Math. Ann. 321, 375–398 (2001) **Mathematische Annalen**

Digital Object Identifier (DOI) 10.1007/s002080100234

Galois representations attached to representations of GU(3)

Andrew H. Knightly

Received: 22 November 1999 / Accepted: 25 July 2000 /
Published online: 23 July 2001 –

Abstract. For a fixed prime $\ell \in \mathbf{Z}$ we compute the ℓ-adic Lie algebra of the image of the ℓ-adic Galois representation ρ attached to a stable cuspidal automorphic representation π of the unitary similitude group $\mathrm{GU}(3)$. This result depends on whether π admits extra twists in the sense defined below. Two cases emerge: orthogonal image and non-orthogonal image. We show that in the orthogonal case there exists a character ν such that $\rho \otimes \nu$ is the Galois representation attached to the unitary adjoint lift of a cuspidal representation of $\mathrm{GL}(2)$.

Introduction

Deligne's construction of a family $\{\rho_\ell\}$ of two-dimensional ℓ-adic Galois representations associated to each classical Hecke eigenform is of importance in number theory as a meeting point of algebraic and analytic theories. The images (in $\mathrm{GL}_2(\mathbf{Q}_\ell)$) of such Galois representations were initially studied by Serre- both those coming from elliptic curves ([S2] and [S3]), and, with Swinnerton-Dyer, those attached to elliptic modular forms with level 1 and Hecke eigenvalues in $\mathbf{Z}$ ([S1] and [S4]). In each case, it was shown that for all but finitely many primes ℓ the image of ρ_ℓ is as large as possible, under a necessary determinant constraint. Among other consequences, this result explains the congruences satisfied by Ramanujan's τ-function.

A more recent accomplishment in this area is in the theory of Picard modular surfaces. A **Picard modular surface** is a Shimura variety attached to a unitary similitude group $\mathrm{GU}(3)$ defined relative to an imaginary quadratic extension of a totally real number field E/F. The main result of the volume [M] expresses the zeta function of a Picard modular surface in terms of automorphic L-functions. A consequence of this is the association to each stable cuspidal cohomological automorphic representation π of $\mathrm{GU}(3)$ a compatible system of three-dimensional ℓ-adic representations of $G = \mathrm{Gal}(\overline{\mathbf{Q}}/E)$. The problem arises to investigate the images of these representations following Serre. The determination of the actual

A.H. Knightly
Department of Mathematics, University of California, Los Angeles, CA 90024
(e-mail: knightly@math.ucla.edu)

This research was partially conducted by the author for the Clay Mathematics Institute.

images in the elliptic modular case relied on an understanding of the structure of the ring of modular forms modulo p. Results of this kind are not yet available for $\mathrm{GU}(3)$, however we may begin the study by examining the Lie algebras of the images, which tell us what small open subgroups of the images look like.

For a fixed prime ℓ, we compute the ℓ-adic Lie algebra of the image of the ℓ-adic representation ρ attached to π. We rule out the "CM" case where the base change of $\pi_\circ = \pi|_{\mathrm{U}(3)}$ to E is automorphically induced from a Hecke character; this case reduces to the abelian theory on $\mathrm{GL}(1)$ which is already well-understood ([S2],[Ri1]). The technique used for the computation is based on methods developed by Momose and Ribet who performed the analogous computations on $\mathrm{GL}(2)$ for elliptic modular forms of weight $k \geq 2$ and arbitrary level ([Mo], [Ri2]). As it turns out, the size of this Lie algebra depends on the following two pieces of information about π, explained in this paper:

(1) whether π admits "extra twists"
(2) whether a twist of $\pi_\circ$ is the unitary adjoint lift of a representation of $\mathrm{GL}_2(\mathbf{A}_F)$.

In Sect. 2 we review those aspects of the construction of ρ_ℓ which will be needed in what follows. The precise connection between ρ and π is discussed, and in Sect. 3 several more properties of ρ are given. Sections 4 and 5 explain the idea of extra twisting of automorphic representations and how this affects the size of the image of ρ. The Lie algebra of this image is computed in Sect. 6 in both of the two cases which emerge: orthogonal image and non-orthogonal image. In Sect. 7, we show that the orthogonal case occurs only when a twist of (the L-packet containing) $\pi_\circ = \pi|_{\mathrm{U}(3)}$ is a unitary adjoint lift from $\mathrm{GL}(2)$. Thus, the study of the images of an orthogonal system $\{\rho_\ell\}$ reduces to the study of systems attached to representations of $\mathrm{GL}(2)$, which are already well-understood, at least in the case $F = \mathbf{Q}$ (cf. [Ri3]). The final section is a discussion of base change and the definition, functoriality, and a descent property of the unitary adjoint lifting from $\mathrm{GL}(2)$ to $\mathrm{U}(3)$. Because this information is used in the arguments of Sect. 7, it may be necessary to read the results of Sect. 8 first.

In [K], we have computed the infinity type of any unitary adjoint lift. In many cases one can show that $\pi_\circ$ is not a unitary adjoint lift because its infinity type is not of the right form, even after twisting. See [K] where this is carried out for the four explicit examples of stable forms given by Finis in [F]. For these examples, none of the systems of Hecke eigenvalues (given in [F]) has the symmetry that extra twisting would impart, and so the associated ℓ-adic Lie algebras are determined.

I would like to thank J. Rogawski and D. Blasius (who suggested this problem) for the many helpful conversations and suggestions which made this paper possible. I would also like to thank the referee for several helpful comments and suggestions.

1. Preliminaries

Let E/F be a purely imaginary extension of number fields of degree 2, with F totally real. The places of F and E will be denoted by v and w respectively. Let $\mathbf{A}_F$ and $\mathbf{A}_E$ be the adeles of F and E, with finite parts $\mathbf{A}_{F,f}$ and $\mathbf{A}_{E,f}$, and infinte parts $\mathbf{A}_{F,\infty}$ and $\mathbf{A}_{E,\infty}$. For any algebraic group H over F, we set $H_\infty = H(\mathbf{A}_{F,\infty})$.

Let W be a 3-dimensional vector space over E. Fix $\Phi \in \mathrm{GL}(W) \cong \mathrm{GL}_3(E)$ satisfying ${}^t\Phi = \overline{\Phi}$, where the bar denotes complex conjugation for E/F. Φ defines a Hermitian inner product on W by $(v,w) = {}^t\overline{v}\,\Phi\, w$, and the **unitary group** $\mathrm{U} = \mathrm{U}(3)$ is the set of automorphisms of W which preserve this inner product. As an algebraic group, U is the F-form of $\mathrm{GL}(3)$ consisting of the fixed points of the following Galois action on $\mathrm{GL}(3)$:

$$\sigma(g) = \begin{cases} g & \text{if } \sigma \in \mathrm{Gal}(\overline{F}/E) \\ \Phi^{-1}\,{}^t\overline{g}^{-1}\Phi & \text{otherwise.} \end{cases}$$

If A is any F-algebra, then

$$\mathrm{U}(A) = \{g \in \mathrm{GL}_{E\otimes_F A}(W \otimes_F A) |\ \Phi^{-1}\,{}^t\overline{g}^{-1}\Phi = g\}.$$

Thus in particular, if v is a place of F which is inert or ramified in E,

$$\mathrm{U}(F_v) = \{g \in \mathrm{GL}_3(E_v) |\ \Phi^{-1}\,{}^t\overline{g}^{-1}\Phi = g\}.$$

Suppose v splits in E, and w_1, w_2 lie over v in E. The bar operation gives a natural isomorphism between E_{w_1} and E_{w_2}, and the map $e \otimes x \mapsto ex \oplus \overline{e}x$ extends to an isomorphism $E \otimes_F F_v \cong E_{w_1} \oplus E_{w_2}$. Consequently, $\mathrm{GL}(W \otimes F_v) \cong \mathrm{GL}_3(E_{w_1}) \oplus \mathrm{GL}_3(E_{w_2})$. If $g \in \mathrm{GL}(W \otimes F_v)$ is identified with (g_1, g_2) under this isomorphism, then $\overline{g}$ is identified with (g_2, g_1). Thus

$$\mathrm{U}(F_v) \cong \mathrm{GL}_3(E_{w_i})$$

since $\mathrm{U}(F_v)$ is by definition the set

$$\begin{aligned}\{(g_1,g_2) \in \mathrm{GL}_3(E_{w_1}) \oplus \mathrm{GL}_3(E_{w_2}) |\ (g_1,g_2) = (\Phi^{-1}\,{}^t g_2^{-1}\Phi, \Phi^{-1}\,{}^t g_1^{-1}\Phi)\} \\ = \{(g_1, \Phi^{-1}\,{}^t g_1^{-1}\Phi)\} \subset \mathrm{GL}_3(E_{w_1}) \oplus \mathrm{GL}_3(E_{w_2}).\end{aligned}$$

The **unitary similitude group** $\mathrm{GU} = \mathrm{GU}(3)$ is the F-form of $\mathrm{GL}(3) \times \mathrm{GL}(1)$ defined by the following Galois action:

$$\sigma(g \times \lambda) = \begin{cases} g \times \lambda & \text{if } \sigma \in \mathrm{Gal}(\overline{F}/E) \\ \overline{\lambda}\Phi^{-1}\,{}^t\overline{g}^{-1}\Phi \times \overline{\lambda} & \text{otherwise.} \end{cases}$$

If v is inert in E, then $\mathrm{GU}(F_v)$ consists of the transformations preserving Φ up to multiples:

$$\mathrm{GU}(F_v) = \{g \in \mathrm{GL}_n(E_v) |\ \Phi^{-1 t}\overline{g}^{-1}\Phi = \lambda^{-1} g \text{ for some } \lambda \in F_v^*\}.$$

Let $g \in \mathrm{GU}(F_v)$. Taking determinants, we see that $\lambda^3 = \overline{\det(g)}\det(g)$. Thus $\lambda \in F_v^*$ is the norm of $\mu = \det(g)\lambda^{-1} \in E_v^*$. It follows that the element $\mu^{-1}g$ lies in $\mathrm{U}(F_v)$. Thus the following decomposition holds:

$$\mathrm{GU}(F_v) = Z(F_v)\,\mathrm{U}(F_v),$$

where Z is the center (i.e. scalar elements) of $\mathrm{GU}(3)$.

When v splits, then arguing as for U, $\mathrm{GU}(F_v) \cong \mathrm{GL}_3(E_{w_i}) \times \mathrm{GL}_1(E_{w_i})$. Clearly in this case we also have

$$\mathrm{GU}(F_v) = Z(F_v)\,\mathrm{U}(F_v),$$

so we obtain the global decomposition

$$\mathrm{GU}(\mathbf{A}_F) = Z(\mathbf{A}_F)\,\mathrm{U}(\mathbf{A}_F).$$

Note that $Z(\mathbf{A}_F) \cong \mathbf{A}_E^*$.

