MODULAR L-VALUES OF CUBIC LEVEL

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In memory of Jonathan Rogawski

Abstract. Using a simple relative trace formula, we compute averages of twisted modular L-values for newforms of cubic level. In the case of Maass forms, we obtain an exact formula. For holomorphic forms of weight $k > 2$, we obtain an asymptotic formula which agrees with the estimate predicted by the Lindelöf hypothesis in the weight and level aspects.

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1. Introduction

A *simple* trace formula is one in which a local discrete series matrix coefficient is used, thereby annihilating the contribution of the continuous spectrum (see Lecture V of [Ge] for a general overview). By choosing the matrix coefficient appropriately, one can also project onto a particular local new vector. For example, using the matrix coefficient attached to a lowest weight vector for the weight k discrete series of $\mathrm{GL}_2(\mathbf{R})$, one isolates the space of holomorphic cusp forms of weight k from the rest of the automorphic spectrum. In essence, this was the method used by Selberg in his formula for the trace of a Hecke operator ([Sel] §4).

In this paper we give a non-archimedean illustration of this technique, using matrix coefficients attached to certain supercuspidal representations of $\mathrm{GL}_2(\mathbf{Q}_p)$. We work with a relative trace formula to compute averages of the form

$$\sum_{u\in\mathcal{F}} \frac{\lambda_n(u)\overline{a_r(u)}\Lambda(s,u,\chi)}{\|u\|^2} B_r(u),$$

where u ranges over the set of newforms of weight k and level N^3 for N square-free and $k > 2$ or $k = 0$, $\lambda_n(u)$ is the associated eigenvalue of the Hecke operator T_n, $a_r(u)$ is the r-th Fourier coefficient, $\Lambda(s,u,\chi)$ is the completed L-function, twisted by a fixed primitive character χ of conductor D prime to N, and $B_r(u)$ is a function of the spectral parameter of u with sufficient decay, which we take to be 1 in the case of holomorphic forms (i.e. when $k > 2$).

We have two main results, one for Maass newforms and one for holomorphic newforms. Each is an explicit version of the relative trace formula introduced by Jacquet in [J]. In broad terms, we start with a kernel function attached to the Hecke operator T_n, and integrate it (against a character) over the group $N \times M$, where N is unipotent and M is diagonal. The unipotent integral gives the Fourier

In the published article, equation (2.23) is incorrect, and consequently so are equations (3.6) and (3.9). All other results, including the main theorems, are unaffected. This is a corrected version of the paper.

coefficient $\overline{a_r(u)}$, and the diagonal integral gives the L-function. The geometric side reduces to the calculation of numerous local orbital integrals.

The result for Maass forms is given in Theorem 5.4 below. A special case of it is the following exact expression for a weighted average of Maass newform L-values:

Theorem 1.1. *Let χ be a primitive Dirichlet character with modulus D. Let $h(iz)$ be any even Paley-Wiener function, and let $h_1(s)$ be the $e^{-2\pi ix}$-twisted spherical transform of the inverse Selberg transform of h (cf. (5.5)). Then there exists a constant $C \geq 1$ depending only on h, such that for all square-free integers $N > C$ prime to D and all complex numbers s,*

$$\sum_{u_j \in \mathcal{F}_+^{new}(N^3)} \frac{\Lambda(s, u_j, \chi)}{\psi(N^3)\|u_j\|^2} h(t_j) K_{it_j}(2\pi) = 2h_1(s) \prod_{p|N} (1 - \tfrac{1}{p}). \tag{1.1}$$

Here, $\mathcal{F}_+^{new}(N^3)$ denotes the set of even Maass newforms on $\Gamma_0(N^3)$ of weight 0 and trivial central character, normalized with first Fourier coefficient $a_1(u) = 1$, t_j is the spectral parameter of u_j, $K_\nu(x)$ is the Bessel function, and $\psi(N^3)$ denotes the index $[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N^3)]$.

Remarks:

(1) It is interesting to note that the right-hand side of (1.1) (and hence also the left-hand side for N sufficiently large) is independent of χ.

(2) Given any $s \in \mathbf{C}$, we can choose h so that $h_1(s)$ is nonzero. Therefore an immediate consequence is the existence of a Maass newform of level N^3 with nonvanishing twisted L-value at s.

(3) We normalize the Petersson norm on page 4 so that it is independent of the choice of level and coincides with the adelic L^2-norm. Many people write $\|u\|^2$ where we have written $\psi(N^3)\|u\|^2$.

The analogous result for holomorphic cusp forms is stated in Theorem 4.1. In that case, we no longer have an exact formula because the archimedean discrete series matrix coefficient is not compactly supported. But the resulting asymptotic formula still gives nonvanishing, as well as a bound for the sum of the central L-values which is as strong as that predicted by the Lindelöf Hypothesis in the weight and level aspects (cf. Corollary 4.3). In Corollary 4.4, we compare the contribution of newforms and oldforms in the analogous sum for the full space of cusp forms of level N^3. When N is prime, the contribution of oldforms becomes negligible as $N \to \infty$, but in the other extreme, if N is the product of the first m primes, the contribution of newforms becomes negligible as $m \to \infty$.

In both of our main results, we project onto the newforms of cubic level by using the simple supercuspidal representations defined by Gross and Reeder [GR]. Matrix coefficients for these representations have previously been used in the trace formula by Gross in [Gr], where, for a simple group over a totally real number field, he computed the multiplicities of cuspidal representations with certain prescribed local behavior in terms of values of modified Artin L-functions at negative integers. The local test vector used by Gross has a very simple matrix coefficient and is ideally suited for counting representations. However, it is not a new vector so it cannot be used for our purpose here.

In [KL4], we defined simple supercuspidal representations for the group $\mathrm{GL}_n(\mathbf{Q}_p)$, showing that they have conductor p^{n+1} and exhibiting the new vector. We then gave an explicit formula for the matrix coefficient attached to the new vector in the case where $n = 2$. Lastly, we showed that every irreducible admissible representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ with conductor p^3 is a simple supercuspidal representation, assuming that its central character is unramified or tamely ramified. In the present paper, at each place $p|N$ we sum the new vector matrix coefficients attached to the $2(p-1)$ distinct simple supercuspidal representations to obtain a test function which projects onto the newforms of level N^3 and annihilates the continuous spectrum.

We restrict to the field $\mathbf{Q}$ throughout for simplicity, but since all of the computations are local, there would be no serious obstruction to working over an arbitrary totally real number field.

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2. Preliminaries

2.1. **Orthogonality of matrix coefficients.** The proposition below, which has been attributed to Langlands, will be a key ingredient in what follows.

Proposition 2.1. *Let G be a unimodular locally compact group with center Z. Let (π, V) be an irreducible unitary square integrable representation of G with formal degree d_π. Let $w \in V$ be a unit vector, and suppose that the function $f(g) = d_\pi \overline{\langle \pi(g)w, w\rangle}$ is absolutely integrable over $\overline{G} = G/Z$. Then for any irreducible unitary representation (ρ, W) of G with the same central character as π (but not necessarily square integrable), the operator $\rho(f)$ is identically zero on W unless $\rho \cong \pi$. Furthermore, $\pi(f)$ is the orthogonal projection operator from V onto $\mathbf{C}w$.*

Remark: The formal degree d_π depends on a choice of Haar measure on $\overline{G}$, as does the operator $\pi(f)$. We must assume that these measures are the same.

Proof. See Corollary 10.29 of [KL1]. □

2.2. **Notation and measure.** Given a prime number p and an integer x, we write $x_p = \mathrm{ord}_p(x)$, so that $x = \prod_p p^{x_p}$.

Let $\mathbf{A}, \mathbf{A}_{\mathrm{fin}}$ be the adeles and finite adeles of $\mathbf{Q}$, and henceforth let $G = \mathrm{GL}(2)$. Write $\overline{G} = G/Z$, where Z is the center of G. We let $Z_p = Z(\mathbf{Q}_p)$ and $Z_\infty = Z(\mathbf{R})$ be the respective centers of $G(\mathbf{Q}_p)$ and $G(\mathbf{R})$. We also set $K_\infty = \mathrm{SO}(2)$ and $K_p = \mathrm{GL}_2(\mathbf{Z}_p)$.

We take Lebesgue measure dx on $\mathbf{R}$, and we use the measure $d^*y = \frac{dy}{|y|}$ on $\mathbf{R}^*$. On $\mathbf{Q}_p$ and $\mathbf{Q}_p^*$ we normalize the Haar measures so that $\mathrm{meas}(\mathbf{Z}_p) = 1$ and $\mathrm{meas}(\mathbf{Z}_p^*) = 1$ respectively. With these choices, the product measure on $\mathbf{A}$ has the property that $\mathrm{meas}(\mathbf{Q}\backslash\mathbf{A}) = 1$. In $\mathbf{A}_{\mathrm{fin}}^*$ we have $\mathrm{meas}(\widehat{\mathbf{Z}}^*) = 1$. We normalize Haar measure on $G(\mathbf{Q}_p)$ by taking $\mathrm{meas}(K_p) = 1$. Likewise in $\overline{G}(\mathbf{Q}_p)$ we take

$\mathrm{meas}(\overline{K_p}) = 1$. On $\overline{G}(\mathbf{A}_{\mathrm{fin}})$, we give $\overline{G}(\widehat{\mathbf{Z}})$ the measure 1. We normalize Haar measure on $\overline{G}(\mathbf{A})$ so that $\mathrm{meas}(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})) = \pi/3$. See [KL1] for further details about this normalization.

