endurant (Ex-A3) or to a void’s region (Ex-A4). The material and immaterial extents of a material endurant may

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1Throughout the paper, axioms are labelled X-Ax, theorems X-Tx, and definitions X-Dx. Definitions may define a concept or relation only for a restricted domain.
overlap because its region and entire void space may overlap. However, the sum of the material and immaterial extents is always the spatial extent.

\[ \text{mat}(x) \lor \text{V}(x) \rightarrow e(x) = r(x) + \text{voidspace}_{\text{all}}(x) \] (the spatial extent of a material endurant or void is the sum of its region and the regions of all voids it hosts)

\[ \text{V}(x) \rightarrow e(x) = r(x) \] (a void’s extent is its region)

\[ \text{mat}(x) \rightarrow \text{P}(r(x), e(x)) \land \text{P}(e(x), \text{ch}(x)) \] (a material endurant’s extent contains its region and is a subregion of its own convex hull)

\[ \text{mat}(x) \rightarrow \text{me}(x) = r(x) \] (a material endurant’s material extent is its region)

\[ \text{mat}(x) \rightarrow \text{ve}(x) = \text{voidspace}_{\text{all}}(x) \] (a material endurant’s immaterial extent is its entire voidspace)

\[ \text{V}(x) \rightarrow \text{ve}(x) = r(x) \] (a void’s immaterial extent is its region)

4.2. Characterization of material-spatial interdependence

In more detail, but still informally, for two physical endurants (either material endurants or voids) to be in a mat-dep relation there must exist some region within the participants shared spatial extent that satisfies the following condition: (a) if it is a part of the material extent of both endurants, then decreasing one’s material extent by subtracting this region from its material extent equally decreases the second endurant’s material extent, and (b) if it is part of the immaterial extent of both endurants, then decreasing one’s immaterial extent by extending its hosts’ matter will equally decrease the other’s immaterial extent. Two physical endurants are thus mat-dep as long as one such region exists, regardless of whether other regions within the shared spatial extent of the participants violate this condition. This characterization covers the three main cases of mat-dep: two material endurants sharing matter, two voids (and their hosts) overlapping, or a void and a part of its host being in strong contact.

Formal characterization of mat-dep begins with our previous work on different kinds of containment relations between physical endurants [10], in which dependent containment is distinguished from detachable containment. Dependent containment in that context refers to the full enclosure of one physical endurant (the containee) by another physical endurant (the container) in some necessary, non-accidental way. Thus, dependent containment is differentiated from detachable containment primarily through a preliminary notion of physical interdependence dep [10], with a superficial formalization that requires the relation to be restricted to physical endurants (Dep-A1) and to be symmetric (Dep-A2), that is, if \( x \) is interdependent with \( y \) then \( y \) is also interdependent with \( x \). To better reflect this intended lack of directionality, we renamed the relation dep here to interdependence. We use the term interdependence in the weakest possible sense: the dependence is either bidirectional or only in one direction, though the direction cannot be inferred from the relation. Bidirectional dependence as in, e.g., [7], is not implied. We add the new axiom Dep-A6 that requires dep to be reflexive for all physical endurants.

\[ \text{fully-phys-contains}(y, x) \leftrightarrow \text{dep-contains}(y, x) \land \text{dep}(y, x) \] (dependent containment is generic containment where \( x \) and \( y \) are interdependent)

\[ \text{dep}(x, y) \rightarrow \text{PED}(x) \land \text{PED}(y) \] (dep is restricted to physical endurants)

\[ \text{dep}(x, y) \rightarrow \text{dep}(y, x) \] (dep is symmetric, i.e. undirected)

\[ \text{PED}(x) \rightarrow \text{dep}(x, x) \] (dep is reflexive for physical endurants)
We now characterize mat-dep as a narrower version of dep (MDep-A1) that involves only material endurants or voids (MDep-A2). As a specialization of dep, mat-dep is also symmetric (MDep-T1) and reflexive for material endurants and voids (MDep-T2).

