Frobenius C^*-algebras and local adjunctions of C^*-correspondences

Tyrone Crisp

University of Maine

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The Dictionary

Spaces	Algebras
X : loc. compact Hausdorff	$C_0(X)$: commutative C^*-algebra of continuous functions $X \to \mathbb{C}$ vanishing at ∞
noncommutative space	noncommutative C^*-algebra
continuous map	???

What is a morphism of non-unital C^*-algebras?

Multiplier algebras

A : C^*-algebra

$M(A) = \{t : A \to A \mid \exists t^* : A \to A \text{ with } t(a)^* b = a^* t^*(b)\}$

$M(A)$ is a unital C^*-algebra

$A \hookrightarrow M(A)$ via $a \mapsto$ [left mult. by a]

A is unital iff $A = M(A)$

Example: $M(\mathrm{K}(H)) = \mathrm{B}(H)$

Example: $M(C_0(X)) = C(\beta X)$, Stone-Cech compactification

The Dictionary, again

Spaces	Algebras
loc. compact Hausdorff	commutative C^*-algebra
noncommutative space	noncommutative C^*-algebra
continuous map	$A \dashrightarrow B$: nondegenerate[†] $*$-homomorphism $A \to M(B)$

[†] $B = AB$

The Dictionary, again

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sheaves	modules...

Hilbert C^*-modules

A : C^*-algebra

Hilbert A-module : right A-module X with inner product $\langle\ |\ \rangle : X \times X \to A$ satisfying Hilbert space axioms

Morphisms:

adjointable
$$\mathsf{L}_A(X, Y) = \{t : X \to Y \mid \exists t^* : Y \to X \text{ s.t.} \langle t(x) \,|\, y\rangle = \langle x \,|\, t^*(y)\rangle\}$$

compact
$$\mathsf{K}_A(X, Y) = \overline{\operatorname{span}}\{|y\rangle\langle x| : x' \mapsto y\langle x \,|\, x'\rangle\}$$

Example: $A = \mathbb{C}$: X is a Hilbert space; $\mathsf{L}_A =$ bounded operators, $\mathsf{K}_A =$ compact operators

Example: A commutative $\Rightarrow X =$ sections of a cts field of Hilbert spaces

What is (or should be) 'the category of Hilbert A-modules'?

The category of Hilbert A-modules

Objects : Hilbert A-modules Morphisms?

$L_A(X, Y)$? Ok, but sometimes (eg K-theory) we want to remember $K_A(X, Y)$

$K_A(X, Y)$? But that's not a category: usually $\mathrm{id} \notin K_A(X, X)$

Idea: consider the non-unital C^*-category[†] K_A of Hilbert A-modules and compact operators

[†] C^*-category : morphisms have $*$, linear structure, norm, $\|t^*t\| = \|t\|^2$ ...and $t^*t \geq 0$

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Non-unital category?!?

Multiplier categories

Theorem [Kandelaki, Vasselli, Antoun-Voigt] : Every nonunital C^*-category C has a multiplier category $M(C)$, and $C \hookrightarrow M(C)$ with equality iff C is unital.

Idea: $t \in M(C)(X, Y)$ is a collection of maps $C(Z, X) \to C(Z, Y)$ and $C(Y, Z) \to C(X, Z)$ (for each object Z) defining what 'composition with t' means.

Example: nonunital C^*-alg $\Leftrightarrow$ non-unital C^*-cat with one object $\rightsquigarrow M_{\text{algebra}} = M_{\text{category}}$.