We let $\theta : \mathbf{A} \longrightarrow \mathbf{C}^*$ be the nontrivial character given locally by

$$\theta_p(x) = \begin{cases} e^{-2\pi ix} & \text{if } p = \infty \quad (x \in \mathbf{R}) \\ e^{2\pi i r_p(x)} & \text{if } p < \infty \quad (x \in \mathbf{Q}_p), \end{cases} \tag{2.1}$$

where $r_p(x) \in \mathbf{Q}$ is the p-principal part of x, a number with p-power denominator characterized up to $\mathbf{Z}$ by $x \in r_p(x) + \mathbf{Z}_p$. The kernel of θ_p is $\mathbf{Z}_p$, and θ is trivial on $\mathbf{Q} \subset \mathbf{A}$.

2.3. **Cusp forms.** Let k be a nonnegative integer. Eventually we will assume further that $k \neq 1, 2$. Let N be a positive integer, and let ω' be a Dirichlet character modulo N satisfying

$$\omega'(-1) = (-1)^k.$$

Define the Hecke congruence subgroups

$$\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) |\, c \in N\mathbf{Z}\},$$

$$\Gamma_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) |\, d \in 1 + N\mathbf{Z}\},$$

and let

$$\psi(N) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + \frac{1}{p}). \tag{2.2}$$

Consider the space of measurable complex-valued functions u on the complex upper half-plane $\mathbf{H}$ which have the following properties:

(1) For all $z \in \mathbf{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$

$$u\Big(\frac{az + b}{cz + d}\Big) = \overline{\omega'(d)}(cz + d)^k u(z). \tag{2.3}$$

(2) u has finite Petersson norm:

$$\|u\|^2 = \frac{1}{\psi(N)} \int_{\Gamma_0(N)\backslash\mathbf{H}} |u(x + iy)|^2 y^k \frac{dx\, dy}{y^2} < \infty.$$

(3) u is holomorphic if $k > 0$.

(4) u is cuspidal: at each cusp of $\Gamma_1(N)$ it has a constant term which vanishes almost everywhere (see e.g. §4.1 of [KL3] for a detailed definition).

We denote this space by $S_k(N, \omega')$ if $k > 0$, and by $L^2_0(N, \omega')$ if $k = 0$. The latter space is infinite-dimensional if nonzero, but it has a basis consisting of Maass forms, i.e. those elements which are eigenfunctions of the Laplacian $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. We write the Laplace eigenvalue as

$$\Delta u = (\tfrac{1}{4} + t^2)u,$$

and refer to t as the **spectral parameter** of u. We know that $t \in \mathbf{R}^* \cup i(-\frac{1}{2}, \frac{1}{2})$, with the number of u with **exceptional** (non-real) parameter being finite.

If u is continuous, condition (1) implies that u has a Fourier expansion of the form

$$u(x + iy) = \sum_{n \neq 0} a_n(u, y) e^{2\pi i n x}.$$

The coefficient $a_n(u,y)$ has the well-known form

$$a_n(u,y) = \begin{cases} a_n(u)e^{-2\pi ny} & \text{if } n,k>0 \\ 0 & \text{if } k>0, n<0 \\ a_n(u)y^{1/2}K_{it}(2\pi|n|y) & \text{if } k=0, \end{cases} \tag{2.4}$$

where K_{it} is the Bessel function and t is the spectral parameter of u.

The weight k Hecke operator T_n is defined by

$$T_n u(z) = n^{\alpha(k)} \sum_{\substack{ad=n \\ a>0}} \sum_{b=0}^{d-1} \overline{\omega'(a)} d^{-k} u(\frac{az+r}{d}),$$

where $\alpha(k) = k-1$ if $k>0$ and $\alpha(k) = -1/2$ if $k=0$. If u is a Hecke eigenform, we denote the eigenvalues by $T_n u = \lambda_n(u)u$. We say that u is a **newform** if its Hecke eigenvalue packet $\{\lambda_p(u)\}_{p\nmid N}$ has an eigenspace that is exactly one-dimensional. In this case, $a_1(u) \neq 0$, and we will normalize so that $a_1(u) = 1$. Under this normalization,

$$a_n(u) = \lambda_n(u) \tag{2.5}$$

for all n. We let

$$\mathcal{F}_k^{new}(N,\omega') = \{\text{newforms } u, \text{ with } a_1(u) = 1\}.$$

We also define $T_{-1}u(x+iy) = u(-x+iy)$. A Maass cusp form is **even** (resp. **odd**) if $T_{-1}u = u$ (resp. $T_{-1}u = -u$). If u is even, then in (2.4) we have $a_{-n}(u) = a_n(u)$, while if u is odd, $a_n(u) = -a_{-n}(u)$. It is a basic fact that $L_0^2(N,\omega')$ has an orthogonal basis consisting of Maass eigenforms which are also eigenfunctions of T_{-1}. We let

$$\mathcal{F}_+^{new}(N,\omega') = \{u \in \mathcal{F}_0^{new}(N,\omega') |\, u \text{ is even}\}.$$

We define the L-function of u by

$$L(s,u) = \sum_{n=1}^{\infty} a_n(u) n^{-s}.$$

This converges absolutely when $\mathrm{Re}(s)$ is sufficiently large. We define the completed L-function by

$$\Lambda(s,u) = \begin{cases} (2\pi)^{-s}\Gamma(s)L(s,u) & k>0 \\ \pi^{-s}\Gamma(\frac{s+\varepsilon+it}{2})\Gamma(\frac{s+\varepsilon-it}{2})L(s,u) & k=0, \end{cases} \tag{2.6}$$

where $\varepsilon = 0$ or 1 according to whether u is even or odd. It has an analytic continuation which satisfies a functional equation relating values at s and $1-s$ when $k=0$, and at s and $k-s$ when $k>0$.

2.4. **Adelic cusp forms.** Let ω be the Hecke character attached to ω' by

$$\omega : \mathbf{A}^* = \mathbf{Q}^*(\mathbf{R}^+ \times \widehat{\mathbf{Z}}^*) \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^* \tag{2.7}$$

where the first two arrows are the canonical projections, and the last arrow is ω'. For $q>0$, let $L^q(\omega) = L^q(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}),\omega)$ denote the space of measurable $G(\mathbf{Q})$-invariant functions $\phi : G(\mathbf{A}) \longrightarrow \mathbf{C}$ which transform under the center by ω, and satisfy $\int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} |\phi(g)|^q dg < \infty$. When $q=2$, we let $L_0^2(\omega) \subset L^2(\omega)$ denote the subspace of cuspidal functions.

Letting

$$K_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\widehat{\mathbf{Z}}) |\, c, d-1 \in N\widehat{\mathbf{Z}}\},$$

we embed $S_k(N, \omega')$ and $L_0^2(N, \omega')$ (taking $k = 0$ in the latter case) isometrically into $L_0^2(\omega)$ by defining

$$\phi_u(\gamma(g_\infty \times g_{\text{fin}})) = j(g_\infty, i)^{-k} u(g_\infty(i)) \tag{2.8}$$

for $\gamma(g_\infty \times g_{\text{fin}}) \in G(\mathbf{Q})(G(\mathbf{R})^+ \times K_1(N)) = G(\mathbf{A})$ and

$$j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = (ad - bc)^{-1/2}(cz + d).$$

In the $k = 0$ case, the map $u \mapsto \phi_u$ is a surjective linear isometry from $L_0^2(N, \omega')$ to $L_0^2(\omega)^{K_\infty \times K_1(N)}$ (the $K_\infty \times K_1(N)$-invariant vectors), (cf. [KL3], Proposition 4.5).

Lemma 2.2. *Let u be a holomorphic Hecke eigenform ($k > 0$) or a Maass eigenform with spectral parameter t ($k = 0$). Then for $r \in \mathbf{Q}$,*

$$\int_{\mathbf{Q}\backslash\mathbf{A}} \phi_u(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))\theta(rx)dx = \begin{cases} a_r(u)K_{it}(2\pi|r|) & \textit{if } r \in \mathbf{Z}, k = 0 \\ e^{-2\pi r}a_r(u) & \textit{if } r \in \mathbf{Z}^+, k > 0 \\ 0 & \textit{otherwise,} \end{cases}$$

where θ is the character defined in (2.1)*. For all $s \in \mathbf{C}$,*

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*} \phi_u(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right))|y|^{s-k'/2}d^*y = \begin{cases} \frac{1}{2}\Lambda(s, u) & \textit{if } k = 0,\ u \textit{ is even} \\ 0 & \textit{if } k = 0,\ u \textit{ is odd} \\ \Lambda(s, u) & \textit{if } k > 0, \end{cases}$$

where $\Lambda(s, u)$ is the completed L-function defined in (2.6) *and* $k' = \begin{cases} k & k > 2 \\ 1 & k = 0. \end{cases}$

Each of the above integrals is absolutely convergent.