(MDep-A1)  \( \text{mat-dep}(x, y) \rightarrow \text{dep}(x, y) \) (mat-dep is a kind of phys. interdependence)
(MDep-A2)  \( \text{mat-dep}(x, y) \rightarrow [\text{mat}(x) \lor \text{V}(x)] \land [\text{mat}(y) \lor \text{V}(y)] \) (mat-dep applies only to material endurants and voids)
(MDep-T1)  \( \text{mat-dep}(x, y) \rightarrow \text{mat-dep}(y, x) \) (mat-dep is a symmetric relation)
(MDep-T2)  \( \text{mat}(x) \lor \text{V}(x) \rightarrow \text{mat-dep}(x, x) \) (mat-dep is reflexive)

Since this study of mat-dep is motivated in part by a lack of precision in [10] regarding the description of physical interdependence used in full physical dependent containment, DepCont-D is now strengthened to DepCont-D’ by using \( \text{mat-dep} \) instead of \( \text{dep} \), to benefit from the increased precision.

(DepCont-D’)  \( \text{dep-contains}(x, y) \rightarrow \text{mat-dep}(x, y) \) (dependent physical containment requires mat-dep)

4.3. Spatial conditions for material-spatial interdependence

Through MDep-A2, mat-dep is limited to ‘bulky’ physical endurants (as opposed to bodiless ones, such as abstract boundaries; see [8, 20] for this distinction), which exhibit a spatial codimension of 0. It excludes, however, unintended dependencies between a material endurant and other dependent places such as abstract surfaces, which are lower-dimensional and as such have a codimension greater than 0 (MDep-T3).

Mat-dep imposes additional strict spatial constraints on the participants. First, the spatial extents of the participants must overlap (MDep-A3). Two bulky physical endurants that have no material or immaterial space in common are sufficiently apart in the sense that they are not spatially overlapping and thus cannot share matter and are, thus, not mat-dep in our narrow sense. Of course, they can still be physically related, for example, by exerting gravitational forces onto each other, or spatially related, for example, by one confining another (such as a pebble in a canyon).

(MDep-T3)  \( \text{mat-dep}(x, y) \rightarrow r(x) = \dim r(y) \land \text{MaxDim}(r(x)) \land \text{MaxDim}(r(y)) \) (only entities of equal spatial dimension and of codimension 0 can be mat-dep)
(MDep-A3)  \( \text{mat-dep}(x, y) \rightarrow \text{PO}(e(x), e(y)) \) (mat-dep requires the participants to have overlapping spatial extents)

However, overlap of spatial extents alone is insufficient for mat-dep. For example, a material endurant may be accidentally located in another material endurant’s voids, so that its removal does not necessarily materially affect either endurant. Even if two endurants’ material extents overlap, they may not be mat-dep; for example, consider again a water body located in the microscopic gaps of a rock body. To accommodate such cases where material endurants spatially overlap but are not mat-dep, we must weaken axiom Dep-A3 from [10], which equated physical interdependence with spatial overlap for material endurants, to Dep-A3’, in which spatial overlap is necessary but not sufficient for physical interdependence.

(Dep-A3’)  \( \text{mat}(x) \land \text{mat}(y) \rightarrow [\text{dep}(x, y) \rightarrow \text{PO}(r(x), r(y))] \) (material endurants that are physically interdependent must overlap)
A second necessary condition for mat-dep requires the participants to be strongly spatially connected, that is, the participants’ regions must either overlap (PO) or be in strong contact (CS). In the case of overlap, the regions must either properly overlap or one region must be a spatial part of the other. Strong contact means that the three-dimensional physical endurants we are interested in must share a surface, not just a curve or a set of isolated points, in order to participate in mat-dep. Notice that these conditions are imposed on the participant’s regions, which in the case of material endurants only encompasses their material extents.

\[(\text{MDep-A4}) \quad \text{mat-dep}(x, y) \rightarrow \text{PO}(r(x), r(y)) \lor \text{CS}(r(x), r(y)) \]  
(mat-dep occurs only between endurants that overlap or are in strong contact)

More specific kinds of mat-dep are definable if the strength of the spatial relation between the participants’ regions is used as discriminator: they are equivalent \(r(x) = r(y)\); one is a subregion of the other \(P(r(x), r(y))\) or \(P(r(y), r(x))\); they properly overlap \(\text{PO}(r(x), r(y)) \land \neg P(r(x), r(y)) \land \neg P(r(y), r(x))\); or they are in strong contact \(\text{CS}(r(x), r(y))\). The feasibility of these cases varies with the materiality of the participants. For example, two material endurants must at least overlap in order to be mat-dep, while a void and its host can never overlap and thus must be in strong contact. These constraints are encompassed by material preconditions described in the next section.