We assume that Φ is chosen to have signature (2,1) at precisely one infinite place $v_\circ$ of F, and signature (3,0) at the other infinite places, so that $\mathrm{GU}(\mathbf{A}_F)$ is quasi-split at $v_\circ$. For each infinite place v of F, let $\mathcal{K}_v$ be a maximal compact subgroup of $\mathrm{SU}(F_v)$. Note that $\mathcal{K}_v = \mathrm{SU}(F_v)$ unless $v = v_\circ$. Set $\mathcal{K}_\infty = \prod \mathcal{K}_v$, where the product is taken over the infinite places of F. The symmetric space attached to $\mathrm{GU}(\mathbf{A}_F)$ is

$$X = \mathrm{GU}_\infty / \mathcal{K}_\infty Z_\infty,$$

which is isomorphic to the unit ball in $\mathbf{C}^2$ (see [Go]).

2. The ℓ-adic representations attached to π_f

Let π be a cuspidal automorphic representation of $\mathrm{GU}(\mathbf{A}_F)$, and let π_f be its restriction to $\mathrm{GU}(\mathbf{A}_{F,f})$. Fix a prime $\ell \in \mathbf{Z}$. We review the construction of the ℓ-adic Galois representation attached to π_f when $\pi_{v_\circ}$ belongs to the discrete series, closely following [BR1] and [R2]. More details and background can be found in these sources, as well as [BR2].

Let $\tau = \otimes\tau_v$ be an absolutely irreducible rational finite-dimensional representation of GU_∞, defined over a number field L, on which Z_∞ acts by a character which is the infinity type of an algebraic Hecke character of E. Let $\mathcal{C}_\infty = \mathcal{C}_\infty(\tau)$ be the (finite) set of representations π_∞ of GU_∞ which satisfy

$$H^2(\mathrm{Lie}(\mathrm{GU}_\infty), \mathcal{K}'_\infty, \pi_\infty \otimes \tau^*) \neq 0,$$

where τ^* is the contragredient of τ, and $\mathcal{K}'_\infty$ is the centralizer of the center of $\mathcal{K}_\infty$ in GU_∞. Such π_∞ are τ-**cohomological**. (For simplicity, one may prefer to consider the case where τ is the trivial representation and $L = \mathbf{Q}$.)

Let $\mathcal{C}' = \mathcal{C}'(\tau)$ be the set consisting of the one-dimensional automorphic representations of GU together with the cuspidal automorphic representations π such that $\pi_\infty \in \mathcal{C}_\infty(\tau)$. Let $\mathcal{C}'_f$ be the set of representations π_f such that $\pi_f \otimes \pi_\infty \in \mathcal{C}'$ for some π_∞. If $\pi \in \mathcal{C}'(\tau)$ is infinite-dimensional, then π_{v_o} is one of the three discrete series representations with the same infinitesimal character as τ_{v_o}, while $\pi_v = \tau_v$ at the other infinite places, and

$$H^2(\mathrm{Lie}(\mathrm{GU}_\infty), \mathcal{K}'_\infty, \pi_\infty \otimes \tau^*) \cong \mathbf{C}.$$

Fix an open compact subgroup $\mathcal{K}$ of $\mathrm{GU}(\mathbf{A}_{F,f})$. We assume that $\mathcal{K}$ is small enough that the associated Shimura variety

$$S_\mathcal{K}(\mathbf{C}) = \mathrm{GU}(F)\backslash(X \times \mathrm{GU}(\mathbf{A}_{F,f}))/\mathcal{K}$$

is non-singular. $S_\mathcal{K}$ is an algebraic variety with a canonical model over E. Let $S^\#$ be the Baily-Borel compactification of $S_\mathcal{K}(E)$. When τ is the trivial representation, let

$$H^2 = IH^2(S^\# \times \mathbf{C}, \mathbf{Q}).$$

More generally, let H^2 be the degree 2 intersection cohomology group of $S^\# \times \mathbf{C}$ with coefficients in the locally constant sheaf $\mathcal{F}(L)$ of vector spaces over L determined by τ^*.

Let $\mathcal{H} = \mathcal{H}_\mathcal{K}(\mathbf{Q})$ be the Hecke algebra of $\mathbf{Q}$-valued compactly supported bi-$\mathcal{K}$-invariant functions on $\mathrm{GU}(\mathbf{A}_{F,f})$. For any $\mathbf{Q}$-algebra A, let $\mathcal{H}(A) = \mathcal{H} \otimes A$. $\mathcal{H}(L)$ is a semisimple algebra with an action on H^2. If π_f is an admissible representation of $\mathrm{GU}(\mathbf{A}_{F,f})$, let $\pi_f^\mathcal{K}$ be the finite-dimensional space of $\mathcal{K}$-fixed vectors of π_f. Recall that $\pi_f \mapsto \pi_f^\mathcal{K}$ gives a bijection between the isomorphism classes of irreducible admissible $\mathrm{GU}(\mathbf{A}_{F,f})$-modules with nonzero $\mathcal{K}$-fixed vectors and the isomorphism classes of irreducible finite-dimensional $\mathcal{H}(\mathbf{C})$-modules.

Fix an embedding $L \hookrightarrow \mathbf{C}$. By the Zucker conjecture, $H^2 \otimes_L \mathbf{C}$ can be identified with the L^2 cohomology, so Matsushima's formula (coupled with the multiplicity one theorem for GU) yields the following decomposition of $H^2 \otimes \mathbf{C}$ into $\mathcal{H}(\mathbf{C})$-isotypic components:

$$H^2 \otimes_L \mathbf{C} \cong \bigoplus_{\pi_f \in \mathcal{C}'_f} \left[\bigoplus_{\pi_\infty \in \Pi_\infty} H^2(\mathrm{Lie}(\mathrm{GU}_\infty), \mathcal{K}'_\infty, \pi_\infty \otimes \tau^*) \right] \otimes \pi_f^\mathcal{K}$$

where $\Pi_\infty = \otimes \Pi_v$ is the product of the infinite components of the global L-packet determined by π_f. If π_f is stable and infinite-dimensional, Π_∞ has 3 members, corresponding to the holomorphic, nonholomorphic, and antiholomorphic members of Π_{v_o}. Thus the summand indexed by π_f is isomorphic to $\mathbf{C}^3 \otimes \pi_f^\mathcal{K}$. (See Sect. 8 below or [R1] §13 for the notion of a stable representation).

Let K be the Galois closure (over $\mathbf{Q}$) of the splitting field for H^2 as an $\mathcal{H}(L)$-module. K is a finite Galois extension of $\mathbf{Q}$ containing L such that each simple $\mathcal{H}(K)$-submodule of $H^2 \otimes_L K$ is absolutely irreducible (i.e. $\mathcal{H}(K)$ acts on $H^2 \otimes_L K$ like a sum of matrix algebras). In particular, $H^2 \otimes_L K$ decomposes just like $H^2 \otimes \mathbf{C}$ above.

Let H^2_ℓ be the étale cohomology group $IH^2_{ét}(S^\# \times \overline{\mathbf{Q}}, \mathcal{F}(L \otimes_{\mathbf{Q}} \mathbf{Q}_\ell))$. By the comparison theorem, $H^2 \otimes \mathbf{Q}_\ell \cong H^2_\ell$, so the $\mathcal{H}(L \otimes \mathbf{Q}_\ell)$-action on $H^2 \otimes \mathbf{Q}_\ell$ transfers to an action on H^2_ℓ. Thus there is an isomorphism of $\mathcal{H}(K \otimes \mathbf{Q}_\ell)$-modules:

$$H^2_\ell \otimes_L K \cong \bigoplus_{\pi_f \in \mathcal{C}'_f} V_{\pi_f} \otimes_{K \otimes \mathbf{Q}_\ell} \pi_f^{\mathcal{K}}(K \otimes \mathbf{Q}_\ell),$$

where $\pi_f^{\mathcal{K}}(K \otimes \mathbf{Q}_\ell) = \pi_f^{\mathcal{K}}(K) \otimes \mathbf{Q}_\ell$ is a simple $\mathcal{H}(K \otimes \mathbf{Q}_\ell)$-module (where $\pi_f^{\mathcal{K}}(K)$ is a K-form of $\pi_f^{\mathcal{K}}$), and V_{π_f} is a free $K \otimes \mathbf{Q}_\ell$-module of rank ≤ 3 (rank 3 occurs when π_f is stable and infinite-dimensional).

The group $\mathrm{Aut}(K)$ acts naturally on $H^2_\ell \otimes K$. This action permutes the summands in the above decomposition, inducing a permutation of $\mathcal{C}'_f$. For $\sigma \in \mathrm{Aut}(K)$, define $\pi_f^\sigma \in \mathcal{C}'_f$ to be image of π_f under the permutation induced by σ. More precisely, $\pi_f^{\mathcal{K}}(K)$ is a finite-dimensional vector space over K on which $\mathcal{H}(K)$ acts irreducibly. Composition with σ gives a different irreducible representation of $\mathcal{H}(K)$, and π_f^σ is the corresponding representation of $\mathrm{GU}(\mathbf{A}_{F,f})$. Locally, σ acts on the (K-rational) entries of the Langlands classes of π_f, i.e. $g(\pi_v^\sigma) = \sigma(g(\pi_v))$. It is easy to confuse π_f^σ with a representation obtained by composing π_f with a Galois action on the adelic entries of the elements of GU (i.e. the action relevant to base change). However, the “base change" action permutes the local Langlands classes, instead of acting on their entries.

The Galois group

$$G = \mathrm{Gal}(\overline{\mathbf{Q}}/E)$$

acts continuously on $H^2_\ell \otimes K$. This action commutes with that of $\mathcal{H}(K \otimes \mathbf{Q}_\ell)$, and so respects the decomposition above. Suppose $\pi_f^{\mathcal{K}}$ is nonzero, and let ρ_{π_f} be the restriction of the action of G to V_{π_f}. Then G acts on the summand $V_{\pi_f} \otimes \pi_f^{\mathcal{K}}(K \otimes \mathbf{Q}_\ell)$ by $\rho_{\pi_f} \otimes 1$.

We assume henceforth that π is stable and infinite-dimensional. The relationship between $\rho = \rho_{\pi_f}$ and π is expressed as an equality of L-series. Let π_E be the base change lift of π to $\mathrm{GL}_3(\mathbf{A}_E) \times \mathrm{GL}_1(\mathbf{A}_E)$. The L-series of π depends only on π_E. Let χ_π be the central character of π, let $\pi_\circ = \pi|_{\mathrm{U}}$, and let $\pi_{\circ E}$ be the base change of $\pi_\circ$ to E (see Sect. 8). Then by Lemma 4.1.1 of [R2],

$$\pi_E = \pi_{\circ E} \otimes \overline{\chi}_\pi$$

as representations of $\mathrm{GL}_3(\mathbf{A}_E) \times \mathrm{GL}_1(\mathbf{A}_E)$, where $\overline{\chi}_\pi$ is the character $z \mapsto \chi_\pi(\overline{z})$. Let w be a finite place of E at which the local representation $\pi_{E,w}$ is unramified. By definition the local L-factor of π_f at w is

$$L_w(s,\pi_f) = L_w(s,\pi_{\circ E}\otimes\overline{\chi}_\pi) = \det(1-\overline{\chi}_\pi(\varpi_w)g(\pi_{\circ E,w})q_w^{-s})^{-1},$$

where q_w is the size of the residue field of E at w, $g(\pi_{\circ E,w})$ is the local Langlands class at w of the representation $\pi_{\circ E}$ of $\mathrm{GL}_3(\mathbf{A}_E)$, and ϖ_w is a prime element at w, identified with the idele $(\dots,1,1,\varpi_w,1,1,\dots)$.