Proof. For a proof of the first statement, see Corollary 12.4 of [KL1] and Lemma 7.1 of [KL3]. For the second, suppose $k = 0$. Using the fundamental domain $\mathbf{R}^+ \times \widehat{\mathbf{Z}}^*$ for $\mathbf{Q}^*\backslash\mathbf{A}^*$, we have

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*} \phi_u(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right))|y|^{s-1/2}d^*y = \int_0^\infty u(iy)y^{s-1/2}\frac{dy}{y}.$$

The result then follows by a well-known classical computation using the Fourier expansion (cf. [Go], pp. 86). The proof when $k > 0$ is similar; see e.g. Lemma 3.1 of [KL2]. □

2.5. **Newforms.** Here we will define a space of adelic newforms, and realize the orthogonal projection onto it as an integral operator.

We wish to study newforms with certain local behavior. Let N be an integer multiple of the conductor of ω with the property that $N_p \geq 2$ for all $p|N$. For each $p|N$, let σ_p be a fixed supercuspidal representation of $G(\mathbf{Q}_p)$ with central character ω_p and conductor p^{N_p}. Let $\widehat{\sigma}$ denote the tuple $\{\sigma_p\}_{p|N}$.

Under the action of $G(\mathbf{A})$ on $L_0^2(\omega)$ by right translation, the space decomposes as a direct sum of irreducible cuspidal representations π. Given a nonnegative integer $k \neq 1, 2$ (i.e. $k \in \{0, 3, 4, 5, \ldots\}$), we define the subspace

$$H_k(\widehat{\sigma}, \omega) = \bigoplus \pi \subset L_0^2(\omega), \tag{2.9}$$

where π ranges through the irreducible cuspidal representations for which:

(1) $\pi_p = \sigma_p$ for all $p|N$.
(2) π_p is unramified for all finite $p \nmid N$.
(3) π_∞ is a spherical principal series representation of $G(\mathbf{R})$ with trivial central character if $k = 0$.
(4) π_∞ is the weight k discrete series representation π_k of $G(\mathbf{R})$ with central character $\left(\begin{smallmatrix} z & \\ & z \end{smallmatrix}\right) \mapsto \operatorname{sgn}(z)^k$ if $k > 2$.

For each such $\pi = \otimes' \pi_p$, define a vector (the "newform") $w_\pi = \otimes w_{\pi_p}$ in the space of π by taking

$$w_{\pi_p} = \begin{cases} \text{unit new vector ([Ca])} & p|N \\ \text{unit unramified vector} & p \nmid N\infty \\ \text{unit spherical vector} & p = \infty, k = 0 \\ \text{unit lowest weight vector} & p = \infty, k > 2, \end{cases}$$

where, in almost every unramified case, the unit vector is the one predetermined by the restricted tensor product. In each case, the vector w_{π_p} is unique up to unitary scaling. Let

$$A_k(\widehat{\sigma}, \omega) = \bigoplus_\pi \mathbf{C} w_\pi \subset H_k(\widehat{\sigma}, \omega). \tag{2.10}$$

This corresponds to a classical space of newforms of level N on the upper half-plane. Letting $\phi_\pi \in L_0^2(\omega)$ denote the function defined by ω_π, the associated cusp form on $\mathbf{H}$ is given by

$$u(x + iy) = y^{-k/2} \phi_\pi(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)_\infty \times 1_{\text{fin}}) \qquad (y > 0). \tag{2.11}$$

This is the inverse of the association (2.8), i.e., $\phi_u = \phi_\pi$.

For $p|N$, define a function $f_p : G(\mathbf{Q}_p) \longrightarrow \mathbf{C}$ by

$$f_p(g) = d_p \overline{\langle \sigma_p(g) w_{\sigma_p}, w_{\sigma_p} \rangle} \qquad (p|N), \tag{2.12}$$

where d_p is the formal degree of the supercuspidal representation σ_p relative to our choice of Haar measure on $\overline{G}(\mathbf{Q}_p)$, and the inner product is $G(\mathbf{Q}_p)$-invariant. Likewise, if $p = \infty$ and $k > 2$ we take

$$f_\infty(g) = d_k \overline{\langle \pi_\infty(g) w_{\pi_\infty}, w_{\pi_\infty} \rangle} \qquad (k > 2), \tag{2.13}$$

where d_k is the formal degree of the discrete series representation $\pi_\infty = \pi_k$. The latter function is supported on the subgroup

$$G(\mathbf{R})^+ = \{g \in G(\mathbf{R}) |\ \det(g) > 0\}.$$

(We rule out $k = 2$ because the function (2.13) is integrable precisely when $k > 2$, and integrability is required by Proposition 2.1.)

For $p \nmid N\infty$, we assume that f_p is a bi-K_p-invariant function on $G(\mathbf{Q}_p)$ with compact support modulo the center, and that for all but finitely many such p, $f_p = \phi_p$ is the function supported on $Z_p K_p$ given by

$$\phi_p(z\kappa) = \overline{\omega_p(z)} \qquad (z \in Z_p,\ \kappa \in K_p). \tag{2.14}$$

Likewise if $p = \infty$ and $k = 0$, we take

$$f_\infty \in C_c^\infty(G(\mathbf{R})^+ // K_\infty) \qquad (k = 0). \tag{2.15}$$

The latter is the space of smooth functions on $G(\mathbf{R})^+$ which are bi-invariant under $Z(\mathbf{R})K_\infty$ and have compact support modulo $Z(\mathbf{R})$. Such a function enables us to project onto the K_∞-invariant space of $L^2(\omega)$, which contains the Maass forms of weight $k=0$.

Proposition 2.3. *With local functions as above, let $f=\prod f_p$ be the associated function on $G(\mathbf{A})$. Let $R(f)$ be the operator on $L^2(\omega)$ defined by*

$$R(f)\phi(x)=\int_{\overline{G}(\mathbf{A})} f(g)\phi(xg)dg.$$

Then $R(f)$ annihilates $L^2_0(\omega)^\perp$. In fact, it factors through the orthogonal projection of $L^2(\omega)$ onto $A_k(\widehat{\sigma},\omega)$, and acts diagonally on the latter space, the vectors w_π being eigenvectors.

Proof. For a proof of the first statement, see Proposition 1.1 of Rogawski's article [Rog]. Now suppose $v\in L^2_0(\omega)$. Since the latter space is a direct sum of cuspidal representations, we may assume that $v\in V_\pi$ for some $\pi=\otimes'\pi_p$. Likewise, we may assume that $v=\otimes' v_p$ is a pure tensor. For the purposes of this proof, let G' denote the restricted direct product $G'=\prod'_{p\nmid N} G(\mathbf{Q}_p)$. Decompose π as

$$\pi=\pi_\infty\otimes\pi'\otimes\bigotimes_{p|N}\pi_p,$$

where π' is a representation of G', and write $v=v_\infty\otimes v'\otimes\bigotimes_{p|N} v_p$ accordingly. Then (e.g. by Proposition 13.17 of [KL1])

$$R(f)v=\pi_\infty(f_\infty)v_\infty\otimes\pi'(f')v'\otimes\bigotimes_{p|N}\pi_p(f_p)v_p.$$

If $p|N$, or $p=\infty$ and $k>2$, then by Proposition 2.1, the above vanishes unless $\pi_p=\sigma_p$ (resp. π_k), and in the latter case $\pi_p(f_p)$ is the orthogonal projection onto $\mathbf{C}w_{\pi_p}$. Because f' is bi-invariant under $\prod_{p\nmid N} K_p$, $\pi'(f')$ has its image in the space $\mathbf{C}w'=\bigotimes'_{p\nmid N}\mathbf{C}w_{\pi_p}\subset V_{\pi'}$, and it annihilates the orthogonal complement of this subspace (see e.g. Lemma 3.10 of [KL3]). The analogous statement holds for $\pi_\infty(f_\infty)$ if $k=0$ for the same reasons. It follows that $R(f)$ annihilates $A_k(\widehat{\sigma},\omega)^\perp$, and acts by scalars on the vectors $w_\pi\in A_k(\widehat{\sigma},\omega)$. □

2.6. **Twisting.** Let D be a positive integer with $\gcd(D,N)=1$, and let χ be a primitive Dirichlet character modulo D. Given a cusp form

$$u(z)=\sum_{n\neq 0} a_n(u,y)e^{2\pi i n x}$$

in $S_k(N,\omega')$ or $L^2_0(N,\omega')$, its twist by χ is the form

$$u_\chi(z)=\sum_{n\neq 0}\chi(n)a_n(u,y)e^{2\pi i n x},$$

which belongs to $S_k(D^2N,\overline{\chi}^2\omega')$ or $L^2_0(D^2N,\overline{\chi}^2\omega')$. If u is a Maass form with spectral parameter t, then so is u_χ. In this section we will define a function f^χ on $G(\mathbf{A}_{\text{fin}})$ for which $R(f^\chi)$ encodes the twisting operation adelically. See §3 of [JK] for more detail. Beware that the nebentypus ψ in [JK] plays the role of $\overline{\omega'}$ here, since we have a complex conjugate in (2.3) which is not present in [JK].