5. The three cases of material-spatial interdependence

Apart from spatial preconditions, material preconditions are also important for mat-dep, and these vary according to the (im)materiality of the participants, resulting in the following three disjunct but jointly exhaustive cases of mat-dep:

1. If both participants are material, then they share some material part, i.e., at a specific time, some amount of matter at least partially constitutes both participants.
2. If one participant is material and the other a void, then there exists a material part of the material participant whose disappearance enlarges the void.
3. If both participants are voids, then they must spatially overlap and have hosts with a common submaterial that is in strong contact to the voids’ shared region.

These three cases roughly correspond, in order, to the three kinds of dependent physical containment formalized in [10]: material containment, materially-contains\((x, y)\), hosting a void, hosts-v\((y, x)\) and similar relations, and immaterial containment, immaterially-contains\((x, y)\). As refinements of \(\text{dep-contains}\), each of them requires mat-dep. But, mat-dep is more general, it can hold between two endurants that are not related by either containment relation.

\[(\text{MDep-T4}) \quad \text{materially-contains}(x, y) \rightarrow \text{mat}(x) \land \text{mat}(y) \land \text{mat-dep}(x, y) \]  
\[(\text{MDep-T5}) \quad \text{hosts-v}(x, y) \rightarrow \text{mat}(x) \land \text{V}(y) \land \text{mat-dep}(x, y) \]  
\[(\text{MDep-T6}) \quad \text{immaterially-contains}(x, y) \rightarrow \text{V}(x) \land \text{V}(y) \land \text{mat-dep}(x, y) \]

5.1. Mat-dep between two material endurants

Two material endurants that do not share any material extent, but only immaterial extent, are not in mat-dep because either of the material endurants could be arbitrarily displaced without changing the other. Thus, the regions of two material participants of mat-dep must overlap (MDep-A5).
Spatial overlap of the participants alone, however, is insufficient to ensure mat-dep. Consider again the case of the rock body and the water that seeps into it after rainfall (e.g. Fig. 2(d)): these spatially overlap at a macroscopic scale, because the microscopic pores in the rock matter that hold the water body are not considered voids in the macroscopic rock body. But they are not mat-dep because they do not share matter at any time or place, and removal of the water from the rock body, by means of seepage, leaves the rock body materially unchanged, and the water could have been located somewhere else altogether. Consequently, for mat-dep between material endurants, we must not only consider their spatial relationship but also their material relationship. One way to express this relies on generic material constitution (compare [15]), expressed through statements of the form: “a physical object of kind ClassA is constituted of matter of kinds ClassB, ClassC, . . .”. In other words, generic constitution allows us to specify the kinds of matter that contribute to a certain kind of material endurant.

As a preliminary step in that direction [9], we limited direct constituents, i.e. constituents of the next lower level of granularity, to certain kinds of matter. For example, a water body’s is only directly constituted by water (if it has any constituent at all), a rock body by rock matter (and at least some rock matter), and a soil body by soil matter (of which at least a portion must be rock matter or organic matter).

Now we want to extend this by expressing that water bodies, rock bodies or soil bodies are only mat-dep on their constituent water, rock matter, and soil matter, respectively.

\[ WB(x) \rightarrow NAPO(x) \land \forall y[DK_1(y, x) \rightarrow Water(y)] \]
\[ RB(x) \rightarrow NAPO(x) \land \exists y[DK_1(y, x)] \land \forall y[DK_1(y, x) \rightarrow RockMatter(y)] \]
\[ SB(x) \rightarrow NAPO(x) \land \exists y[P(r(y), r(x)) \land (RockMatter(y) \lor OrganicMatter(y))] \]

While in the example only generic kinds of direct constituents are captured, such restrictions are flexible enough to accommodate multiple levels of constitution. It is still consistent to say that a water body, for example, is also constituted of the \( H_2O \) molecules located in any amount of water in the water body. Furthermore, the condition on the constituents of water bodies makes no claim about whether a water body is constituted of any water at all. But a water body not constituted of any water, e.g. a dry lake or a dry aquifer, is not mat-dep with any water.