To define the local factor of the L-series attached to ρ, fix once and for all an embedding

$$\varepsilon:\overline{\mathbf{Q}}\hookrightarrow\overline{\mathbf{Q}}_\ell,$$

and extend it to an isomorphism $\iota:\mathbf{C}\longrightarrow\mathbf{C}_\ell$, where $\mathbf{C}_\ell$ is the completion of $\overline{\mathbf{Q}}_\ell$. The representation ρ is unramified at almost all places of E. Let w be such a place, and let Fr_w be any Frobenius element at w. By definition

$$L_w(s,\rho) = \det(1-\iota^{-1}(\rho(\mathrm{Fr}_w))q_w^{-s})^{-1},$$

which is the reciprocal of a polynomial of degree 3 in q_w^{-s} with coefficients in K. Here we interpret $\rho(\mathrm{Fr}_w)$ as a 3-by-3 matrix with $\overline{\mathbf{Q}}_\ell$-entries by fixing a $K\otimes\mathbf{Q}_\ell$-basis for V_{π_f} and applying the map $k\otimes x\mapsto\varepsilon(k)x$ to the matrix entries of $\rho(\mathrm{Fr}_w)$. This local factor is independent of ℓ for $(\ell,w)=1$.

Theorem 2.1 (cf. [BR1], 1.9.1 and 2.2.1). *Let π and ρ be as above, viewing ρ as a three-dimensional $\overline{\mathbf{Q}}_\ell$-representation, using ε. If π_f is stable and $\dim(\pi_f)=\infty$, then*

(1) $L_w(s,\rho)=L_w(s-1,\pi_f)$, for almost all places w of E.
(2) One of the following two statements holds:
   (a) $\rho|_H$ is irreducible for any open subgroup $H\subset G=\mathrm{Gal}(\overline{\mathbf{Q}}/E)$.
   (b) There exist a cubic extension L/E and an algebraic Hecke character Ψ of L such that $\rho\cong\mathrm{Ind}_L^E(\Psi)$.

The case where ρ is induced is the case where $\pi_{\circ E}$ is automorphically induced from a Hecke character of L, analogous to complex multiplication in the case of elliptic modular forms. Henceforth we remove this case from consideration.

The equality of L-factors in the theorem is equivalent to the statement that

(1) $$\rho(\mathrm{Fr}_w)\sim q_w\overline{\chi}_\pi(\varpi_w)g(\pi_{\circ E,w})$$

for almost every place w of E ("$\sim$" denotes conjugacy). Because ι is fixed throughout, we always write $\rho(\mathrm{Fr}_w)$ instead of $\iota^{-1}(\rho(\mathrm{Fr}_w))$.

3. Properties of ρ

For each stable infinite-dimensional $\pi_f \in \mathcal{C}'_f$, we have the continuous irreducible representation

$$\rho_{\pi_f} : G \longrightarrow \mathrm{Aut}(V_{\pi_f}) \cong \mathrm{GL}_3(K \otimes \mathbf{Q}_\ell).$$

Fix such a π_f, and set $V = V_{\pi_f}$ and $\rho = \rho_{\pi_f}$. Let $\overline{V} = V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}_\ell$. Using ε, we regard K as a subfield of $\overline{\mathbf{Q}}_\ell$. The decomposition $K \otimes \overline{\mathbf{Q}}_\ell = \prod_{\sigma \in \mathrm{Aut}(K)} \overline{\mathbf{Q}}_\ell$ induces the decomposition

$$\overline{V} = \prod_{\sigma \in \mathrm{Aut}(K)} V_\sigma.$$

Each V_σ is a three-dimensional $\overline{\mathbf{Q}}_\ell$-vector space.

Let ρ_σ be the representation of G on V_σ. More precisely, given a basis for V, ρ_σ is the composite

$$G \xrightarrow{\rho} \mathrm{GL}_3(K \otimes \mathbf{Q}_\ell) \hookrightarrow \mathrm{GL}_3(K \otimes \overline{\mathbf{Q}}_\ell) \rightarrow \mathrm{GL}_3(\overline{\mathbf{Q}}_\ell) = \mathrm{Aut}(V_\sigma),$$

where the last arrow is the projection given by $k \otimes x \mapsto \sigma(k)x = \varepsilon(\sigma(k))x$. For any open subgroup $H \subset G$, ρ_σ restricts to an irreducible representation of H. In fact, ρ_σ is identical to the representation of G on $V_{\pi_f^\sigma}$, using ε to identify the latter representation with a map into $\mathrm{GL}_3(\overline{\mathbf{Q}}_\ell)$ as in Theorem 2.1.

Equation (1) tells us that

$$\det(\rho(\mathrm{Fr}_w)) = q_w^3 \overline{\chi}_\pi(\varpi_w)^3 \chi_{\pi_{oE}}(\varpi_w),$$

for almost every prime w of E. Here $\det$ is computed relative to $\overline{\mathbf{Q}}_\ell$. For any open subgroup $H \subset G$, let M_H be the field generated by $\det(\rho(\mathrm{Fr}_w))$ for $\mathrm{Fr}_w \in H$. Let

$$M = \bigcap_{H \subset G} M_H.$$

Then $M \subset K$. Let M_λ be the image of $M \otimes \mathbf{Q}_\ell$ under the map $m \otimes x \mapsto \varepsilon(m)x$. By continuity and the Cebotarev density theorem, $\det(\rho)$ takes values in M_λ on small open subgroups. Similarly, $\det(\rho_\sigma(G)) \subset M^*_{\lambda'}$ for some $\lambda'|\ell$ in M on such subgroups. It follows that if we compute $\det(\rho)$ relative to $K \otimes \mathbf{Q}_\ell$, its image is contained in $M \otimes \mathbf{Q}_\ell$ on small open subgroups.

For example, if $\overline{\chi}_\pi^3 \chi_{\pi_{oE}}$ gives rise (by the class field theory isomorphism, which we regard as fixed throughout) to a Galois character of finite order, then $M = \mathbf{Q}$, and $\det(\rho)$ is $\mathbf{Q}_\ell^*$-valued on sufficiently small open subgroups.

4. Extra twisting of representations

Definition 4.1. *Let $\Gamma = \Gamma(\pi_f) \subset \mathrm{Aut}(K)$ be the set of $\sigma \in \mathrm{Aut}(K)$ such that there exists a finite order Hecke character χ_σ of E satisfying*

$$(\pi_f^\sigma)_E \cong (\pi_f)_E \otimes \chi_\sigma,$$

where $(\pi_f^\sigma)_E$ and $(\pi_f)_E$ are the base change lifts of π_f^σ and π_f to E.

If $\sigma \in \Gamma$, and χ_σ is nontrivial, π_f is said to **admit an extra twist by** σ. (Because we chose K as the splitting field for all of H^2, some automorphisms of K may belong to Γ for the trivial reason that they fix π_f.)

Under our assumption that $\pi_{\circ E}$ is not automorphically induced, χ_σ is unique if it exists. This follows from the fact that a cuspidal representation of $\mathrm{GL}_3(\mathbf{A}_E)$ is automorphically induced if and only if it is isomorphic to a nontrivial twist of iteslf ([AC], chapter 3). By the strong multiplicity-one theorem, an equivalent formulation of the property defining Γ is

$$\overline{\chi}_{\pi_f^\sigma}(\varpi_w) g(\pi_{\circ E,w}^\sigma) \sim \overline{\chi}_{\pi_f}(\varpi_w) g(\pi_{\circ E,w}) \chi_\sigma(\varpi_w)$$

for almost all finite places w of E, where $\pi_{\circ E}^\sigma$ is the base change of $\pi_f^\sigma|_{\mathrm{U}}$. The relationship between $g(\pi_{\circ,v})$ and $g(\pi_{\circ E,w})$ is given explicitly in [R2] §4.2. Using this, it is immediate that the relationship $g(\pi_{\circ,v}^\sigma) = \sigma(g(\pi_{\circ,v}))$ extends to the base change, i.e. $g(\pi_{\circ E,w}^\sigma) = \sigma(g(\pi_{\circ E,w}))$.

Comparing the traces of $g(\pi_{\circ E,w}^\sigma)$ and $g(\pi_{\circ E,w})\chi_\sigma(\varpi_w)$, we find that χ_σ takes values in K. It follows that Γ is a subgroup of $\mathrm{Aut}(K)$ since:

(1) If $\sigma, \tau \in \Gamma$, then $(\pi_f^\tau)_E \cong (\pi_f)_E \otimes \chi_\tau$, and so applying σ to both sides we see that $(\pi_f^{\sigma\tau})_E \cong (\pi_f^\sigma)_E \otimes \sigma\chi_\tau \cong (\pi_f)_E \otimes \chi_\sigma \sigma\chi_\tau$, so $\sigma\tau \in \Gamma$.
(2) If $\sigma \in \Gamma$, then $(\pi_f^\sigma)_E \cong (\pi_f)_E \otimes \chi_\sigma$, so allowing σ^{-1} to act on both sides, we have $(\pi_f)_E \cong (\pi_f^{\sigma^{-1}})_E \otimes \sigma^{-1}\chi_\sigma$, which shows $\sigma^{-1} \in \Gamma$.

The class field theory isomorphism allows us to identify χ_σ with a character of G, which we also denote by χ_σ. Using the relationship between ρ and π_f and the above remarks, we see that $\sigma \in \Gamma$ if and only if for almost all w,

$$\begin{aligned}\rho_\sigma(\mathrm{Fr}_w) &\sim q_w \overline{\chi}_{\pi_f^\sigma}(\varpi_w) g(\pi_{\circ E,w}^\sigma) \\ &\sim q_w \overline{\chi}_\pi(\varpi_w) g(\pi_{\circ E,w}) \chi_\sigma(\varpi_w) \\ &\sim \rho(\mathrm{Fr}_w)\chi_\sigma(\mathrm{Fr}_w).\end{aligned}$$

Here we write ρ for ρ_{id}, where $id \in \mathrm{Aut}(K)$ is the identity. Now by the Cebotarev density theorem, we have:

$$\sigma \in \Gamma \quad \Leftrightarrow \quad (\pi_f^\sigma)_E \cong (\pi_f)_E \otimes \chi_\sigma \quad \Leftrightarrow \quad \rho_\sigma \cong \rho \otimes \chi_\sigma.$$

5. The spaces $\mathrm{End}_H V$

Let H be a subgroup of G. The term "H-module" will always refer to a vector space (over $\mathbf{Q}_\ell$ or $\overline{\mathbf{Q}}_\ell$) with an H-action. Endomorphisms will always be vector space endomorphisms. Thus for example, $\mathrm{End}_H V = \mathrm{End}_{\mathbf{Q}_\ell[H]} V$ and $\mathrm{End}_H V_\sigma = \mathrm{End}_{\overline{\mathbf{Q}}_\ell[H]} V_\sigma$.