We let $\chi^* : \mathbf{A}^* \longrightarrow \mathbf{C}^*$ be the Hecke character attached to χ as in (2.7) (but using D in place of N).[1] We let χ_p be the local component of χ^*. It is a character of $\mathbf{Q}_p^*$, and when $p|D$ it can be viewed as a primitive character of the group $(\mathbf{Z}/p^{D_p}\mathbf{Z})^*$. The Gauss sum attached to χ is

$$\tau(\chi) = \sum_{m\in(\mathbf{Z}/D\mathbf{Z})^*} \chi(m)e^{2\pi i m/D}.$$

If we set

$$\tau(\chi)_p = \chi_p(\tfrac{D}{p^{D_p}})\tau(\chi_p), \tag{2.16}$$

then $\tau(\chi) = \prod_{p|D} \tau(\chi)_p$ (cf. [JK], (3.10)).

For each prime $p|D$, we define a local test function $f_p^\chi : G(\mathbf{Q}_p) \longrightarrow \mathbf{C}$ by

$$f_p^\chi(x) = \begin{cases} \frac{\overline{\omega_p(z)}\,\overline{\chi_p(m)}}{\tau(\overline{\chi})_p} & \text{if } x = zg \text{ for } z \in Z_p \text{ and } g \in \left(\begin{smallmatrix} 1 & -m/D \\ 0 & 1 \end{smallmatrix}\right)K_p \text{ for } m \in (\mathbf{Z}_p/D\mathbf{Z}_p)^* \\ 0 & \text{otherwise.} \end{cases}$$

For the primes $p|N$, we take f_p^χ to be the function supported on $Z_pK_1(N)_p$ given by

$$f_p^\chi(z\kappa) = \frac{\overline{\omega_p(z)}}{\mathrm{meas}(\overline{K_1(N)_p})} = \frac{\psi_p(N)}{\omega_p(z)}, \tag{2.17}$$

where $\psi_p(N) = [K_p : K_1(N)_p] = p^{N_p}(1 + \frac{1}{p})$. Lastly, for $p \nmid DN$, we take f_p^χ to be the function defined in (2.14). Now let $f^\chi = \prod_{p<\infty} f_p^\chi$, and define the operator

$$R(f^\chi)\phi(x) = \int_{\overline{G}(\mathbf{A}_{\mathrm{fin}})} f^\chi(g)\phi(xg)dg \qquad (\phi \in L^2(\omega)). \tag{2.18}$$

We call this the **twisting operator of level** N attached to χ.

Proposition 2.4. *For $y \in \mathbf{R}^+ \times \widehat{\mathbf{Z}}^* \cong \mathbf{Q}^*\backslash\mathbf{A}^*$ and u a holomorphic or Maass cusp form of level N and nebentypus ω',*

$$R(f^\chi)\phi_u(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)) = \chi^*(y)\phi_{u_\chi}(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)).$$

Proof. See Proposition 3.2 of [JK]. That result is stated for holomorphic cusp forms, but the proof carries over verbatim to the case of Maass forms. □

Given two functions $f_1, f_2 \in L^1(\overline{G}(\mathbf{A}_{\mathrm{fin}}), \overline{\omega})$, we define their convolution by

$$f_1 * f_2(x) = \int_{\overline{G}(\mathbf{A}_{\mathrm{fin}})} f_1(g)f_2(g^{-1}x)dg = \int_{\overline{G}(\mathbf{A}_{\mathrm{fin}})} f_1(xg^{-1})f_2(g)dg.$$

Then $f_1 * f_2 \in L^1(\overline{G}(\mathbf{A}_{\mathrm{fin}}), \overline{\omega})$, and it is straightforward to show that $R(f_1 * f_2) = R(f_1)R(f_2)$ as operators on $L^2(\omega)$.

Proposition 2.5. *Let $f = f_\infty \times f_{\mathrm{fin}}$ be a function on $G(\mathbf{A})$ of the type defined in Section 2.5, with the property that for all $p|D$, f_p is the function* (2.14)*. Then*

$$R(f_\infty \times (f^\chi * f_{\mathrm{fin}})) = R(f^\chi)R(f). \tag{2.19}$$

As a result, the above operator factors through the orthogonal projection of $L^2(\omega)$ onto $A_k(\widehat{\sigma}, \omega)$ by Proposition 2.3.

[1]Thus we use two sets of notation: ω' and χ are Dirichlet characters and ω, χ^* are the associated Hecke characters. This was done in order to conform to notation in papers we reference.

Proof. As mentioned above, $R(f^\chi * f_{\rm fin}) = R(f^\chi)R(f_{\rm fin})$. The local components of the convolution are given as follows:

$$(f^\chi * f_{\rm fin})_p = f_p^\chi * f_p = \begin{cases} f_p^\chi & \text{if } p|D \\ f_p & \text{if } p\nmid D. \end{cases} \tag{2.20}$$

Indeed, if $p \nmid DN$, then the assertion is immediate because f_p is bi-K_p-invariant and f_p^χ is the identity element of the local Hecke algebra of bi-K_p-invariant functions. Similarly, the case $p|D$ follows easily by the right K_p-invariance of f_p^χ and our assumption that f_p is given by (2.14). If $p|N$, then for $k \in K_1(N)_p$, by (2.12) we have

$$f_p(k^{-1}x) = d_p\overline{\langle \sigma_p(x)w_{\sigma_p}, \sigma_p(k)w_{\sigma_p}\rangle} = f_p(x),$$

since w_{σ_p} is fixed by $K_1(N)_p$. Thus by (2.17),

$$f_p^\chi * f_p(x) = \int_{\overline{K_1(N)_p}} f_p^\chi(k) f_p(k^{-1}x)dk = f_p(x)\int_{\overline{K_1(N)_p}} f_p^\chi(k)dk = f_p(x),$$

as claimed.

In view of (2.20), we may apply Proposition 2.3 to both sides of the proposed equality (2.19) to see that they each vanish on $L_0^2(\omega)^\perp$. Therefore it suffices to show that they agree on $L_0^2(\omega)$. Let (π, V_π) be a cuspidal representation in $L_0^2(\omega)$. Given $v = v_\infty \otimes v_{\rm fin} \in V_\pi$, by (2.18) we have

$$\begin{aligned} R(f^\chi)v = \int_{\overline{G}(\mathbf{A}_{\rm fin})} f^\chi(g)\pi(1_\infty \times g)v\, dg &= \int_{\overline{G}(\mathbf{A}_{\rm fin})} v_\infty \otimes f^\chi(g)\pi_{\rm fin}(g)v_{\rm fin}\, dg \\ &= v_\infty \otimes \pi_{\rm fin}(f^\chi)v_{\rm fin}. \end{aligned}$$

For details justifying the movement of the tensor outside the integral, see Lemma 13.16 of [KL1]. Applying the above identity with $R(f)v$ in place of v, the result follows:

$$\begin{aligned} R(f^\chi)R(f)v = \pi_\infty(f_\infty)v_\infty \otimes \pi_{\rm fin}(f^\chi)\pi_{\rm fin}(f_{\rm fin})v_{\rm fin} &= \pi_\infty(f_\infty)v_\infty \otimes \pi_{\rm fin}(f^\chi * f_{\rm fin})v_{\rm fin} \\ &= R(f_\infty \times (f^\chi * f_{\rm fin}))v. \end{aligned}$$

For a justification of the last step, see e.g. Proposition 13.17 of [KL1]. □

2.7. **A particular choice of function.** The above discussion is rather general, and we will now define a very specific function f as in §2.5, designed to project onto the newforms of cubic level and then act as a Hecke operator. For our main test function in the trace formula, we will then take $F = f_\infty \times (f^\chi * f_{\rm fin})$, with f^χ a twisting operator defined as above.

Henceforth we take $N > 1$ to be a square-free integer. We make the following assumption in all that follows:

(**) ω' is a Dirichlet character of modulus N^3 whose conductor divides N.

As before, we let ω be the associated Hecke character.

For each $p|N$, the conductor of ω_p divides p. Therefore by Proposition 7.2 of [KL4], there are exactly $2(p-1)$ irreducible admissible representations of $G(\mathbf{Q}_p)$ of conductor p^3 and central character ω_p, up to isomorphism. These are the simple supercuspidal representations, which are parametrized naturally by the pairs (t,ζ) with $t \in (\mathbf{Z}/p\mathbf{Z})^*$ and $\zeta \in \mathbf{C}$ satisfying $\zeta^2 = \omega_p(tp)$. The construction depends on the choice of a nontrivial character of $\mathbf{Z}/p\mathbf{Z}$, which we fix to be $x \mapsto \theta_p(\frac{x}{p})$.