Example: $M(\mathsf{K}_A) = \mathsf{L}_A$

Functors between non-unital C^*-categories

C, D : non-unital C^*-cats (eg $C = \mathsf{K}_A$, $D = \mathsf{K}_B$)

A nondegenerate $*$-functor $C \dashrightarrow D$ is a functor $\mathcal{F} : C \to M(D)$ (objects $\to$ objects, morphisms $\to$ multipliers, preserves $\circ$) satisfying $\mathcal{F}(t)^* = \mathcal{F}(t^*)$, and

$$D(\mathcal{F}X, \mathcal{F}Y) = \mathcal{F}C(Y, Y) \circ D(\mathcal{F}X, \mathcal{F}Y) = D(\mathcal{F}X, \mathcal{F}Y) \circ \mathcal{F}C(X, X)$$

Example: nonunital C^*-algs $\rightsquigarrow$ morphisms from the dictionary

Theorem [Blecher, '97]: The nondegenerate $*$-functors $\mathsf{K}_A \dashrightarrow \mathsf{K}_B$ are (up to unitary natural isomorphism) the functors $X \mapsto X \otimes_A F$ where F is a C^*-correspondence[†] from A to B.

In particular, $\mathsf{K}_A \cong \mathsf{K}_B$ (unitarily equivalent via nondegenerate $*$-functors) iff A and B are Morita equivalent.

[†]Hilbert B-module + morphism $A \dashrightarrow \mathsf{K}_B(F)$

Local adjunctions

A, B : C^*-algebras $\qquad {}_AF_B,\ {}_BG_A$: C^*-correspondences

A local adjunction between F and G is a natural isomorphism

$$\mathrm{K}_B(X \otimes_A F, Y) \xrightarrow{\cong} \mathrm{K}_A(X, Y \otimes_B G)$$

for all Hilbert A-modules X and Hilbert B-modules Y. ('Natural' means with respect to all adjointable maps.)

Theorem [Clare-C-Higson]: local adjunction $\Leftrightarrow$ conjugate-linear completely bounded isomorphism $\varphi : F \to G$ satisfying $\varphi(afb) = b^*\varphi(f)a^*$ $\Leftrightarrow$ structure of a bi-Hilbertian bimodule of finite numerical index on F [Kajiwara-Pinzari-Watatani].

Local adjunctions are 2-sided: $\varphi : F \to G \rightsquigarrow \varphi^{-1} : G \to F$

Examples of local adjunctions

$$\mathsf{K}_B(X \otimes_A F, Y) \xrightarrow{\cong} \mathsf{K}_A(X, Y \otimes_B G) \iff \varphi : F \xrightarrow{*} G$$

□ ${}_A F_B$ a Morita equivalence, $G = \mathsf{K}_B(F, B)$, $\varphi(f) = \langle f|$

□ $A \hookrightarrow B$ nondegenerate subalgebra, $\varepsilon : B \to A$ conditional expectation such that $\mathrm{const} \cdot \varepsilon - \mathrm{id}_B \geq 0$
$\rightsquigarrow F = {}_A B_B$ with $\langle b_1 \mid b_2 \rangle = b_1^* b_2$, $\quad G = {}_B B_A$ with $\langle b_1 \mid b_2 \rangle = \varepsilon(b_1^* b_2)$, $\quad \varphi : F \xrightarrow{b \mapsto b^*} G$. [Frank-Kirchberg]

□ G real reductive group, $P = LN$ parabolic subgroup $\rightsquigarrow$ locally adjoint pair of C^*-correspondences ${}_{C_r^*(G)} C_r^*(G/N)_{C_r^*(L)}$ and ${}_{C_r^*(L)} C_r^*(N \backslash G)_{C_r^*(G)}$ (parabolic induction and restriction) [CCH]

Local adjunctions and adjunctions

Theorem [KPW]: Let $\varphi : F \to G$ be a local adjunction. The adjunction isos extend to natural isos

$$\mathsf{L}_B(X \otimes_A F, Y) \xrightarrow{\cong} \mathsf{L}_A(X, Y \otimes_B G)$$

if and only if A acts on F by B-compact operators, and B acts on G by A-compact operators.

Example: cond exp $\varepsilon : B \to A$ with finite quasi-basis $(x_1, \ldots, x_n, y_1, \ldots, y_n \in B$ with $b = \sum_i x_i \varepsilon(y_i b) = \sum_i \varepsilon(b x_i) y_i)$

Theorem [CCH]: Let $\varphi : F \to G$ be a local adjunction. For all Hilbert space representations $A \to \mathsf{B}(H)$ and $B \to \mathsf{B}(K)$ we have natural isos

$$\mathsf{B}_A(F \otimes_B K, H) \xrightarrow{\cong} \mathsf{B}_B(K, G \otimes_B H).$$

Question: are local adjunctions like adjunctions?