Recall from Sect. 3 that if H is an open subgroup of G, then H acts irreducibly on V_σ by way of ρ_σ. For reference, we record the following basic fact.

Lemma 5.1 (Schur's Lemma). *Let H be an open subgroup of G. Then for any $\sigma \in \mathrm{Aut}(K)$, $\mathrm{End}_H V_\sigma = \overline{\mathbf{Q}}_\ell$.*

As we will see in the next section, the Lie algebra of the image of ρ is characterized using its commutant in $\mathrm{End}\, V$. The reason for introducing Γ is to compute this commutant over $\overline{\mathbf{Q}}_\ell$.

Lemma 5.2. *Let $\sigma, \tau \in \mathrm{Aut}(K)$, and let H be any open normal subgroup of G. Then $V_\sigma \cong V_\tau$ as H-modules if and only if $\tau^{-1}\sigma \in \Gamma$ and $H \subset \ker \chi_{\tau^{-1}\sigma}$.*

Proof. Suppose $V_\sigma \cong V_\tau$ as H-modules. Then there exists an isomorphism $A \in \mathrm{Isom}_{\overline{\mathbf{Q}}_\ell}(V_\tau, V_\sigma) \cong \mathrm{GL}_3(\overline{\mathbf{Q}}_\ell)$ such that

$$\rho_\sigma(h) = A\rho_\tau(h)A^{-1}$$

for all $h \in H$. For any $g \in G$, define

$$\phi(g) = \rho_\tau(g)^{-1}A^{-1}\rho_\sigma(g)A.$$

Clearly $\phi(h) = 1$ for all $h \in H$. We claim that in fact $\phi(g)$ is a scalar for all $g \in G$. This follows because one computes directly that

$$\phi(g)^{-1}\rho_\tau(h)\phi(g) = \rho_\tau(h) \text{ for all } g \in G, h \in H,$$

using the normality of H. Hence $\phi(g) \in \mathrm{End}_H V_\tau = \overline{\mathbf{Q}}_\ell$ as claimed. Thus $\rho_\sigma \cong \rho_\tau \otimes \phi$, which is equivalent to $\tau^{-1}\sigma \in \Gamma$.

Conversely, if $\tau^{-1}\sigma \in \Gamma$ and $H \subset \ker \chi_{\tau^{-1}\sigma}$, then $\rho_\sigma \cong \rho_\tau \otimes \phi$, for $\phi = \tau\chi_{\tau^{-1}\sigma}$, and ϕ is trivial on H. This says that $V_\sigma \cong V_\tau$ as H-modules. □

Proposition 5.3. *Let H be an open subgroup of G which is contained in the kernels of all the characters χ_γ ($\gamma \in \Gamma$). Then $\mathrm{End}_H \overline{V} \cong M_a(\overline{\mathbf{Q}}_\ell)^b$ where $a = \#\Gamma$ and $b = \#(\mathrm{Aut}(K)/\Gamma)$.*

Proof. We may assume that H is normal, since G has a basis of open normal subgroups, and if the result holds for two normal subgroups, it clearly holds for all intermediate subgroups.

Regard $\mathrm{End}_H \overline{V}$ as the matrix block sum of the sets $\mathrm{Hom}_H(V_{\sigma_i}, V_{\sigma_j})$. By the irreducibility of the V_σ's and Lemma 5.1, each of these sets is equal either to

$\overline{\mathbf{Q}}_\ell$ or to 0, according to whether or not V_{σ_i} and V_{σ_j} are isomorphic H-modules. Lemma 5.2 shows that they are isomorphic if and only if $\sigma_i \equiv \sigma_j \mod \Gamma$. This says that $\operatorname{End}_H \overline{V} \cong \prod_{\mathrm{Aut}(K)/\Gamma} M_a(\overline{\mathbf{Q}}_\ell)$ as claimed. □

Thus if H_1 and H_2 are any sufficiently small open subgroups of G, we see that $\operatorname{End}_{H_1} V = \operatorname{End}_{H_2} V$. This follows because the nested $\mathbf{Q}_\ell$-vector spaces $\operatorname{End}_{H_i} V \subset \operatorname{End}_{H_1 \bigcap H_2} V$ are actually equal since they are equal over $\overline{\mathbf{Q}}_\ell$ by Proposition 5.3, and tensoring by $\overline{\mathbf{Q}}_\ell$ preserves the codimension. Define

$$\mathfrak{X} = \operatorname{End}_H V,$$

for any "sufficiently small" H as in the proposition.

6. The Lie algebra $\mathfrak{g}$ of $\rho(G)$

Let $\mathfrak{g}$ be the ℓ-adic Lie algebra of the image $\rho(G)$ of ρ. Our goal is a description of $\mathfrak{g}$. An endomorphism of V is in the commutant of every sufficiently small open subgroup of G if and only if it is in the commutant of the Lie algebra $\mathfrak{g}$. Hence $\operatorname{End}_{\mathfrak{g}} V = \mathfrak{X}$, and so $\mathfrak{g} \subset \operatorname{End}_{\mathfrak{X}} V$. It is clear that $K \otimes \mathbf{Q}_\ell \subset \operatorname{End}_H V = \mathfrak{X}$. Thus

$$\mathfrak{g} \subset \operatorname{End}_{\mathfrak{X}} V \subset \operatorname{End}_{K\otimes\mathbf{Q}_\ell} V \cong \mathfrak{gl}_3(K \otimes \mathbf{Q}_\ell).$$

Furthermore, as we saw in Sect. 3, $\det(\rho)$ (computed relative to $K \otimes \mathbf{Q}_\ell$) is $M \otimes \mathbf{Q}_\ell$-valued on such H. Hence

$$\mathfrak{g} \subset \{m \in \operatorname{End}_{\mathfrak{X}} V \mid \operatorname{tr}(m) \in M \otimes \mathbf{Q}_\ell\}.$$

Note that the trace of an element of $\operatorname{End}_{\mathfrak{X}} V$ (always computed relative to $K \otimes \mathbf{Q}_\ell$) lies *a priori* in $K \otimes \mathbf{Q}_\ell$.

This is nearly enough information to determine $\mathfrak{g}$. Let $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \overline{\mathbf{Q}}_\ell$. Then

$$\bar{\mathfrak{g}} \subset \operatorname{End}_{K\otimes\overline{\mathbf{Q}}_\ell} \overline{V} = \prod_{\sigma\in\mathrm{Aut}(K)} \operatorname{End} V_\sigma.$$

The irreducibility of ρ on open subgroups of G implies that $\bar{\mathfrak{g}}$ is a reductive Lie algebra. Let $\tilde{\mathfrak{g}}$ be its semisimple part. Let $\tilde{\mathfrak{g}}_\sigma$ denote the projection of $\tilde{\mathfrak{g}}$ to $\operatorname{End} V_\sigma$. $\tilde{\mathfrak{g}}_\sigma$ is the semisimple part of the Lie algebra of the image of the representation ρ_σ. Because ρ_σ remains irreducible on open subgroups of G, $\tilde{\mathfrak{g}}_\sigma$ acts irreducibly on V_σ. Thus $\tilde{\mathfrak{g}}_\sigma$ is a simple subalgebra of $\mathfrak{gl}(V_\sigma) \cong \mathfrak{gl}_3(\overline{\mathbf{Q}}_\ell)$. There are only two nonzero isomorphism classes of simple Lie subalgebras of $\mathfrak{gl}_3(\overline{\mathbf{Q}}_\ell)$:

$$\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell) \ \text{ and } \ \mathfrak{sl}_3(\overline{\mathbf{Q}}_\ell).$$

Because $\tilde{\mathfrak{g}}_\sigma$ acts irreducibly, the only possibility in the first case is that $\tilde{\mathfrak{g}}_\sigma$ is a copy of $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$ embedded in $\mathfrak{gl}_3(\overline{\mathbf{Q}}_\ell)$ as the Lie algebra of an orthogonal group.

Furthermore, because the action of $\mathrm{Aut}(K)$ on the representations ρ_σ preserves the dimension of the images, if one $\tilde{\mathfrak{g}}_\sigma$ is orthogonal, then all $\tilde{\mathfrak{g}}_\sigma$ are orthogonal. So there are two distinct classes of π_f's:

(1) ρ is **Orthogonal**: $\tilde{\mathfrak{g}}_\sigma = \mathfrak{so}(V_\sigma) \cong \mathfrak{so}_3(\overline{\mathbf{Q}}_\ell)$ for all $\sigma \in \mathrm{Aut}(K)$.
(2) ρ is **Non-orthogonal**: $\tilde{\mathfrak{g}}_\sigma = \mathfrak{sl}(V_\sigma) \cong \mathfrak{sl}_3(\overline{\mathbf{Q}}_\ell)$ for all $\sigma \in \mathrm{Aut}(K)$.

By the compatibility of the system $\{\rho_\ell\}$, this classification is independent of ℓ.

In the orthogonal case, for each $\sigma \in \mathrm{Aut}(K)$, $\rho_\sigma(H)$ is contained in an orthogonal similitude group $\mathrm{GO}_3(\overline{\mathbf{Q}}_\ell)$ for some sufficiently small open normal subgroup H of G. This implies that $\rho_\sigma(G) \subset \mathrm{GO}_3(\overline{\mathbf{Q}}_\ell)$ because $\rho_\sigma(G)$ normalizes $\rho_\sigma(H)$ and $\mathrm{GO}_3(\overline{\mathbf{Q}}_\ell)$ is its own normalizer in $\mathrm{GL}_3(\overline{\mathbf{Q}}_\ell)$. This condition on the ρ_σ's is equivalent to $\rho(G) \subset \mathrm{GO}_3(K \otimes \mathbf{Q}_\ell)$.

Thus $\mathfrak{g} \subset \mathfrak{h}$, where

$$\mathfrak{h} = \begin{cases} \{m \in \mathrm{End}_{\mathfrak{X}} V \cap \mathfrak{go}(V) \mid \mathrm{tr}(m) \in M \otimes \mathbf{Q}_\ell\} & \text{in the orthogonal case} \\ \{m \in \mathrm{End}_{\mathfrak{X}} V \mid \mathrm{tr}(m) \in M \otimes \mathbf{Q}_\ell\} & \text{in the non-orthogonal case,} \end{cases}$$

and where $\mathfrak{go}(V) \cong \mathfrak{go}_3(K \otimes \mathbf{Q}_\ell)$ is the Lie algebra of the orthogonal group $\mathrm{GO}(V)$.

Theorem 6.1. $\mathfrak{g} = \mathfrak{h}$.

Before proving the theorem, we remark that it is reasonable to expect that in most cases π will not admit any extra twists. In such a circumstance, Γ is trivial (provided that $\mathrm{Aut}(K)$ acts faithfully on π_f's orbit in $\mathcal{C}'_f$), and $\mathrm{End}_{\mathfrak{g}} \overline{V} = \overline{\mathbf{Q}}_\ell^n$ by Proposition 5.3, where $n = [K : \mathbf{Q}]$. Hence $\mathfrak{X} = K \otimes \mathbf{Q}_\ell$ since its $\mathbf{Q}_\ell$-dimension is n, and so Theorem 6.1 tells us that if ρ is non-orthogonal, we have

$$\mathfrak{g} = \{m \in \mathrm{End}_{K \otimes \mathbf{Q}_\ell} V \mid \mathrm{tr}(m) \in M \otimes \mathbf{Q}_\ell\}.$$

The proof of Theorem 6.1 uses the following basic lemma.