Let $\sigma = \sigma_{t,\zeta}$ be the supercuspidal representation indexed by (t, ζ). It is defined precisely in [KL4], but all that we need here is the formula for its matrix coefficient

$$f^\sigma = d_\sigma \overline{\langle \sigma(g) w_\sigma, w_\sigma \rangle},$$

where the formal degree d_σ is taken relative to the Haar measure for which meas$(\overline{K}_p)$ is 1, and w_σ is a unit new vector as before. By Theorem 7.1 of [KL4],

$$f^\sigma = f_1^\sigma + f_2^\sigma, \tag{2.21}$$

where f_1^σ and f_2^σ have disjoint support, and for $z \in Z_p$, are given by Kloosterman sums:

$$f_1^\sigma(zg) = \frac{p+1}{2\omega_p(z)} \sum_{w \in (\mathbf{Z}/p\mathbf{Z})^*} \theta_p\Big(\frac{-bw - \frac{tc}{a} w^{-1}}{p}\Big) \tag{2.22}$$

for $g = \left(\begin{smallmatrix} a & bp^{-1} \\ cp^2 & d \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \mathbf{Z}_p^* & \frac{1}{p}\mathbf{Z}_p \\ p^2\mathbf{Z}_p & 1+p\mathbf{Z}_p \end{smallmatrix}\right)$, and

$$f_2^\sigma(zg) = \frac{(p+1)\zeta}{2\omega_p(zd)} \sum_{w \in (\mathbf{Z}/p\mathbf{Z})^*} \overline{\omega_p(w)} \theta_p\Big(\frac{-\frac{c}{a}w - \frac{tb}{d}w^{-1}}{p}\Big) \tag{2.23}$$

for $g = \left(\begin{smallmatrix} c & dp^{-2} \\ ap & b \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \mathbf{Z}_p & \frac{1}{p^2}\mathbf{Z}_p^* \\ p\mathbf{Z}_p^* & \mathbf{Z}_p \end{smallmatrix}\right)$. The function f^σ vanishes outside the set

$$Z_p \cdot \begin{pmatrix} \mathbf{Z}_p^* & p^{-1}\mathbf{Z}_p \\ p^2\mathbf{Z}_p & 1+p\mathbf{Z}_p \end{pmatrix} \bigcup Z_p \cdot \begin{pmatrix} \mathbf{Z}_p & p^{-2}\mathbf{Z}_p^* \\ p\mathbf{Z}_p^* & \mathbf{Z}_p \end{pmatrix}.$$

Fix an integer $n > 0$ with $\gcd(n, DN) = 1$. Let

$$M(n)_p = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbf{Z}_p) |\, ad - bc \in n\mathbf{Z}_p^*\}.$$

Define, for $p|n$, the local Hecke operator $f_p^n : G(\mathbf{Q}_p) \longrightarrow \mathbf{C}$, supported on $Z_p M(n)_p$, by

$$f_p^n(zg) = \omega_p(z) \qquad (z \in Z_p, g \in M(n)_p). \tag{2.24}$$

This plays the role of the classical Hecke operator $T_{p^{n_p}}$.

Finally, we let f_∞ be the matrix coefficient (2.13) if $k > 2$, or a spherical function as in (2.15) if $k = 0$.

With these choices, we define the global function $f : G(\mathbf{A}) \longrightarrow \mathbf{C}$ by

$$f = f_\infty \times \prod_{p|N} (\sum_{(t,\zeta)} f^{\sigma_{t,\zeta}}) \prod_{p|n} f_p^n \prod_{p \nmid nN} \phi_p,$$

where, in the case $p \nmid nN$, ϕ_p is the unramified function supported on $Z_p K_p$ defined in (2.14). We remark that at the places $p|N$,

$$\sum_{(t,\zeta)} f^{\sigma_{t,\zeta}} = \sum_{(t,\zeta)} f_1^{\sigma_{t,\zeta}}$$

since from the definition (2.23) it follows easily that for each t, $f_2^{\sigma_{t,\zeta}} + f_2^{\sigma_{t,-\zeta}} = 0$. Nevertheless, we will compute the contribution of f_2^σ to the local orbital integrals in the trace formula that follows, since these do not vanish individually and may be of interest in other applications.

The function f defined above is a finite sum of functions of the type considered in §2.5. Thus any new vector w_π belonging to the space

$$A_k(N^3, \omega) \stackrel{\text{def}}{=} \bigoplus_{\widehat{\sigma}} A_k(\widehat{\sigma}, \omega) \tag{2.25}$$

is an eigenvector of $R(f)$. Here, $\widehat{\sigma}$ runs through all tuples $\{\sigma_p\}_{p|N}$ of simple supercuspidal representations ($\sigma_p = \sigma_{t,\zeta}$) with central character ω_p, and $A_k(\widehat{\sigma}, \omega)$ is the space defined in (2.10).

Proposition 2.6. *Given a new vector $w_\pi \in A_k(N^3, \omega)$, let u be the associated newform. Then $R(f)w_\pi = \lambda_f(u)w_\pi$, for*

$$\lambda_f(u) = \begin{cases} n^{1-k/2}\lambda_n(u) & \text{if } k > 2 \\ n^{1/2}h(t)\lambda_n(u) & \text{if } k = 0, \end{cases} \tag{2.26}$$

where $\lambda_n(u)$ is the eigenvalue of the classical Hecke operator T_n acting on u, and in the $k = 0$ case, t is the spectral parameter of u and $h(t)$ is the Selberg transform of f_∞ (defined in (5.3) below).

Proof. We may write

$$R(f)w_\pi = \pi_\infty(f_\infty)w_\infty \otimes \pi'(f')w' \bigotimes_{p|N} \pi_p(f_p)w_p \bigotimes_{p|n} \pi_p(f_p)w_p,$$

where the $'$ indicates the contribution of the primes $p \nmid Nn\infty$ as in the proof of Proposition 2.3. If $p|N$, then

$$\pi_p(f_p)w_p = \sum_\sigma \pi_p(f^\sigma)w_p = w_p$$

by Proposition 2.1, since exactly one of the representations σ is isomorphic to π_p. Likewise if $p \nmid nN$, then $\pi_p(f_p)w_p = \pi_p(\phi_p)w_p = w_p$ by the definition of ϕ_p. Hence

$$R(f)w_\pi = \pi_\infty(f_\infty)w_\infty \otimes w' \bigotimes_{p|N} w_p \bigotimes_{p|n} \pi_p(f_p)w_p.$$

Now

$$\pi_\infty(f_\infty)w_\infty = \begin{cases} w_\infty & \text{if } k > 2 \text{ (by Prop. 2.1 above)} \\ h(t)w_\infty & \text{if } k = 0 \text{ (by Prop. 3.9 of [KL3]).} \end{cases}$$

From the product over the places $p|n$, we get the scalar $\sqrt{n}\lambda_n(u)$ if $k = 0$ (see Lemma 4.6 of [KL3]), and $n^{1-k/2}\lambda_n(u)$ if $k > 2$ (see Proposition 13.6 of [KL1]). $\square$

To incorporate twisting, we consider the function

$$F = f_\infty \times (f^\chi * f_{\text{fin}}) = f_\infty \times \prod_{p|N} \Big(\sum_{(t,\zeta)} f^{\sigma_{t,\zeta}}\Big) \prod_{p|D} f_p^\chi \prod_{p|n} f_p^n \prod_{p\nmid nDN} \phi_p, \tag{2.27}$$

where f^χ is the twisting operator of level N^3 as defined in Section 2.6 (where the level was denoted by N rather than N^3 used here). The second equality in (2.27) follows from (2.20). We will use the above as our test function in the trace formula. The kernel of the operator $R(F)$ is

$$K(x, y) = \sum_{\gamma \in \overline{G}(\mathbf{Q})} F(x^{-1}\gamma y). \tag{2.28}$$

Proposition 2.7. *Let $\mathcal{F}_k^{new}(N^3, \omega')$ be the set of newforms of weight k, level N^3 and central character ω'. Then the above kernel function has the spectral form*

$$K(x, y) = \sum_{u \in \mathcal{F}_k^{new}(N^3, \omega')} \frac{\lambda_f(u) R(f^\chi)\phi_u(x)\overline{\phi_u(y)}}{\|u\|^2} \tag{2.29}$$

for $\lambda_f(u)$ as in (2.26)*. The kernel function is continuous on $G(\mathbf{A})\times G(\mathbf{A})$ and the above equality is valid for all points (x, y).*

Proof. First, note that by Propositions 2.5 and 2.6,

$$R(F)\phi_u = \lambda_f(u)R(f^\chi)\phi_u.$$

Therefore the RHS of (2.29) is the same as

$$\sum_\phi \frac{R(F)\phi(x)\overline{\phi(y)}}{\|\phi\|^2} \tag{2.30}$$

where ϕ ranges through an orthogonal basis for $A_k(N^3,\omega)$ (defined in (2.25)). In fact we may even allow ϕ to range over an orthogonal basis for the whole space $L^2(\omega)$ since $R(F)$ annihilates $A_k(N^3,\omega)^\perp$. The restriction of $R(F)$ to the cuspidal subspace is well-known to be Hilbert-Schmidt, and since $R(F)$ vanishes on $L^2_0(\omega)^\perp$ $R(F)$ is itself Hilbert-Schmidt. (In fact it has finite rank when $k > 2$, but not when $k = 0$.) Hence its kernel is equal to (2.30), proving that (2.29) holds almost everywhere.

The continuity of (2.28) is trivial when $k = 0$, since in that case the defining sum is locally finite, F having compact support modulo the center and $\overline{G}(\mathbf{Q})$ being discrete and closed in $\overline{G}(\mathbf{A})$. When $k > 2$, f_∞ is not compactly supported, so the continuity is not trivial, but a proof is given in Proposition 18.4 of [KL1].