Units and counits

Suppose we have an adjunction on L_A and L_B:

$$(\star) \qquad \mathsf{L}_B(X \otimes_A F, Y) \xrightarrow{\cong} \mathsf{L}_A(X, Y \otimes_B G).$$

□ put (1) $X = A,\ Y = F,$ and (2) $X = G,\ Y = B$:

$$(1) \qquad \mathsf{L}_B(F, F) \xrightarrow{\cong} \mathsf{L}_A(A, F \otimes_B G), \quad \mathrm{id}_F \longmapsto \eta \text{ unit}$$

$$(2) \qquad \mathsf{L}_B(G \otimes_A F, B) \xrightarrow{\cong} \mathsf{L}_A(G, G), \quad \text{counit } \varepsilon \longleftarrow \mathrm{id}_G$$

□ $F \xrightarrow{af \mapsto \eta(a) \otimes f} F \otimes_B G \otimes_A F \xrightarrow{f_1 \otimes g \otimes f_2 \mapsto f_1 \varepsilon(g \otimes f_2)} F$ is the identity (and similarly for G)

□ (unit,counit) as above determines an adjunction $(\star)$

□ ε^* and η^* are the unit and counit of another adjunction

□ this 'unit/counit' picture of adjunctions is often more useful than the 'isomorphisms between Homs' picture.

Units and counits for local adjunctions?

Now suppose we have a local adjunction:

$$\mathsf{K}_B(X \otimes_A F, Y) \xrightarrow{\cong} \mathsf{K}_A(X, Y \otimes_B G) \quad \Longleftrightarrow \quad \varphi : F \xrightarrow{*} G$$

Q: $\exists$ unit $\eta \in \mathsf{K}_A(A, F \otimes_B G)$ and counit $\varepsilon \in \mathsf{K}_B(G \times_A F, B)$?

A: only if we have an actual adjunction on L_A and L_B.

But we do have something like a unit and counit:

☐ $F \otimes_B G \cong \mathsf{K}_B(F)$ (cb iso of operator spaces) and we have

$$\eta : A \dashrightarrow \mathsf{K}_B(F) \qquad \text{(action morphism)}$$

☐ $G \otimes_A F \cong \mathsf{K}_A(G)$ and we have a completely positive map

$$\varepsilon : \mathsf{K}_A(G) \to B, \qquad |g_1\rangle\langle g_2| \mapsto \langle \varphi^{-1}(g_1) \,|\, \varphi^{-1}(g_2)\rangle$$

$\Longrightarrow$ η and ε don't exist as adjointable operators (natural transformations between nondegenerate $*$-functors)...

...but they do exist in C^*-algebra theory.

Goal: Find the right 2-category for studying Hilbert C^*-modules.

In search of a 2-category

2-category: objects, morphisms, and 2-morphisms (morphisms between morphisms)

Example: categories, functors, and natural transformations

Example: non-unital C^*-categories, nondegenerate $*$-functors, natural transformations

Example (almost): C^*-algebras, C^*-correspondences, and adjointable bimodule maps (this is a bicategory)

Problem: Find a 2-category with objects C^*-categories; morphisms nondegenerate $*$-functors; and some kind of 2-morphisms such that

☐ isomorphisms $\mathsf{K}_A \cong \mathsf{K}_B \iff$ Morita equivalences

☐ adjunctions between K_A and $\mathsf{K}_B \iff$ local adjunctions (or some other sufficiently flexible notion)

Evidence that a good 2-category exists: Frobenius algebras

A : ring (with unit). A Frobenius algebra over A is:

- ☐ a ring C
- ☐ a ring homomorphism $\eta : A \to C$
- ☐ an A-bimodule map $\varepsilon : C \to A$

such that $C \otimes_A C$ with mult. $(c_1 \otimes c_2) \cdot (c_3 \otimes c_4) = c_1 \varepsilon(c_2 c_3) \otimes c_4$ has a multiplicative identity.