Goursat's Lemma (Lie Algebra Version): *Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be simple Lie algebras, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{s}_1 \times \mathfrak{s}_2$ such that the projections p_i of $\mathfrak{g}$ to $\mathfrak{s}_i$ are both surjective. Then either $\mathfrak{g}$ is all of $\mathfrak{s}_1 \times \mathfrak{s}_2$ or $\mathfrak{g}$ is the graph of an isomorphism $\mathfrak{s}_1 \cong \mathfrak{s}_2$.*

Proof. Let $N \times 0 = \ker(p_2)$. Then N is an ideal of $\mathfrak{s}_1$. To see this, let $n \in N$ and $x \in \mathfrak{s}_1$ be arbitrary. Then there exists $y \in \mathfrak{s}_2$ such that $(x, y) \in \mathfrak{g}$ since p_1 is surjective. It is immediate that $N \times 0$ is an ideal of $\mathfrak{g}$. Hence $N \times 0$ contains $[(n, 0), (x, y)] = ([n, x], [0, y]) = ([n, x], 0)$, and so $[n, x] \in N$ as claimed. Thus by the simplicity of $\mathfrak{s}_1$ either $N = 0$ or $N = \mathfrak{s}_1$. In the former case, $\mathfrak{g} \cong \mathfrak{s}_2$, and we also must have $\mathfrak{g} \cong \mathfrak{s}_1$ since we now know that $\mathfrak{g}$ is simple. So $p_2 \circ p_1^{-1}$ is an isomorphism $\mathfrak{s}_1 \cong \mathfrak{s}_2$. In the latter case, we see that $\mathfrak{g} = \mathfrak{s}_1 \times \mathfrak{s}_2$, as required. □

Proof. (Theorem 6.1). Let $\overline{\mathfrak{h}} = \mathfrak{h} \otimes \overline{\mathbf{Q}}_\ell$. Then

$$\overline{\mathfrak{g}} \subset \overline{\mathfrak{h}} \subset \mathrm{End}_{K \otimes \overline{\mathbf{Q}}_\ell} \overline{V} = \prod_{\sigma \in \mathrm{Aut}(K)} \mathrm{End}\, V_\sigma.$$

It suffices to show that $\overline{\mathfrak{g}} = \overline{\mathfrak{h}}$ since $\mathfrak{g}$ is a $\mathbf{Q}_\ell$-subspace of $\mathfrak{h}$ and change of base preserves the codimension. Let $\tilde{\mathfrak{h}} = \{m \in \overline{\mathfrak{h}} |\ \mathrm{tr}(m) = 0\}$ be the semisimple part of $\overline{\mathfrak{h}}$. We only need to prove that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}}$, since the abelian parts of $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{h}}$ are both equal to $M \otimes \overline{\mathbf{Q}}_\ell$. This will follow once we verify the conditions of the following lemma for $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{h}}$ with $\mathfrak{s}_\sigma = \mathfrak{sl}(V_\sigma)$ in the non-orthogonal case, and $\mathfrak{s}_\sigma = \mathfrak{so}(V_\sigma)$ in the orthogonal case.

Lemma 6.2. *Let Σ be a finite set, and for each $\sigma \in \Sigma$ let $\mathfrak{s}_\sigma$ be a finite-dimensional simple Lie algebra over a field of characteristic 0. Let $\mathfrak{g}$ and $\mathfrak{h}$ be subalgebras of $\prod_\sigma \mathfrak{s}_\sigma$, with $\mathfrak{g} \subseteq \mathfrak{h}$. Suppose that*

(1) $\mathfrak{h}$ maps onto each factor $\mathfrak{s}_\sigma$.
(2) $\mathfrak{g}$ and $\mathfrak{h}$ have equal images in $\mathfrak{s}_\sigma \times \mathfrak{s}_\tau$, for $\sigma \neq \tau$.

Then $\mathfrak{g}$ and $\mathfrak{h}$ are equal.

Proof. [Ri2], 4.6 □

By construction, $\tilde{\mathfrak{g}}_\sigma = \tilde{\mathfrak{h}}_\sigma = \mathfrak{s}_\sigma$. Thus condition 1 is automatic. Now for $\sigma \neq \tau$ in $\mathrm{Aut}(K)$, let $\tilde{\mathfrak{g}}_{\sigma,\tau}$ (resp. $\tilde{\mathfrak{h}}_{\sigma,\tau}$) be the image of $\tilde{\mathfrak{g}}$ (resp. $\tilde{\mathfrak{h}}$) in $\mathfrak{s}_\sigma \times \mathfrak{s}_\tau$. We need to verify condition 2, i.e. that $\tilde{\mathfrak{g}}_{\sigma,\tau} = \tilde{\mathfrak{h}}_{\sigma,\tau}$. For this we apply Goursat's Lemma to see that either $\tilde{\mathfrak{g}}_{\sigma,\tau} = \mathfrak{s}_\sigma \times \mathfrak{s}_\tau$ or else $\tilde{\mathfrak{g}}_{\sigma,\tau}$ is the graph of an isomorphism $\mathfrak{s}_\sigma \cong \mathfrak{s}_\tau$.

We can now show in either case that $\tilde{\mathfrak{g}}_{\sigma,\tau} = \tilde{\mathfrak{h}}_{\sigma,\tau}$. First we point out that $\tilde{\mathfrak{g}}_{\sigma,\tau}$ (resp. $\tilde{\mathfrak{h}}_{\sigma,\tau}$) is the graph of an isomorphism $\mathfrak{s}_\sigma \cong \mathfrak{s}_\tau$ if and only if $V_\sigma \cong V_\tau$ as $\overline{\mathfrak{g}}$-modules (resp. $\overline{\mathfrak{h}}$-modules). To see this for example in the orthogonal case, choose bases for V_σ and V_τ in such a way that the bottom arrow in the following diagram is the identity map:

$$\begin{array}{ccc} \mathfrak{s}_\sigma & \xrightarrow{\sim} & \mathfrak{s}_\tau \\ \downarrow & & \downarrow \\ \mathfrak{so}_3(\overline{\mathbf{Q}}_\ell) & \xrightarrow{id.} & \mathfrak{so}_3(\overline{\mathbf{Q}}_\ell) \end{array}$$

where the top arrow is a lift to $\tilde{\mathfrak{g}}_{\sigma,\tau}$ followed by the projection to $\mathfrak{s}_\tau$. The identification of these two bases then gives a $\overline{\mathfrak{g}}$-isomorphism between V_σ and V_τ. Conversely, if φ is a $\overline{\mathfrak{g}}$-isomorphism from V_σ to V_τ, define a map from $\mathfrak{s}_\sigma$ to $\mathfrak{s}_\tau$ by lifting an element of $\mathfrak{s}_\sigma$ to $\tilde{\mathfrak{g}}$ and projecting to $\mathfrak{s}_\tau$. Because $\mathfrak{s}_\sigma$ and $\mathfrak{s}_\tau$ are simple, it just suffices to check that this map is well-defined. Suppose $X, Y \in \tilde{\mathfrak{g}}$ both project to $X_\sigma \in \mathfrak{s}_\sigma$, and let X_τ and Y_τ be their projections to $\mathfrak{s}_\tau$. Then for any $v \in V_\tau$, $X_\tau v = \varphi(X_\sigma \varphi^{-1}(v)) = \varphi(Y_\sigma \varphi^{-1}(v)) = Y_\tau v$. Hence X_τ and Y_τ are the same endomorphism of V_τ, so our map is well-defined.

If V_σ and V_τ are *not* isomorphic $\overline{\mathfrak{g}}$-modules, we may conclude that $\tilde{\mathfrak{g}}_{\sigma,\tau}$ (and hence $\tilde{\mathfrak{h}}_{\sigma,\tau}$) equals $\mathfrak{s}_\sigma \times \mathfrak{s}_\tau$.

On the other hand, if $V_\sigma \cong V_\tau$ as $\overline{\mathfrak{g}}$-modules, let $\varphi : V_\sigma \to V_\tau$ be an intertwining operator for $\overline{\mathfrak{g}}$. Then φ extends to an element of $\mathrm{End}_{\overline{\mathfrak{g}}}\overline{V} = \overline{\mathfrak{X}}$. This shows that φ also commutes with $\overline{\mathfrak{h}}$, since

$$\overline{\mathfrak{h}} \subset \mathrm{End}_{\overline{\mathfrak{X}}}\overline{V}$$

implies that

$$\mathrm{End}_{\overline{\mathfrak{h}}}\overline{V} \supset \mathrm{End}_{\mathrm{End}_{\overline{\mathfrak{X}}}\overline{V}}\overline{V} = \overline{\mathfrak{X}}.$$

This last equality follows immediately from the fact that $\overline{\mathfrak{X}} \cong \mathrm{M}_a(\overline{\mathbf{Q}}_\ell)^b$ (Proposition 5.3). Thus V_σ and V_τ are isomorphic $\overline{\mathfrak{g}}$-modules only if they are isomorphic $\overline{\mathfrak{h}}$-modules. Hence, using the remarks made above, in the case that $\tilde{\mathfrak{g}}_{\sigma,\tau} \cong \mathfrak{s}_\sigma$, we see that $\tilde{\mathfrak{h}}_{\sigma,\tau} \cong \mathfrak{s}_\sigma$, and so $\tilde{\mathfrak{g}}_{\sigma,\tau} = \tilde{\mathfrak{h}}_{\sigma,\tau}$.

This completes the verification of the two conditions of the lemma, and proves the theorem.

7. The orthogonal case

We now show that the study of the orthogonal case essentially reduces to the study of Galois representations attached to representations of $\mathrm{GL}_2(\mathbf{A}_F)$. It is more convenient here to work on the group U than on GU. In fact the representation theories of these groups do not differ in any significant way since $\mathrm{GU} = Z \cdot \mathrm{U}$.

Let $\pi_f \in \mathcal{C}'_f$, and let ρ be the associated ℓ-adic representation (always viewed in this section as a map into $\mathrm{GL}_3(\overline{\mathbf{Q}}_\ell)$, using ε as usual). Let $\pi_\circ = \pi_f|_{\mathrm{U}}$, and assume as usual that $\pi_{\circ E}$ is not automorphically induced. We can associate to $\pi_\circ$ an ℓ-adic Galois representation $\rho_\circ$ as follows. By Theorem 2.1, $L(s,\rho) = L(s-1, \pi_{\circ E} \otimes \overline{\chi}_\pi)$. Because $\overline{\chi}_\pi$ is an algebraic Hecke character of E, we may identify it with a character of G, which we also denote by $\overline{\chi}_\pi$. Let $\rho_\circ = \rho \otimes \overline{\chi}_\pi{}^{-1}$. Because twisting by a character does not affect the irreducibility of a representation, $\rho_\circ$ satisfies property 2(a) in Theorem 2.1. Note that $\rho_\circ$ also satisfies $L(s,\rho_\circ) = L(s-1,\pi_\circ)$, and thus its isomorphism class is independent of the choice of π restricting to $\pi_\circ$. Hence it is meaningful to discuss the ℓ-adic representation attached to $\pi_\circ$.