Because such logical statements are specific to each kind of material endurant, the general case is best expressed in the form of axiom schemata rather than a single universally applicable axiom. To facilitate the expression of generic constitution in a compact way, e.g. by domain experts, we introduce a finite set of ordered pairs of formulas

\[ GenConst = \{(\Phi_1(x), \Psi_1(x)), \ldots, (\Phi_n(x), \Psi_n(x))\} \]
where the formulae $\Phi_i(x)$ and $\Psi_i(x)$ are finite boolean combinations of unary relations that constrain the classes used to specify generic constitution: $\Phi_i(x)$ defines the class of the constituted kind and $\Psi_i(x)$ the class of the constituent kind. For example, we can state $GenConst = \{(WB(x), Water(x)), (RB(x), RockMatter(x)), (SB(x), RockMatter(x) \lor OrganicMatter(x))\}$. By instantiating the axiom schemata MDep-A6 and MDep-A7 with pairs of formulae from $GenConst$, we can specify the desired set of first-order axioms that describe generic constitution of the kinds of interest for a particular domain. MDep-A6 requires all $\Phi$ and $\Psi$ to be subclasses of $mat$, whereas the instances of MDep-A7 specify necessary and sufficient conditions for the kinds of materials that can compose each material endurant of the constituted kind. Any material that is not of the constituent kind will be excluded from being mat-dep with a material endurant of the constituted kind. For each $(\Phi_i, \Psi_i) \in GenConst$ the following are axioms:

\[
\text{MDep-A6} \quad \Phi_i(x) \lor \Psi_i(x) \rightarrow mat(x)
\]

(constituted and constituent kinds are specializations of mat)

\[
\text{MDep-A7} \quad \Phi_i(x) \land mat(y) \rightarrow \left[\text{mat-dep}(x, y) \iff PO(r(x), r(y)) \land \exists z[\Psi_i(z) \land P(r(z), r(x)) \land r(z) \cdot r(y)]\right]
\]

(two material endurants are mat-dep iff they spatially overlap and share a part that is of a constituent kind of $x$ and $y$)

We intend the pairs in $GenConst$ to be specified by domain experts; they may vary depending on the domain. For example, a hydrologist may consider only water as generic constituent of water bodies whereas a marine biologist may additionally allow organic matter such as plankton as constituents. Moreover, the statements may differ in the level of granularity they are concerned with. While a hydrologist may not be interested in anything more fine-grained than water (as matter), a chemist may state that water matter is constituted of $H_2O$ molecules. Both statements are consistent and result in knowledge about generic constitution that crosses multiple levels of granularity. In the example, water bodies would automatically be generically constituted of $H_2O$ molecules as well.

Generic constitution also refines the submaterial relation from [10]. Previously, any material endurant spatially included in another material endurant was considered to be a submaterial thereof. Now we can be more precise: a material endurant is a submaterial of another material endurant iff it is (1) spatially contained and (2) all its constituent parts must also constitute the second endurant (SubMat-D'). We use the formula

\[
\Upsilon_{ij}(x, y) \triangleq \Phi_i(x) \land \Phi_j(y) \land \forall z[\Psi_i(z) \land P(r(z), r(x)) \rightarrow \Psi_j(z) \land P(r(z), r(y))]
\]

defined for all pairs $(\Phi_i, \Psi_i), (\Phi_j, \Psi_j) \in GenConst$ to express the second condition. Material endurants in a submaterial relation are also mat-dep (MDep-T7).

\[
\text{SubMat-D'} \quad \text{submaterial}(x, y) \iff P(r(x), r(y)) \land \bigwedge_{i,j=1}^n \Upsilon_{ij}(x, y)
\]

(a submaterial $x$ of $y$ is located in a subregion of $y$ and is of some kind $\Phi_i$ such that every constituent part of it is also a constituent part of $y$ for some kind $\Phi_j$ of $y$)

\[
\text{MDep-T7} \quad \text{submaterial}(x, y) \rightarrow \text{mat-dep}(x, y)
\]

(a submaterial is mat-dep)
5.2. Mat-dep between a material endurant and a void

For a material and a void endurant to be mat-dep, the void’s region must overlap the material endurant’s spatial extent by overlapping either its material or immaterial extent. One of the most common cases of mat-dep between a material and a void endurant is the void-hosting relation (MDep-T8) in which the material endurant hosts the void. For example, the rock body (RB) in Fig. 2(b) hosts the tunnel T₁.