Theorem [Morita]: (1) If ${}_A F_B$ is an A-B bimodule such that the functor $\otimes_A F : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ has a two-sided adjoint, then $\mathrm{End}_B(F)$ is a Frobenius algebra over A.

(2) Every Frobenius algebra arises in this way.

More generally:

Theorem [Lauda]: Every 2-sided adjunction in a 2-category gives rise to a Frobenius algebra in a monoidal category, and vice versa.

Frobenius C^*-algebras

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A : C^*-algebra. A Frobenius C^*-algebra over A is:

- □ a C^*-algebra C
- □ a morphism $\eta : A \dashrightarrow C$
- □ a cp A-bimodule map $\varepsilon : C \to A$

such that $C \otimes^{\mathrm{h}}_A C$ with mult. $(c_1 \otimes c_2) \cdot (c_3 \otimes c_4) = c_1 \varepsilon(c_2 c_3) \otimes c_4$ has a bounded approximate identity.

$\otimes^{\mathrm{h}}$ = Haagerup tensor product: $C \otimes^{\mathrm{h}}_A C = \overline{\mathrm{span}}\{c_1 * c_2\} \subset C *_A C$

Frobenius C^*-algebras and local adjunctions

Theorem [arXiv:2108.08345]

(1) If ${}_A F_B$ is a C^*-correspondence such that the functor $\otimes_A F : \mathsf{K}_A \to \mathsf{K}_B$ has a local adjoint, then $\mathsf{K}_B(F)$ is a Frobenius C^*-algebra over A.

(2) Every Frobenius C^*-algebra arises in this way. [And this gives a bijection of isomorphism classes for the natural notions of isomorphism on each side.]

Idea (1): Let $\varphi : F \to G$ be a local adj. and let $C = \mathsf{K}_B(F)$. Then

$$C \otimes^{\mathsf{h}}_A C \to \mathsf{K}_A(F \otimes_B G), \quad |f_1\rangle\langle f_2| \otimes |f_3\rangle\langle f_4| \longmapsto |f_1 \otimes \varphi(f_2)\rangle\langle f_3 \otimes \varphi(f_4)|$$

is an isomorphism of Banach algebras [Blecher], and the C^*-algebra $\mathsf{K}_A(F \otimes_B G)$ has a bounded approximate identity. □

Frobenius C^*-algebras and local adjunctions

Theorem: (2) Frobenius C^*-algebras come from local adjunctions.

Idea (2): C, $\eta : A \dashrightarrow C$, $\varepsilon : C \to A$ Frobenius C^*-algebra over A.

Set ${}_AF_C = {}_AC_C$, $\langle c_1 \mid c_2 \rangle = c_1^* c_2$; $\quad G = {}_CC_A$, $\langle c_1 \mid c_2 \rangle = \varepsilon(c_1^* c_2)$.

Key estimate: $\mathrm{id} : F \to G$ is completely bounded from below.

Algebra: for all $c \in C$, $a \in A$ one has

$$(ca)^*(ca) = \lim_\lambda \langle c \otimes \eta(a) \mid cax_\lambda \rangle_{G \otimes_A F}$$

where x_λ is a bai for $C \otimes_A^{\mathrm{h}} C$

Cauchy-Schwartz:

$$\left\| \left\langle c \otimes \eta(a) \;\middle|\; cay_\lambda \right\rangle \right\| \leq \|ca\|_G \cdot \|\varepsilon\| \cdot \|ca\|_C \cdot \|x_\lambda\|_{C \otimes_A^{\mathrm{h}} C}$$

and all of this applies to matrices with the same constants.

Conclusion

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☐ adjunctions between K_A and $\mathsf{K}_B \iff$ local adjunctions.

Theorem [Lauda]: If this existed, then local adjunctions would correspond to Frobenius algebras.

Theorem: Local adjunctions do correspond to something that looks a lot like a Frobenius algebra.

Conclusion: C^*-algebras and their modules don't fit perfectly with category theory . . . but there is hope that they can be made to.

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Thanks!