MAT 463 : Intro to Abstract Algebra I

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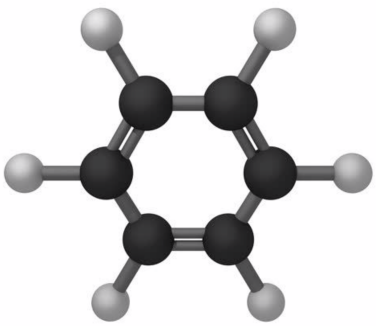

Question: In this molecule, the six inner atoms are carbon, and the six outer atoms are hydrogen. How many different molecules can you make by replacing some of these hydrogen atoms by chlorine atoms?

Idea: There are six spaces, and two different atoms that we can put in each space. So the total number of ways of assigning atoms to spaces is $2^6 = 64$.

But: Different assignments of atoms to spaces can lead to different pictures of the same molecule, since rotating the picture, or flipping it over, doesn't change the molecule. So our count of 64 is too high. We need a smarter way of counting, that takes the *symmetries* of the picture into account.

How many symmetries are there? Keeping the picture face-up, you can rotate through six different angles (60°, 120°, ..., 360°). You also flip the picture over, around one of six different axes (12 & 6 o'clock, 1 & 7 o'clock, ..., 5 & 11 o'clock). 6 rotations + 6 flips = 12 symmetries all together.

You can combine these operations, one after another. The net effect on the picture will always be the same as one of the 12 operations that we already found.

Example: Flip the picture over around the 12 & 6 o'clock axis. Which of the 12 symmetries should you do next, so that the overall effect on the picture is the same as if you had just rotated it 60° clockwise? As an algebraic equation: if $f =$ 12 & 6 o'clock flip, and $r = 60^\circ$ rotation, which symmetry x solves "x after $f = r$"?

Group theory is a branch of mathematics that uses algebra to study symmetries of mathematical and physical objects. A key insight is that combining symmetries is a lot like multiplying matrices.

MAT 463 is an introduction to group theory and related algebraic structures.

If you found that $x =$ flip around the 1 & 7 o'clock axis, then you have already solved your first group theory problem. Take MAT 463 if you want to solve more.

(There are 13 different molecules. Perhaps you didn't need group theory for that. But what if your choices for the six outer atoms are hydrogen, chlorine, fluorine, and bromine?)