More generally, any submaterial of the material endurant—not just the whole material endurant—may host the void (MDep-T9). In this case, the void is either within the material endurant’s void space or the void is hosted by a material part of the material endurant. The first case is exemplified by the rock body and the gap between the rock matter in Fig. 2(d), which are mat-dep because the rock matter is a constituent and, thus, a submaterial of the rock body while also hosting the gap. The mat-dep relation between the rock body RB and the void V₁ in Fig. 2(b) exemplifies the second case: while RB cannot host V₁, it has in RB₁ a material part that hosts V₁. A void and a material endurant may be mat-dep even when no host of the void is a submaterial of the material participant: this requires the existence of a material part that (1) is a submaterial of the material participant and of some host of the void and that (2) spatially overlaps or is in strong contact to the void (MDep-T10). For example, the lagoon void in Fig. 2(c) is not hosted by a submaterial of the coral reef but is hosted by a larger material endurant (i.e. bedrock + dead corals + live corals) of which the coral reef is a submaterial. But they are still mat-dep because the lagoon is in strong contact to the coral reef. Removal of certain parts of the coral reef necessarily enlarges the lagoon void. Axiom MDep-A8 formalizes this general case, which implies the specialized cases.

In general, the material and void participants may either spatially overlap or be in strong contact, that is, neither overlap nor strong contact is implied. Only in the most specialized case, in void-hosting, we know that the participants cannot spatially overlap and thus must be in strong contact (MDep-T11).

(MDep-A8) \[ \text{mat}(x) \land V(y) \rightarrow \left[ \text{mat-dep}(x,y) \leftrightarrow \exists z, h_y \left[ \text{submaterial}(z,x) \land \text{hosts-v}(h_y,y) \land [C_S(r(z),r(y)) \lor PO(r(z),r(y))] \right] \right] \]

(a material endurant \( x \) and a void endurant \( y \) are mat-dep iff some host \( h_y \) of the void shares with \( x \) a submaterial \( z \) that overlaps or is in strong contact with the void \( y \))

(MDep-T8) \[ \text{hosts-v}(x,y) \rightarrow \text{mat-dep}(x,y) \] (a void and its host are mat-dep)

(MDep-T9) \[ \text{hosts-v}(h_y,y) \land \text{submaterial}(h_y,x) \rightarrow \text{mat-dep}(y,x) \] (if a submaterial \( h_y \) of an endurant \( x \) hosts a void \( y \), then the endurant \( x \) is mat-dep with the void \( y \))

(MDep-T10) \[ \text{hosts-v}(h_y,y) \land \text{submaterial}(x,h_y) \land [C_S(r(x),r(y)) \lor PO(r(x),r(y))] \rightarrow \text{mat-dep}(y,x) \] (if a submaterial \( x \) of \( h_y \) is in strong contact or overlaps a void \( y \) hosted by \( h_y \), then \( x \) is mat-dep with the void \( y \))

(MDep-T11) \[ \text{hosts-v}(x,y) \rightarrow \text{mat-dep}(x,y) \land C_S(x,y) \] (hosting a void requires the two mat-dep participants to be in strong contact)

This analysis shows that a material endurant hosting a void is not the only way a material endurant can dependently contain a void. Other, more general, cases of mat-dep can arise between a material endurant and a void, such as a void being hosted by a submaterial of the material endurant, and these yield new special cases of \( \text{dep-contains} \) that need to be added to the classification of dependent containment in [10].
5.3. Mat-dep between two voids

Voids are not made up of any material: they have empty material extents. Hence, for voids to be mat-dep, their immaterial extents, i.e. their regions, must overlap (MDep-T12).

\[(\text{MDep-T12})\quad V(x) \land V(y) \land \text{mat-dep}(x, y) \to \text{PO}(r(x), r(y))\]