Now suppose ρ is orthogonal so that its image in $\mathrm{GL}_3(\overline{\mathbf{Q}}_\ell)$ is contained in an orthogonal similitude group. By possibly replacing ρ with an isomorphic representation, we shall assume that this orthogonal group is defined by the standard bilinear form given by the identity matrix. Thus the statement that ρ is orthogonal is equivalent to the statement that ρ is self-dual up to a twist: $\rho \cong \rho^* \otimes \nu$ for some Galois character ν. Then $\rho_\circ$ is also orthogonal because $\rho_\circ$ and ρ only differ by a twist. Theorem 7.3 below shows that a twist of the L-packet containing $\pi_\circ$ is the unitary adjoint lift of a cuspidal representation of $\mathrm{GL}(2)_F$. Thus by functoriality, up to a twist by a character any orthogonal ρ comes from a representation of $\mathrm{GL}_2(\mathbf{A}_F)$.

For details on the unitary adjoint lifting, and a summary of the properties of the other functorial liftings discussed in this section please refer to Sect. 8. For basic information on automorphic induction, see [R3]. We repeatedly use the fact that a cuspidal representation of $\mathrm{GL}_3(\mathbf{A}_E)$ is automorphically induced from a Hecke character of a three-dimensional extension of E if and only if it isomorphic to a nontrivial twist of itself ([AC], chapter 3).

Lemma 7.1. *Suppose τ_1 and τ_2 are automorphic representations of $\mathrm{GL}(2)$ whose adjoint lifts to $\mathrm{GL}(3)$ differ by a twist: $Ad(\tau_1) \cong Ad(\tau_2) \otimes \omega$ for some Hecke character ω. Then either $\omega = 1$ or $Ad(\tau_i)$ are automorphically induced from Hecke characters.*

Proof. By the fact that any adjoint lift is self-dual, we have

$$Ad(\tau_2) \otimes \omega \cong Ad(\tau_2)^* \otimes \omega^{-1} \cong Ad(\tau_2) \otimes \omega^{-1}.$$

Hence $Ad(\tau_2) \cong Ad(\tau_2) \otimes \omega^2$. So either $Ad(\tau_2)$ is automorphically induced, or else $\omega^2 = 1$. In the latter case, taking central characters in the initial condition, we see that $\chi_{Ad(\tau_1)} = \chi_{Ad(\tau_2)}\omega^3$. But the central character of any adjoint lift is trivial. Hence $\omega^3 = 1$ and $\omega^2 = 1$, and so $\omega = 1$. □

Lemma 7.2. *Let Π and Π' be stable L-packets on U and let π and π' be their respective base change lifts to E. Suppose $\pi' \cong \pi \otimes \mu$ for some Hecke character μ of E. Then either there exists a character χ of $\mathrm{U}(\mathbf{A}_F)$ such that $\Pi' \cong \Pi \otimes \chi$ or else π and π' are automorphically induced.*

Proof. As the base change of Π', π' satisfies $\pi' \cong \overline{\pi}'^*$. This implies that

$$\pi \otimes \mu \cong (\overline{\pi \otimes \mu})^* \cong \pi \otimes \overline{\mu}^{-1},$$

which gives $\pi \otimes \mu\overline{\mu} \cong \pi$. So either $\mu\overline{\mu} = 1$ or else π is automorphically induced. In the first case, let $\mu_F = \mu|_{I_F}$, where I_F denotes the ideles of F. Then $\mu_F\overline{\mu}_F = \mu_F^2 = 1$. In fact μ_F is actually trivial: taking central characters of the relationship $\pi' \cong \pi \otimes \mu$, we have $\chi_{\pi'} = \chi_\pi \mu^3$. Because π and π' descend to U, their central characters have trivial restrictions to I_F. Hence $\mu_F^3 = 1$ and $\mu_F^2 = 1$, and so $\mu_F = 1$. This, together with the condition $\mu\overline{\mu} = 1$, is the criterion for descending μ to a character χ of U. Then $(\Pi \otimes \chi)_E = \pi \otimes \mu \cong \pi'$. By the injectivity of this base change lifting, $\Pi' \cong \Pi \otimes \chi$. □

Theorem 7.3. *Let $\pi_\circ$ be a cuspidal automorphic representation of $\mathrm{U}(\mathbf{A}_F)$ belonging to a stable L-packet Π, with associated ℓ-adic Galois representation ρ. Assume that the image of ρ is contained in an orthogonal group. Then either*

$(\pi_\circ)_E$ is automorphically induced from a Hecke character, or else there is some character ψ of $\mathrm{U}(\mathbf{A}_F)$ such that $\Pi \otimes \psi$ is the unitary adjoint lift of a cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_F)$.

Proof. Let $\pi = (\pi_\circ)_E$. Then π is cuspidal. Assume that π is not automorphically induced. Because ρ is orthogonal, it is self-dual up to a twist: $\rho \cong \rho^* \otimes \nu$ for some Galois character ν. We also write ν for the associated Hecke character of E. For almost every place w of E, $\rho(\mathrm{Fr}_w) \sim q_w g(\pi_w)$. Putting these together, we have $q_w g(\pi_w) \sim q_w^{-1} g(\pi_w)^{-1} \nu(\varpi_w)$, for almost all w, and hence by strong multiplicity-one

$$\pi \cong \pi^* \otimes |\cdot|^2_{\mathbf{A}_F} \nu,$$

where $|\cdot|_{\mathbf{A}_F}$ is the the adelic norm (composed with $\det$). Taking central characters in this equation gives

$$\chi_\pi = \chi_\pi^{-1} |\cdot|^6_{\mathbf{A}_F} \nu^3,$$

and so $|\cdot|^2_{\mathbf{A}_F} \nu = (\chi_\pi |\cdot|^{-2}_{\mathbf{A}_F} \nu^{-1})^2$ is a square. Setting $\mu = \chi_\pi^{-1} |\cdot|^2_{\mathbf{A}_F} \nu$, we see that $\pi \cong \pi^* \otimes (\mu^{-1})^2$ and so

$$\pi \otimes \mu \cong (\pi \otimes \mu)^*,$$

i.e. a twist of π is self-dual. We may also assume that $\pi \otimes \mu$ has trivial central character. (Otherwise, letting ω be the central character of $\pi \otimes \mu$, $\pi \otimes \mu\omega^{-1}$ is self-dual with trivial central character.) The image of the adjoint lifting from $\mathrm{GL}(2)$ to $\mathrm{GL}(3)$ is the set of self-dual representations of $\mathrm{GL}(3)$ with trivial central characters (see Sect. 8). Thus there exists a cuspidal representation τ of $\mathrm{GL}_2(\mathbf{A}_E)$ whose adjoint lift $Ad(\tau)$ is $\pi \otimes \mu$.

We wish to show that $\pi \otimes \mu$ is the base change lift of an L-packet on U. This holds if $(\pi \otimes \mu)^* \cong \overline{\pi \otimes \mu}$ and the central character ω of $\pi \otimes \mu$ has trivial restriction to the ideles I_F of F. We are already assuming that $\omega = 1$, so the second condition is automatic. To verify the first condition, note that as a base change from U, π satisfies $\pi^* \cong \overline{\pi}$. Because the $\mathrm{Gal}(E/F)$ action on representations commutes with the adjoint lifting, we see that

$$Ad(\overline{\tau}) = \overline{\pi \otimes \mu} \cong \pi^* \otimes \overline{\mu} \cong \pi \otimes \mu^2 \overline{\mu} \cong (\pi \otimes \mu) \otimes \mu\overline{\mu} = Ad(\tau) \otimes \mu\overline{\mu}.$$

Lemma 7.1 applied to τ and $\overline{\tau}$ tells us that $\mu\overline{\mu} = 1$ (since we assume that π, and hence $\pi \otimes \mu$, is not automorphically induced), and so by the above equation,

$$\overline{\pi \otimes \mu} \cong \pi \otimes \mu \cong (\pi \otimes \mu)^*$$

as needed. Thus we may write $\pi \otimes \mu = (\Pi')_E$ for some stable cuspidal L-packet Π' of representations of U.

Because $\pi \otimes \mu$ is an adjoint lift, Π' is the unitary adjoint lift of a cuspidal representation of $\mathrm{GL}(2)_F$ by Proposition 8.3 below. The fact that Π and Π' differ by a twist follows by Lemma 7.2. □

8. Appendix: The unitary adjoint lifting from $\mathbf{GL(2)}$ to $\mathbf{U(3)}$

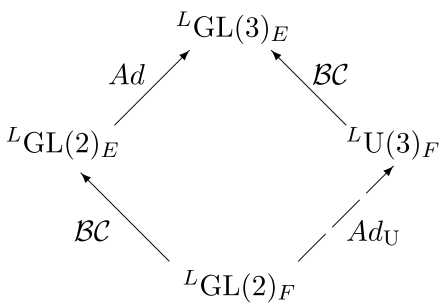
The lifting of representations described here is a special case of the following general conjecture:

Langlands functoriality conjecture: *Let G and G' be reductive groups over F, and let $\rho : {}^LG \longrightarrow {}^LG'$ be an L-map between their L-groups. Then given an L-packet $\Pi = \otimes \Pi_v$ of automorphic representations of $G(\mathbf{A}_F)$, there exist a finite set S of places of F and an L-packet $\rho(\Pi) = \otimes \Pi'_v$ of automorphic representations of $G'(\mathbf{A}_F)$ such that for $v \notin S$, Π_v and Π'_v are unramified local L-packets, and $g(\pi'_v) \sim \rho(g(\pi_v))$, where π_v and π'_v are the unique unramified elements of these local packets, with Langlands classes $g(\pi_v)$ and $g(\pi'_v)$ respectively.*

Remarks:

(1) See [BR2] for definitions of L-groups, L-maps, Langlands classes, and a description of the conjectural partition of the set of irreducible admissible local representations of G into finite sets called L-packets. The existence of L-packets is established for $\mathrm{U}(n)$ for $n \leq 3$ in [R1].
(2) Conjugacy in the L-group $\widehat{G} \rtimes \mathcal{W}_F$ means conjugacy by an element of the dual group factor.
(3) The lifting $\rho(\Pi)$ is unique if strong multiplicity-one holds for L-packets on G'. For example, if $G' = \mathrm{GL}(n)$, then L-packets are singletons and the strong multiplicity-one theorem holds. For U, strong multiplicity-one for L-packets also holds ([R1], Theorem 13.3.5).
(4) When confusion is unlikely, we denote a Langlands class $g(\pi_v)$ by its projection to the dual group.

In this section we define an L-map $Ad_{\mathrm{U}} : {}^L\mathrm{GL}(2) \longrightarrow {}^L\mathrm{U}$ and verify the conjecture for Ad_{U} in the case which is needed in Sect. 7. The construction of Ad_{U} amounts to filling in the following diagram:



The definitions of the adjoint lifting Ad from $\mathrm{GL}(2)$ to $\mathrm{GL}(3)$ and the base change liftings $\mathcal{BC}$ for $\mathrm{GL}(2)$ and U are recalled below.