On the other hand, the continuity of the RHS of (2.29) is trivial when $k > 2$ since in that case it is a finite sum of continuous functions. When $k = 0$, a proof is given in Corollary 6.12 of [KL3]. In all cases, it follows that (2.29) is valid everywhere. □

3. A relative trace formula

Our goal is to compute the relative trace formula given by the integral

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}\int_{\mathbf{Q}\backslash\mathbf{A}} K(\begin{pmatrix} y & \\ & 1\end{pmatrix}, \begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix})\overline{\theta(rx)}\,\overline{\chi^*(y)}|y|^{s-k'/2}dx\,d^*y, \tag{3.1}$$

where $k' = k$ if $k > 2$ and $k' = 1$ if $k = 0$.

On the spectral side we evaluate the double integral using (2.29).

Proposition 3.1. *The integral*

$$\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}\int_{\mathbf{Q}\backslash\mathbf{A}} \sum_{u\in\mathcal{F}_k^{new}(N^3,\omega')}\left|\frac{\lambda_f(u)R(f^\chi)\phi_u((\begin{smallmatrix} y & \\ & 1\end{smallmatrix}))\overline{\phi_u((\begin{smallmatrix} 1 & x\\ 0 & 1\end{smallmatrix}))}}{\|u\|^2}\overline{\theta(rx)}\,\overline{\chi^*(y)}|y|^{s-k'/2}\right|dx\,d^*y$$

is convergent for all $s\in\mathbf{C}$ when $k > 2$, and in some right half-plane when $k = 0$. Hence for such s, (3.1) *is equal to*

$$\sum_{u\in\mathcal{F}_k^{new}(N^3,\omega')}\frac{\lambda_f(u)\overline{a_r(u)}\Lambda(s,u,\chi)}{\|u\|^2}P_r(u), \tag{3.2}$$

where $\lambda_f(u)$ is given in (2.26) *and* $P_r(u) = \begin{cases}\frac{1}{2}K_{it}(2\pi|r|) & \textit{if } k=0 \textit{ and } u \textit{ is even}\\ 0 & \textit{if } k=0 \textit{ and } u \textit{ is odd}\\ e^{-2\pi r} & \textit{if } k>0.\end{cases}$

Proof. By Proposition 2.4, we have

$$R(f^\chi)\phi_u(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)) = \chi^*(y)\phi_{u_\chi}(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right))$$

for all $y \in \mathbf{R}^+ \times \widehat{\mathbf{Z}}^* \cong \mathbf{Q}^*\backslash\mathbf{A}^*$. Therefore, whenever the double integral in the statement of the proposition is convergent, (3.1) is equal to

$$\sum_{u\in\mathcal{F}_k^{new}(N^3,\omega')} \frac{\lambda_f(u)}{\|u\|^2}\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}\phi_{u_\chi}(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right))|y|^{s-k'/2}d^*y\int_{\mathbf{Q}\backslash\mathbf{A}}\overline{\phi_u(\left(\begin{smallmatrix} 1 & x\\ 0 & 1\end{smallmatrix}\right))\theta(rx)}dx,$$

which is equal to (3.2) by Lemma 2.2. Each of the above integrals is absolutely convergent, so the first assertion of the proposition is immediate when $k > 2$ since the sum over u is finite in that case. For the non-holomorphic case, we refer to Proposition 5.1 below. □

For the geometric side, we use the expression (2.28) and formally unfold (3.1) to obtain

$$\sum_\delta\int_{\mathbf{A}^*}\int_{\mathbf{A}} F(\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\delta\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix})\overline{\theta(rx)}\,\overline{\chi^*(y)}|y|^{s-k'/2}dx\,d^*y, \tag{3.3}$$

where δ ranges over $\overline{M}(\mathbf{Q})\backslash\overline{G}(\mathbf{Q})/N(\mathbf{Q})$. (See §7 of [JK] for details.) By the Bruhat decomposition, the elements

$$1,\quad \left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right),\quad \{\left(\begin{smallmatrix} a & -1\\ 1 & 0\end{smallmatrix}\right) |\, a\in\mathbf{Q}^*\}$$

form a set of representatives for these double cosets.

Proposition 3.2. *The convergence*

$$\sum_\delta\int_{\mathbf{A}^*}\int_{\mathbf{A}}\left|F(\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\delta\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix})\overline{\theta(rx)}\,\overline{\chi^*(y)}|y|^{s-k'/2}\right|dx\,d^*y < \infty$$

is valid for all s when $k = 0$, and for $1 < \mathrm{Re}(s) < k-1$ when $k > 2$. Hence the spectral side (3.2) *is equal to the geometric side* (3.3) *when $1 < \mathrm{Re}(s) < k-1$ if $k > 2$, and when $\mathrm{Re}(s)$ is sufficiently large if $k = 0$.*

Proof. We will show in the proof of Proposition 5.3 below that when $k = 0$, the integrand vanishes identically for all but finitely many δ. Since F also has compact support modulo the center in this case, the remaining integrals are absolutely convergent. When $k > 2$, the proof is essentially identical to that of Proposition 7.1 of [JK], in view of the proof of Proposition 4.5 below. □

We let $I_\delta(s)$ denote the double integral attached to δ in (3.3). This orbital integral can be computed locally. The archimedean integral in the case $k = 0$ will be considered in §5 below. In the holomorphic case $k > 2$, the archimedean orbital integral was computed in [KL2] and [JK]. The non-archimedean local calculations at places $p \nmid N$ were carried out in [JK]. Thus it remains here to compute the local integrals at places $p|N$. The results will be given in (3.5), (3.7) and (3.10) below.

3.1. **Orbital integrals for $p|N$.** To simplify notation in this section, we will write k rather than k'. Suppose $p|N$, and let $\sigma = \sigma_{t,\zeta}$ be a supercuspidal representation of conductor p^3 and central character ω_p. Define

$$J_\delta(s, f^\sigma) = J_\delta(s, f_1^\sigma) + J_\delta(s, f_2^\sigma),$$

as in (2.21), where for $i=1,2$,

$$J_\delta(s,f_i^\sigma)=\int_{\mathbf{Q}_p^*}\int_{\mathbf{Q}_p} f_i^\sigma(\begin{pmatrix} y & \\ & 1\end{pmatrix}\delta\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix})\overline{\theta_p(rx)}\,\chi_p(y)|y|_p^{k/2-s}dx\,d^*y.$$

Then replacing y by y^{-1} in (3.3), we see that

$$I_\delta(s)_p=\sum_\sigma J_\delta^\sigma(s).$$

Proposition 3.3. *Let $\delta=1$, so that*

$$J_1(s,f^\sigma)=\int_{\mathbf{Q}_p^*}\int_{\mathbf{Q}_p} f^\sigma(\begin{pmatrix} y & xy\\ 0 & 1\end{pmatrix})\overline{\theta_p(rx)}dx\,\chi_p(y)|y|_p^{k/2-s}d^*y.$$

Then

$$J_1(s,f^\sigma)=J_1(s,f_1^\sigma)=\begin{cases}\frac{p(p+1)}{2} & \text{if } p\nmid r\\ 0 & \text{if } p|r,\end{cases} \tag{3.4}$$

and

$$I_1(s)_p=\begin{cases}p^3-p & \text{if } p\nmid r\\ 0 & \text{otherwise.}\end{cases} \tag{3.5}$$

Proof. By (2.23), the matrix $\left(\begin{smallmatrix} y & xy\\ 0 & 1\end{smallmatrix}\right)$ never belongs to $\mathrm{Supp}(f_2^\sigma)$, so $J_1(s,f^\sigma)=J_1(s,f_1^\sigma)$. Note that $\left(\begin{smallmatrix} y & xy\\ 0 & 1\end{smallmatrix}\right)\in\mathrm{Supp}(f_1^\sigma)=Z_p\begin{pmatrix}\mathbf{Z}_p^* & p^{-1}\mathbf{Z}_p\\ 0 & 1\end{pmatrix}$ if and only if $y\in\mathbf{Z}_p^*$ and $x\in p^{-1}\mathbf{Z}_p$. We substitute $u=y\in\mathbf{Z}_p^*$, and replace yx by $p^{-1}x$, so now $x\in\mathbf{Z}_p$. Then dx becomes $p\,dx$, and

$$\begin{aligned}J_1(s,f_1^\sigma)&=p\int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p} f_1^\sigma(\begin{pmatrix}u & p^{-1}x\\ 0 & 1\end{pmatrix})\overline{\theta_p(\frac{ru^{-1}x}{p})}dxd^*u\\ &=\frac{p(p+1)}{2}\sum_{w\in(\mathbf{Z}/p\mathbf{Z})^*}\int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p}\theta_p(\frac{-xw}{p})\theta_p(\frac{-ru^{-1}x}{p})dxd^*u.\end{aligned}$$

The integral over $\mathbf{Z}_p$ is equal to

$$\int_{\mathbf{Z}_p}\theta_p(\frac{(-w-ru^{-1})x}{p})dx=\begin{cases}1 & \text{if } u\equiv -rw^{-1} \bmod p\\ 0 & \text{otherwise.}\end{cases}$$