(mat-dep between two voids requires them to spatially overlap)

In [10], we introduced a necessary and sufficient condition for two voids being physically interdependent, namely when they spatially overlap and have spatially nested material hosts (Dep-A5). The voids \(V_1\) and \(V_2\) in Fig. 2(a) overlap and have nested hosts \(RB\) and \(RB_1\) and are indeed mat-dep. But these criteria are no longer sufficient. Once Dep-A3 is weakened to Dep-A3', spatial nesting of the material hosts of two voids no longer guarantees mat-dep (or dep), because we removed this direction of the implication from Dep-A3'. Thus, the criteria for mat-dep between voids requires adjustment as well: in MDep-A9, which replaces Dep-A5, spatial nesting between the hosts’ regions is replaced with the condition that the hosts share a submaterial that is in strong contact to the region shared by the voids. This takes into account the generic kinds of matter that constitute the hosts, as well as the spatial relevance of the shared matter to the voids. Then, for example, the pores in the soil body (hosted by its organic and rock matter) and the pores in the rock body (hosted by its rock matter) in Fig. 2(d) are mat-dep, because their hosts share rock particles and those particles are in strong contact to the region shared by the two voids. As a counterexample, if you assume that the water body in Fig. 2(d) has a hole where the rock is located, then that hole and the rock body’s cavity are not mat-dep despite their overlap and the overlap of their hosts, because their hosts (a rock body and a water body) do not share any matter.

\[(\text{Dep-A5})\quad V(x) \land V(y) \to [\text{dep}(x, y) \iff \text{PO}(r(x), r(y)) \land \exists h_x, h_y [\text{hosts-v}(h_x, x) \land \text{hosts-v}(h_y, y) \land \text{mat}(h_x) \land \text{mat}(h_y) \land (\text{P}(r(h_x), r(h_y)) \lor \text{P}(r(h_y), r(h_x)))]\]

(voids are dependent iff they overlap and have spatially nested material hosts)

\[(\text{MDep-A9})\quad V(x) \land V(y) \to [\text{mat-dep}(x, y) \iff \text{PO}(r(x), r(y)) \land \exists h_x, h_y, z [\text{hosts-v}(h_x, x) \land \text{hosts-v}(h_y, y) \land \text{submaterial}(z, h_x) \land \text{submaterial}(z, h_y) \land C_S((z), r(x) \land r(y)))]\]

(voids are mat-dep iff they overlap and have hosts that share a submaterial that is in strong contact to the regions shared by the voids)

Notice that our discussion only covers the case of direct interdependence between two voids, which requires the voids to overlap. It does not express the much weaker case of indirect interdependence, in which two non-overlapping voids of the same host (or hosts in a submaterial relation) disappear if their shared host is destroyed.

A Simple Void and its Host Form a Hole-less Region As an ancillary result of our investigation into mat-dep, a new condition on so-called simple, that is, internally connected, voids (V-A28) has been revealed: they must be tightly attached to their host in the sense that there is no spatial region “in between” the host and the void. More precisely, all simple voids are tightly attached to some (smallest) host of the void such that the sum of the void’s region and the host’s region form a hole-less region, i.e., a region of genus 0. This is enforced by the complement of this sum being internally connected.

\[(\text{V-A28})\quad V_S(x) \to \exists y [\text{hosts-v}(y, x) \land ICon((r(x) + r(y))')]\]
6. Discussion and conclusions

Mat-dep is a type of dependence in which the physical extents of related entities are necessarily dependent. Through the formal characterization of mat-dep we have shown that it can complement other foundational relations, such as containment, void-hosting, and constitution, by providing an augmented physical characterization that is largely absent from previous formal ontology developments. Mat-dep has proven to be vital to our development of physical containment, and is expected to play a significant role in ongoing work on granularity in material constitution. There it will help characterize the relation between, for example, a statue or rock or their matter, and the minerals that constitute them. In doing so, mat-dep will also help further refine an important but largely neglected notion in physical ontologies: that the material extents of an entity vary between granularity levels, e.g. allowing a rock to absorb water, due to gaps in matter at one granularity level (minerals) invisible at coarser levels (the rock).

References