Base change for $\mathrm{GL}(2)$ *from* F *to* E

The functoriality conjecture applies to L-maps between algebraic groups over the same field. So to obtain a lifting of representations from $\mathrm{GL}_2(\mathbf{A}_F)$ to $\mathrm{GL}_2(\mathbf{A}_E)$ we need to consider a reductive group which over F looks like $\mathrm{GL}(2)_E$, namely the restriction of scalars $\widetilde{G} = \mathrm{Res}_F^E(\mathrm{GL}(2))$. By definition, $\widetilde{G}(A) = \mathrm{GL}_2(E \otimes_F A)$ for any F-algebra A.

Fix an embedding $F \hookrightarrow \mathbf{C}$, and let Σ be the set of F-embeddings $E \hookrightarrow \mathbf{C}$. The L-group of $\widetilde{G}$ depends only on its dual group

$$\widehat{\widetilde{G}}(\mathbf{C}) = \prod_{\sigma \in \Sigma} \mathrm{GL}_2(\mathbf{C}) = \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C}).$$

The Galois group $\mathrm{Gal}(E/F)$ acts on Σ, and this gives a natural action of $\mathrm{Gal}(E/F)$ on $\widehat{\widetilde{G}}$ which permutes the coordinates. The L-group ${}^L\widetilde{G}$ is defined as $\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C}) \rtimes \mathcal{W}_F$, where the Weil group $\mathcal{W}_F$ acts through its projection to $\mathrm{Gal}(E/F)$. The L-group of $\mathrm{GL}(2)$ is $\mathrm{GL}_2(\mathbf{C}) \times \mathcal{W}_F$, and the base change L-map is

$$\mathcal{BC} : \mathrm{GL}_2(\mathbf{C}) \times \mathcal{W}_F \longrightarrow [\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})] \rtimes \mathcal{W}_F$$

defined by the diagonal embedding: $(g, \sigma) \mapsto (g, g, \sigma)$. The lifting of automorphic representations corresponding to this L-map is due to Saito, Shintani and Langlands, and holds more generally for E/F cyclic of prime degree.

There is a correspondence between the local Langlands classes of a representation of $\widetilde{G}(\mathbf{A}_F)$ and those of the same representation viewed as a representation of $\mathrm{GL}_2(\mathbf{A}_E)$ (cf. [BR2] §3.5). Using the latter perspective, the relationship between a representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ and its base change lift π_E is $g(\pi_{E,w}) \sim g(\pi_v)^{d_w}$ for almost all places v of F and $w|v$ in E, where $d_w = [E_w : F_v]$.

If π is an automorphic representation of $\mathrm{GL}_2(\mathbf{A}_E)$, let $\overline{\pi}$ be the representation given by $g \mapsto \pi(\overline{g})$. The $\mathrm{Gal}(E/F)$ action $\pi \mapsto \overline{\pi}$ has the effect of interchanging the local components π_w and $\pi_{\overline{w}}$ (cf. [R3] §15). π is the base change lift of an automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ if and only if it is fixed by this action, i.e. if and only if $\pi \cong \overline{\pi}$.

The adjoint lifting from $\mathrm{GL}(2)$ *to* $\mathrm{GL}(3)$

$\mathrm{GL}_2(\mathbf{C})$ acts by conjugation on the 3-dimensional complex vector space $\mathfrak{sl}_2(\mathbf{C})$ of 2×2 matrices with trace 0. For each choice of basis for $\mathfrak{sl}_2$ this action yields a map

$$Ad : \mathrm{GL}_2(\mathbf{C}) \longrightarrow \mathrm{GL}_3(\mathbf{C}).$$

The image of Ad in $\mathrm{GL}_3(\mathbf{C})$ is the orthogonal group $\mathrm{SO}_3(\mathbf{C})$ defined relative to the symmetric bilinear form on $\mathfrak{sl}_2(\mathbf{C})$,

$$X \cdot Y = \mathrm{tr}(XY).$$

For our purposes, we compute Ad relative to the basis

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{-i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

for $\mathfrak{sl}_2(\mathbf{C})$. Then the form $\mathrm{tr}(XY)$ is represented by the matrix

$$\Phi_3 = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

Because we are interested in the adjoint lifting for groups over E, working over the ground field F, we again use restriction of scalars, and the adjoint L-map from ${}^L\mathrm{GL}(2)_E$ to ${}^L\mathrm{GL}(3)_E$ is defined as:

$$Ad : [\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})] \rtimes \mathcal{W}_F \longrightarrow [\mathrm{GL}_3(\mathbf{C}) \times \mathrm{GL}_3(\mathbf{C})] \rtimes \mathcal{W}_F$$

sending $(g, g, \sigma) \mapsto (Ad(g), Ad(g), \sigma)$. Functoriality for this L-map was proven by Gelbart and Jacquet ([GJ]).

Let π be a representation of $\mathrm{GL}_3(\mathbf{A}_E)$. From the fact that

$$Ad(\begin{pmatrix} a & \\ & d \end{pmatrix}) = \begin{pmatrix} \frac{a}{d} & & \\ & 1 & \\ & & \frac{d}{a} \end{pmatrix} \sim \begin{pmatrix} \frac{d}{a} & & \\ & 1 & \\ & & \frac{a}{d} \end{pmatrix}$$

it follows that if π is an adjoint lift, it satisfies $g(\pi_w) \sim g(\pi_w)^{-1}$ for almost every place w. Thus $\pi \cong \pi^*$, where π^* is the representation contragredient to π. It is also clear that π must have trivial central character. In fact the image of the adjoint lifting is precisely the set of automorphic representations with trivial central characters which are self-dual (see table (1) in [GRS]). Furthermore if π is cuspidal, then any descent to $\mathrm{GL}(2)$ is also cuspidal.

The kernel of Ad is the set of scalar matrices. This implies that the fibers of the adjoint lifting are twist classes of representations, i.e. π_1 and π_2 have the same adjoint lift if and only if $\pi_1 \cong \pi_2 \otimes \chi$ for some Hecke character χ.

Base change for U

The L-group of U is $\mathrm{GL}_3(\mathbf{C}) \rtimes \mathcal{W}_F$, with action of $\mathcal{W}_F$ factoring through $\mathrm{Gal}(E/F)$ given by $c(g) = \Phi_3^{-1}\,{}^t g^{-1} \Phi_3$ for complex conjugation c, where $\Phi_3 = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$. Define a map from ${}^L\mathrm{U}$ to ${}^L\mathrm{Res}_F^E(\mathrm{GL}(3))$:

$$\mathcal{BC} : \mathrm{GL}_3(\mathbf{C}) \rtimes \mathcal{W}_F \longrightarrow [\mathrm{GL}_3(\mathbf{C}) \times \mathrm{GL}_3(\mathbf{C})] \rtimes \mathcal{W}_F$$

by $(g,\sigma) \mapsto (g, \Phi_3\,{}^t g^{-1} \Phi_3^{-1}, \sigma)$. The transfer of L-packets corresponding to this L-map exists and is injective ([R1], chapter 13).

Remark. Let $\widetilde{G} = \mathrm{Res}_F^E(\mathrm{GL}(3))$, and let $\widetilde{\mathrm{U}} = \mathrm{Res}_F^E(\mathrm{U})$. Of course the two groups are isomorphic. Define a map

$$H : {}^L\widetilde{\mathrm{U}} \longrightarrow {}^L\widetilde{G}$$

by $(x,y,\sigma) \mapsto (x, \Phi_3^{-1}\,{}^t y^{-1} \Phi_3, \sigma)$. H is an isomorphism of L-groups. The L-map $\mathcal{BC}$ defined above is $H \circ \psi$, where $\psi : {}^L\mathrm{U} \longrightarrow {}^L\widetilde{\mathrm{U}}$ is the standard base change L-map for U as defined in [R1].

An L-packet Π on U is **discrete** if some member of Π occurs in the discrete spectrum of U (cf. [R2] §2). A discrete L-packet Π is **stable** if it is not the functorial transfer of a discrete representation of the endoscopic subgroup $H = \mathrm{U}(2) \times \mathrm{U}(1)$. Π is **cuspidal** if every discrete element of Π is cuspidal.

Theorem 8.1 ([R1], Theorem 13.3.3). *The base change lifting from* U *to* $\mathrm{GL}_3(\mathbf{A}_E)$ *defines a bijection between the set of stable L-packets Π on* U *and the set of discrete automorphic representations π of* $\mathrm{GL}(3)_E$ *which satisfy* $\overline{\pi} \cong \pi^*$ *and* $\chi_\pi|_{I_F} = 1$, *where χ_π is the central character of π. Furthermore, a discrete L-packet Π is infinite-dimensional $\Leftrightarrow$ Π is cuspidal $\Leftrightarrow$ $\mathcal{BC}(\Pi)$ is cuspidal.*

Definition of Ad_{U}

Define an L-map from ${}^L\mathrm{GL}(2)_F$ to ${}^L\mathrm{U}$:

$$Ad_{\mathrm{U}} : \mathrm{GL}_2(\mathbf{C}) \times \mathcal{W}_F \longrightarrow \mathrm{GL}_3(\mathbf{C}) \rtimes \mathcal{W}_F$$

by $(g,\sigma) \mapsto (Ad(g), \sigma)$. Ad_{U} is a homomorphism because

$$\Phi_3^{-1}\,{}^t Ad(g)^{-1} \Phi_3 = Ad(g).$$

This equality holds since $Ad(g)$ preserves the bilinear form $\mathrm{tr}(XY)$ which is given by the matrix Φ_3. This also insures the commutativity of the diagram since

$$\begin{aligned} Ad(\mathcal{BC}(g,\sigma)) &= (Ad(g), Ad(g), \sigma) \\ &= (Ad(g), \Phi_3^{-1}\,{}^t Ad(g)^{-1} \Phi_3, \sigma) = \mathcal{BC}(Ad_{\mathrm{U}}(g,\sigma)). \end{aligned}$$

We now verify the functoriality of Ad_{U} in the case needed for this paper. The same proof will work in the general case provided one verifies that the above descent condition $\overline{\pi} \cong \pi^*$ and $\chi_\pi|_{I_F} = 1$ is valid for all automorphic representations of $\mathrm{GL}(3)_E$, not just the cuspidal ones.

Proposition 8.2. *Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_F)$, let $\pi_E = \mathcal{BC}(\pi)$, and let $\pi_3 = Ad(\pi_E)$. Suppose that π_3 is cuspidal. Then there exist a stable cuspidal L-packet $\Pi = Ad_{\mathrm{U}}(\pi)$ on $\mathrm{U}(\mathbf{A}_F)$ and a finite set S of places of F, including all places where π and Π are ramified, such that for all $v \notin S$, $Ad_{\mathrm{U}}(g(\pi_v)) \sim g(\pi'_v)$, where π'_v is the unique unramified element of the local L-packet Π_v. Π is called the* **unitary adjoint lift** *of π.*

Proof. As a cuspidal representation of $\mathrm{GL}_3(\mathbf{A}_E)$, π_3 is the base change lift of an L-packet of representations of $\mathrm{U}(\mathbf{A}_F)$ if and only if $\overline{\pi}_3 \cong \pi_3^*$ and $\chi_{\pi_3}|_{I_F} = 1$. The second condition is automatic since $\chi_{\pi_3} = 1$, π_3 being an adjoint lift. The condition $\overline{\pi}_3 \cong \pi_3^*$ only needs to be verified locally at almost every place of E by the strong multiplicity-one theorem for $\mathrm{GL}(3)$.