In particular, this vanishes if $p|r$. Assuming $p\nmid r$,

$$J_1(s,f_1^\sigma)=\frac{p(p+1)}{2}\sum_{w\in(\mathbf{Z}/p\mathbf{Z})^*}\int_{-rw^{-1}+p\mathbf{Z}_p}d^*u=\frac{p(p+1)}{2}\,\mathrm{meas}(\mathbf{Z}_p^*),$$

which proves (3.4). The number of pairs (t,ζ) is $2(p-1)$. Since (3.4) is independent of σ (the parameters (t,ζ) not appearing in (3.4)),

$$I_1(s)_p=2(p-1)J_1(s,f^\sigma),$$

and (3.5) follows. □

Proposition 3.4. *Let* $\delta = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, *so that*

$$J_\delta(s, f^\sigma) = \int_{\mathbf{Q}_p^*} \int_{\mathbf{Q}_p} f^\sigma(\begin{pmatrix} 0 & -y \\ 1 & x \end{pmatrix})\overline{\theta_p(rx)}dx\, \chi_p(y)|y|_p^{k/2-s} d^*y.$$

Then

$$J_\delta(s, f^\sigma) = J_\delta(s, f_2^\sigma) = \begin{cases} \dfrac{(p^3)^{k/2-s} p(p+1)\omega_p(-rp^2)}{2\zeta\chi_p(p^3)} & \text{if } p \nmid r \\ 0 & \text{otherwise,} \end{cases} \tag{3.6}$$

and

$$I_\delta(s)_p = 0. \tag{3.7}$$

Proof. By (2.22), the matrix $\left(\begin{smallmatrix} 0 & -y \\ 1 & x \end{smallmatrix}\right)$ never belongs to $\mathrm{Supp}(f_1^\sigma)$, so $J_\delta(s, f^\sigma) = J_\delta(s, f_2^\sigma)$. Note that $\left(\begin{smallmatrix} 0 & -y \\ 1 & x \end{smallmatrix}\right) \in \mathrm{Supp}(f_2^\sigma)$ if and only if

$$\begin{pmatrix} 0 & -py \\ p & px \end{pmatrix} \in \begin{pmatrix} \mathbf{Z}_p & p^{-2}\mathbf{Z}_p^* \\ p\mathbf{Z}_p^* & \mathbf{Z}_p \end{pmatrix}.$$

In this case, we may write $y = -p^{-3}u$ for $u \in \mathbf{Z}_p^*$, and $x' = px \in \mathbf{Z}_p$. Then $dx' = p^{-1}dx$, and dropping the $'$ from the notation, we have

$$J_\delta(s, f_2^\sigma) = p \int_{\mathbf{Z}_p^*} \int_{\mathbf{Z}_p} f_2^\sigma(\begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix} \begin{pmatrix} 0 & p^{-2}u \\ p & x \end{pmatrix})\theta_p(\frac{-rx}{p})dx\, \chi_p(p^{-3})(p^3)^{k/2-s} d^*u$$

$$= \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)} \sum_{w \in (\mathbf{Z}/p\mathbf{Z})^*} \overline{\omega_p(w)} \int_{\mathbf{Z}_p^*} \overline{\omega_p(u)} \int_{\mathbf{Z}_p} \theta_p(\frac{-\frac{tx}{uw}}{p})\theta_p(\frac{-rx}{p})dx\, d^*u$$

by (2.23). Replacing u by $(-uw)^{-1}$, the above is

$$= \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)} \sum_{w \in (\mathbf{Z}/p\mathbf{Z})^*} \overline{\omega_p(w)} \int_{\mathbf{Z}_p^*} \omega_p(-uw) \int_{\mathbf{Z}_p} \theta_p(\frac{(tu-r)x}{p})dx\, d^*u.$$

Observe that w is eliminated, and the sum over w contributes $p-1$. Furthermore,

$$\int_{\mathbf{Z}_p} \theta_p(\frac{(tu-r)x}{p})dx = \begin{cases} 1 & \text{if } u \in t^{-1}r + p\mathbf{Z}_p \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it vanishes if $p|r$. Assuming $p \nmid r$,

$$J_\delta(s, f^\sigma) = \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)}(p-1)\int_{t^{-1}r+p\mathbf{Z}_p} \omega_p(-u)d^*u$$

$$= \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)}\omega_p(-t^{-1}r).$$

Equality (3.6) now follows, using the fact that $\frac{\zeta\omega_p(p)}{\omega_p(t)} = \frac{\zeta\omega_p(p)^2}{\omega_p(pt)} = \frac{\omega_p(p)^2}{\zeta}$. For fixed t, if we sum (3.6) over $\pm\zeta$, we get 0. It follows that $I_\delta(s)_p = \sum_\sigma J_\delta(s, f^\sigma) = 0$. $\square$

Proposition 3.5. *For* $a \in \mathbf{Q}^*$, *let* $\delta_a = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$, *so that*

$$J_{\delta_a}(s, f^\sigma) = \int_{\mathbf{Q}_p^*} \int_{\mathbf{Q}_p} f^\sigma(\begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix})\overline{\theta_p(rx)}dx\chi_p(y)|y|_p^{k/2-s} d^*y.$$

Then $J_{\delta_a}(s, f_1^\sigma)$ vanishes unless $a \in p^2\mathbf{Z}_p$ and $p \nmid r$. In this case, writing $a = p^{a_p}a_0$ for $a_0 \in \mathbf{Z}_p^* \cap \mathbf{Q}^*$, we have

$$J_{\delta_a}(s, f_1^\sigma) = \begin{cases} \dfrac{|a|_p^{2s-k}p(p+1)\omega_p(p^{a_p})}{2\chi_p(a^2)}\theta_p(\dfrac{ta}{rp^3} - \dfrac{r}{a}) & \text{if } a_0 \equiv 1 \bmod p \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

The integral $J_{\delta_a}(s, f_2^\sigma)$ vanishes unless $a \in p^2\mathbf{Z}_p$. For such a,

$$J_{\delta_a}(s, f_2^\sigma) = \begin{cases} \dfrac{(p^3)^{k/2-s}p(p+1)\omega_p(-p^2r)}{2\chi_p(p^3)\zeta}\theta_p(-\dfrac{ta}{rp^3}) & \text{if } p \nmid r \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Finally, $I_{\delta_a}(s)_p$ vanishes unless $p \nmid r$ and $a = p^{a_p}a_0$ for $a_p \geq 2$ and $a_0 \equiv 1 \bmod p\mathbf{Z}_p$. If these conditions are satisfied, then

(3.10)
$$I_{\delta_a}(s)_p = \frac{|a|_p^{2s-k}p(p+1)\omega_p(p^{a_p})\theta_p(-\frac{r}{a})}{\chi_p(a^2)}\Delta_p(a), \quad \text{for } \Delta_p(a) = \begin{cases} p-1, & a_p > 2 \\ -1, & a_p = 2. \end{cases}$$

Proof. We start by computing $J_{\delta_a}(s, f_1^\sigma)$. From (2.22) we see that the determinant of any matrix in the support of f_1^σ is of the form $(p^m)^2u$ for some $m \in \mathbf{Z}$ and $u \in \mathbf{Z}_p^*$ (the square factor coming from the center). Since $\det\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)\delta_a\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right) = y$, it follows that we may assume $y = \frac{u}{p^{2\ell}}$ for some $\ell \in \mathbf{Z}$ and $u \in \mathbf{Z}_p^*$, and that

$$\begin{pmatrix} p^\ell & \\ & p^\ell \end{pmatrix}\begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix} = \begin{pmatrix} p^{-\ell}au & p^{-\ell}u(xa-1) \\ p^\ell & p^\ell x \end{pmatrix} \in \begin{pmatrix} \mathbf{Z}_p^* & p^{-1}\mathbf{Z}_p \\ p^2\mathbf{Z}_p & 1+p\mathbf{Z}_p \end{pmatrix}.$$

This implies $a_p = \ell \geq 2$, and that $p^\ell x = 1 + px'$ for some $x' \in \mathbf{Z}_p$. Then $p^{-\ell}dx = p^{-1}dx'$. Making this substitution, we find that $J_{\delta_a}(s, f_1^\sigma)$ is equal to

$$\frac{(p^{\ell-1})\omega_p(p^\ell)(p^{2\ell})^{k/2-s}}{\chi_p(p^{2\ell})}\int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p} f_1^\sigma(\begin{pmatrix} a_0u & \frac{1}{p^\ell}u((1+px')a_0-1) \\ p^\ell & 1+px' \end{pmatrix})\overline{\theta_p(\tfrac{r(1+px')}{p^\ell})}dx'd^*u.$$

In order that the integrand be nonzero, we need $p^{-\ell}((1+px')a_0 - 1) \in p^{-1}\mathbf{Z}_p$, i.e.

$$1 + px' \equiv a_0^{-1} \bmod p^{\ell-1}\mathbf{Z}_p.$$

This is only possible if $a_0 \equiv 1 \bmod p$. Assuming the latter condition holds, we set $1 + px' = a_0^{-1} + p^{\ell-1}x''$, so $p^{-1}dx' = p^{1-\ell}dx''$. Then, writing x in place of x'', the double integral becomes

$$\frac{p\omega_p(p^\ell)(p^{2\ell})^{k/2-s}}{\chi_p(p^{2\ell})}\int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p} f_1^\sigma(\begin{pmatrix} a_0u & p^{-1}xua_0 \\ p^\ell & a_0^{-1}+p^{\ell-1}x \end{pmatrix})\overline{\theta_p(\tfrac{r(a_0^{-1}+p^{\ell-1}x)}{p^\ell})}dx\, d^*u.$$