Let v be any place of F which lies outside the exceptional sets for all of the lifts of π discussed here. Write $\begin{pmatrix} a & \\ & d \end{pmatrix}$ for the class of $g(\pi_v)$. Let w be any place of E lying over v. Set

$$d_w = [E_w : F_v] = \begin{bmatrix} 1 \text{ if } v \text{ splits in } E \\ 2 \text{ if } v \text{ is inert in } E. \end{bmatrix}$$

Then by the property of the base change lifting,

$$g(\pi_{E,w}) \sim g(\pi_v)^{d_w} \sim \begin{pmatrix} a^{d_w} & \\ & d^{d_w} \end{pmatrix}.$$

This is clearly independent of the choice of $w|v$ in E. Hence

$$g(\pi_{3,w}) \sim \begin{pmatrix} \frac{a}{d} & & \\ & 1 & \\ & & \frac{d}{a} \end{pmatrix}^{d_w}$$

is also independent of the choice of $w|v$. This shows that $\overline{\pi}_3 \cong \pi_3$ since for almost every place w of E, $g(\overline{\pi}_{3,w}) = g(\pi_{3,\overline{w}}) = g(\pi_{3,w})$. On the other hand, $\pi_3 \cong \pi_3^*$ since π_3 is an adjoint lift. Putting these statements together, we see that $\pi_3^* \cong \overline{\pi}_3$.

Hence there exists a stable cuspidal L-packet Π of automorphic representations of $\mathrm{U}(\mathbf{A}_F)$ such that $\mathcal{BC}(\Pi) = \pi_3$. Define $Ad_{\mathrm{U}}(\pi) = \Pi$. Let S be the finite set of places v of F which are exceptional either for the base change of Π to E or for any of the lifts of π discussed here. For each $v \notin S$, let π'_v be the unique unramified element of Π_v. Viewing π_3 as a representation of $\mathrm{Res}_F^E(\mathrm{GL}(3))$ over $\mathbf{A}_F$, the relationship $g(\pi'_v) \sim Ad_{\mathrm{U}}(g(\pi_v))$ follows immediately by comparing the Langlands classes $g(\pi_{3,v}) \sim \mathcal{BC}(g(\pi'_v))$. □

The next task is to give a descent condition for Ad_{U} for stable L-packets. We first remark that the center of U consists of the unitary scalar matrices and may be indentified with $\mathrm{U}(1)$. A property of L-packets is that all elements of an L-packet have the same central character.

Proposition 8.3. *Let Π be a stable cuspidal L-packet on* U *with central character χ_Π. Let $\pi = \mathcal{BC}(\Pi)$. Then the following are equivalent:*

(1) Π is the unitary adjoint lift of a cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_F)$
(2) $\Pi \cong \Pi^*$ and $\chi_\Pi = 1$
(3) π is the adjoint lift of a cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_E)$.

Proof. (1) ⇒ (2): Suppose $\Pi = Ad_{\mathrm{U}}(\tau)$. Let v be a place of F at which both Π and τ are unramified. Write $\begin{pmatrix} a & \\ & d \end{pmatrix}$ for the class of $g(\tau_v)$. Then $g(\Pi_v) = \begin{pmatrix} \frac{a}{d} & & \\ & 1 & \\ & & \frac{d}{a} \end{pmatrix} \times w_v$, where $w_v \in \mathcal{W}_F$ projects to Fr_v in $\mathrm{Gal}(E/F)$. This shows both that $\Pi_v \cong \Pi_v^*$, and that the unramified element of Π_v has trivial central character. Hence $\Pi \cong \Pi^*$ and $\chi_\Pi = 1$ by strong multiplicity-one for L-packets on $\mathrm{U}(n)$ for $n \leq 3$.

(2) ⇒ (3): Let v be a place of F where Π is unramified. If v splits in E, let w_1 and w_2 lie over v in E. Then $\mathrm{U}(F_v) \cong \mathrm{GL}_3(E_{w_1})$, and the unramified element of Π_v may be viewed as an unramified representation of $\mathrm{GL}_3(E_{w_1})$. Then (as described in [R2], §4.2) we can take $g(\pi_{w_1}) = g(\Pi_v)$ and $g(\pi_{w_2}) = g(\Pi_v)^{-1}$. Thus $\pi_{w_i} \cong \pi_{w_i}^*$ if and only if $\Pi_v \cong \Pi_v^*$. If v is inert in E, then the condition $\pi_v \cong \pi_v^*$ is automatic since $\overline{\pi}_v = \pi_v$, and $\overline{\pi}_v \cong \pi_v^*$. Hence $\Pi \cong \Pi^*$ implies π is self-dual. By the injectivity of the base change for $\mathrm{U}(1)$, we have $(\chi_\Pi)_E = \chi_\pi = 1$. Thus the conditions are met for π to be the adjoint lift of a cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_E)$.

(3) ⇒ (1): Suppose there exists a cuspidal representation τ of $\mathrm{GL}_2(\mathbf{A}_E)$ whose adjoint lift $Ad(\tau)$ is π. We wish to descend τ further to a representation of $\mathrm{GL}_2(\mathbf{A}_F)$. For this we need to verify that $\tau \cong \overline{\tau}$. We first check that τ and $\overline{\tau}$ are in the same fiber of the adjoint lifting. As a base change lift from U, π satisfies $\pi^* \cong \overline{\pi}$, and as an adjoint lift it satisfies $\pi^* \cong \pi$. The $\mathrm{Gal}(E/F)$ action on representations commutes with the adjoint lifting, so

$$Ad(\overline{\tau}) = \overline{\pi} \cong \pi^* \cong \pi = Ad(\tau).$$

The fibers of Ad are twist classes of representations, so the above implies that $\overline{\tau} \cong \tau \otimes \omega$ for some Hecke character ω. Lapid and Rogawski have classified all such representations:

Theorem 8.4. ([LR]) *Let ω be a Hecke character of a number field E, let σ be an automorphism of E with fixed field F, and let τ be a cuspidal representation* *of* $\mathrm{GL}_2(\mathbf{A}_E)$ *such that* $\sigma(\tau) \cong \tau \otimes \omega$. *Let* K *be the extension of* F *attached to the restriction* ω_F *of* ω *to* I_F. *Then* $[K : F] \leq 2$ *and*

(1) If $K = F$ *(i.e.* ω_F *is trivial), then* $\tau \otimes \psi$ *is the base change lift of a cuspidal representation of* $\mathrm{GL}_2(\mathbf{A}_F)$, *where* ψ *is any Hecke character such that* $\psi^{1-\sigma} = \omega$.
(2) If K/F *is a quadratic extension, then* $L = KE$ *is a quadratic extension of* E, *and there exists a Hecke character* θ *of* L *such that* $\tau = \mathrm{AI}_L^E(\theta)$.

In our situation, τ is not automorphically induced from a Hecke character because $Ad(\tau) = \pi$ is cuspidal (cf. [R3] §13). Hence replacing τ by $\tau \otimes \psi$, we may assume that $\tau \cong \overline{\tau}$, so there exists a cuspidal representation τ_F of $\mathrm{GL}_2(\mathbf{A}_F)$ whose base change is τ. By the commutativity of the diagram, and the injectivity of the U base change lifting, it is clear that $Ad_{\mathrm{U}}(\tau_F) = \Pi$.

References

[AC] Arthur, J., Clozel, L. Simple algebras, base change, and the advanced theory of the trace formula. Princeton University Press, Princeton, NJ, 1989

[BJ] Borel, A., Jacquet, H. Automorphic forms and automorphic representations. Proc. Symp. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979

[BR1] Blasius, D., Rogawski, J. Tate classes and arithmetic quotients of the two-ball, In: [M].

[BR2] Blasius, D., Rogawski, J. Zeta functions of Shimura varieties, In: [Mot], 525–571

[F] Finis, T. Some computational results on Hecke eigenvalues of modular forms on a unitary group. Manuscripta Math., **96**(2), 149–180 (1998)

[Go] Gordon, B. Canonical models of Picard modular surfaces. In: [M]

[GJ] Gelbart, S., Jacquet, H. A relation between automorphic representations of $\mathrm{GL}(2)$ and $\mathrm{GL}(3)$. Ann. Sci. École Norm. Sup. **11**(4), 471–542 (1978)

[GRS] Ginzburg, D., Rallis, S., Soudry, D. Self-dual automorphic GL_n modules and construction of a backward lifting from GL_n to classical groups. Internat. Math. Res. Not. **14**, 687–701 (1997)

[K] Knightly, A. Representations of unitary groups and associated Galois representations. Thesis, UCLA, 2000

[LR] Lapid, E., Rogawski, J. On twists of cuspidal representations of $\mathrm{GL}(2)$. Forum Math. **10**, 175–197 (1998)

[M] Langlands, R., Ramakrishnan, U. (eds.) Zeta functions of Picard modular surfaces. Montreal Press 1992

[Mo] Momose, F. On the l-adic representations attached to modular forms. Univ. Tokyo Jour. Fac. Sci. (28), pp. 89–109, 1981

[Mot] Motives (Seattle, WA, 1991). Proc. Sympos. Pure Math. **55**(2), Amer. Math. Soc., Providence, RI, 1994

[Ri1] Ribet, K. Galois representations attached to eigenforms with nebentypus. Modular functions of one variable V, Springer Lecure Notes, 601, Springer, pp. 17–52, 1977

[Ri2] Ribet, K. Twists of modular forms and endomorphisms of abelian varieties. Math. Ann. **253**, 43–62 (1980)

[Ri3] Ribet, K. On l-adic representations attached to modular forms II. Glasgow Math. J. **27**, 185–194 (1985)

[R1] Rogawski, J. Automorphic representations of unitary groups in three variables. Princeton University Press, Princeton, NJ, 1990

[R2] Rogawski, J. Analytic expression for the number of points mod p. In: [M]

[R3] Rogawski, J. Functoriality and the Artin conjecture. Proc. Sympos. Pure Math. 61, Amer. Math. Soc., Providence, RI, 1997

[S1] Serre, J-P. Une interprétation des congruences relatives à la fonction τ de Ramanujan. Séminaire Delange-Pisout-Poitou, 1967/1968, exposé 14

[S2] Serre, J-P. Abelian ℓ-adic representations and elliptic curves. revised reprint of the 1968 edition, A. K. Peters, Ltd., Wellesley, MA, 1998

[S3] Serre, J-P. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. Invent. Math. **15**(4), 259–331 (1972)

[S4] Serre, J-P. Congruences et formes modulaires (d'après H. P. F. Swinnerton-Dyer). Séminaire Bourbaki 1971/72, 416