After replacing u by ua_0^{-1}, this becomes

$$\frac{(p^{2\ell})^{k/2-s}p(p+1)\omega_p(p^\ell)}{2\chi_p(p^{2\ell})}\theta_p(\frac{-r}{p^\ell a_0})$$
$$\times \sum_{w\in(\mathbf{Z}/p\mathbf{Z})^*}\int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p}\theta_p(\frac{-xuw - tp^{\ell-2}(uw)^{-1}}{p})\theta_p(\frac{-rx}{p})dx\, d^*u.$$

Replacing u by $-uw^{-1}$, we eliminate w, and the sum contributes a factor of $(p-1)$. The integral over x is then

$$\int_{\mathbf{Z}_p} \theta_p(\frac{(u-r)x}{p})dx = \begin{cases} 1 & \text{if } u \in r + p\mathbf{Z}_p \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it vanishes if $p|r$. Assuming $p \nmid r$, the sum over w of the double integral thus becomes

$$(p-1)\int_{r+p\mathbf{Z}_p} \theta_p(\frac{tp^{\ell-2}u^{-1}}{p})d^*u = \theta_p(\frac{tp^{\ell-2}}{rp}).$$

Hence

$$J_{\delta_a}(s, f_1^\sigma) = \frac{(p^{2\ell})^{k/2-s}p(p+1)\omega_p(p^\ell)}{2\chi_p(p^{2\ell})}\theta_p(\frac{tp^{\ell-2}}{rp} - \frac{r}{a}),$$

which establishes (3.8).

Now consider

$$J_{\delta_a}(s, f_2^\sigma) = \int_{\mathbf{Q}_p^*}\int_{\mathbf{Q}_p} f_2^\sigma(\begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix})\overline{\theta_p(rx)}dx\chi_p(y)|y|_p^{k/2-s}d^*y.$$

By (2.23), the integrand is nonzero precisely when

$$\begin{pmatrix} p & \\ & p \end{pmatrix}\begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix} = \begin{pmatrix} pya & py(xa-1) \\ p & px \end{pmatrix} \in \begin{pmatrix} \mathbf{Z}_p & p^{-2}\mathbf{Z}_p^* \\ p\mathbf{Z}_p^* & \mathbf{Z}_p \end{pmatrix}.$$

Taking the determinant, this says in particular that $p^2y \in p^{-1}\mathbf{Z}_p^*$, so we may write $y = \frac{u}{p^3}$ for $u \in \mathbf{Z}_p^*$. Setting $px = x'$, $p^{-1}dx = dx'$,

$$J_{\delta_a}(s, f_2^\sigma) = \frac{(p^3)^{k/2-s}p\,\omega_p(p)}{\chi_p(p^3)}\int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p} f_2^\sigma(\begin{pmatrix} \frac{ua}{p^2} & \frac{u}{p^2}(\frac{xa}{p}-1) \\ p & x \end{pmatrix})\theta_p(\frac{-rx}{p})dx\,d^*u.$$

From the upper left entry, the integrand is nonzero only if $a_p \geq 2$. Assuming the latter, we also have $\frac{xa}{p} - 1 \in \mathbf{Z}_p^*$, so the upper right entry belongs to $p^{-2}\mathbf{Z}_p^*$ as required. Hence by (2.23), the above is

$$= \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)}\sum_{w\in(\mathbf{Z}/p\mathbf{Z})^*}\overline{\omega_p(w)}\int_{\mathbf{Z}_p^*}\overline{\omega_p(u)}\theta_p(-\frac{uaw}{p^3})$$

$$\int_{\mathbf{Z}_p}\overline{\omega_p(\tfrac{xa}{p}-1)}\theta_p(\frac{-txu^{-1}(\frac{xa}{p}-1)^{-1}w^{-1}}{p})\theta_p(\frac{-rx}{p})dx\,d^*u.$$

Note that $\omega_p(\frac{xa}{p}-1) = \omega_p(-1)$ since $p^2|a$. For the same reason,

$$\theta_p(\frac{-txu^{-1}(\frac{xa}{p}-1)^{-1}w^{-1}}{p}) = \theta_p(\frac{tu^{-1}w^{-1}x}{p}).$$

Therefore the above integral over $\mathbf{Z}_p$ equals

$$\omega_p(-1)\int_{\mathbf{Z}_p}\theta_p\Big(\frac{(-r+tu^{-1}w^{-1})x}{p}\Big)dx = \begin{cases} \omega_p(-1) & \text{if } u \in tr^{-1}w^{-1} + p\mathbf{Z}_p \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $J_{\delta_a}(s, f_2^\sigma) = 0$ if $p|r$. Assuming $p \nmid r$, $J_{\delta_a}(s, f_2^\sigma)$ equals

$$\frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(-p)}{2\chi_p(p^3)}\sum_{w\in(\mathbf{Z}/p\mathbf{Z})^*}\overline{\omega_p(w)}\int_{tr^{-1}w^{-1}+p\mathbf{Z}_p}\overline{\omega_p(u)}\theta_p(-\frac{uaw}{p^3})d^*u$$

$$= \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(-p)}{2\chi_p(p^3)} \sum_{w\in(\mathbf{Z}/p\mathbf{Z})^*} \overline{\omega_p(w)}\,\overline{\omega_p(tr^{-1}w^{-1})}\theta_p(-\frac{tr^{-1}a}{p^3})\frac{1}{p-1}$$

$$= \frac{(p^3)^{k/2-s}p(p+1)\zeta\omega_p(-p)}{2\chi_p(p^3)}\overline{\omega_p(tr^{-1})}\theta_p(-\frac{tr^{-1}a}{p^3}).$$

Equation (3.9) follows upon using $\frac{\zeta\omega_p(p)}{\omega_p(t)} = \frac{\omega_p(p)^2}{\zeta}$.

Since $J_{\delta_a}(s, f_2^{\sigma_{t,\zeta}}) + J_{\delta_a}(s, f_2^{\sigma_{t,-\zeta}}) = 0$, we see that

$$I_{\delta_a}(s)_p = \sum_{(t,\zeta)} J_{\delta_a}(s, f_1^{\sigma}) = \frac{|a|_p^{2s-k}p(p+1)\omega_p(p)^{a_p}}{\chi_p(a^2)}\theta_p(-\frac{r}{a})\Delta_p(a)$$

assuming $p\nmid r$, $a \in p^2\mathbf{Z}_p$, and $a_0 \equiv 1 \bmod p$, where

$$\Delta_p(a) = \sum_{t\in(\mathbf{Z}/p\mathbf{Z})^*} \theta_p(\frac{tp^{a_p-2}}{rp}) = \begin{cases} p-1 & \text{if } a_p > 2 \\ -1 & \text{if } a_p = 2. \end{cases} \qquad \square$$

3.2. **Summary of local results.** We summarize here the contribution of $\delta = 1$, which turns out to be the main term. By (7.7) and (7.8) of [JK], and (3.5) above, we have

(3.11)

$$I_1(s)_p = \begin{cases} \chi_p(r) & \text{if } p|D \\ p(p+1)(p-1) & \text{if } p|N \\ (p^{n_p})^{\frac{k'}{2}-s} \sum_{d_p=0}^{\min(r_p,n_p)} (p^{d_p})^{2s-k'+1}\overline{\omega_p(\frac{p^{d_p}}{p^{n_p}})}\chi_p(\frac{p^{2d_p}}{p^{n_p}}) & \text{if } p|n \\ \frac{2^{k-1}(2\pi r)^{k-s-1}}{(k-2)!e^{2\pi r}}\Gamma(s) & \text{if } p=\infty, k>2 \\ 1 & \text{if } p\nmid NDn\infty. \end{cases}$$

The local integrals for $\delta = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ are irrelevant, since those at places dividing N vanish by (3.7).

We will discuss the local integrals for $\delta = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in §4.1 below.

4. Results for holomorphic cusp forms

In this section, we will prove the following.

Theorem 4.1. *Let $r, n, D, N, k \in \mathbf{Z}^+$ with N square-free, $k > 2$, $(rn, ND) = 1$, and $(D, N) = 1$. Let ω' be a Dirichlet character modulo N, with $\omega'(-1) = (-1)^k$, and let χ be a primitive Dirichlet character modulo D. Then for all $s = \sigma + i\tau$ in the strip $1 < \sigma < k-1$,*

$$\sum_{u\in\mathcal{F}_k^{new}(N^3,\omega')} \frac{\lambda_n(u)\overline{a_r(u)}\Lambda(s,u,\chi)}{\psi(N^3)\|u\|^2} = F + E, \tag{4.1}$$

where

$$F = \frac{2^{k-1}(2\pi rn)^{k-s-1}}{(k-2)!}\Gamma(s)\prod_{p|N}(1-\tfrac{1}{p})\sum_{d|(n,r)} d^{2s-k+1}\overline{\omega'(\tfrac{n}{d})}\chi(\tfrac{rn}{d^2})$$

is the main term, and the error term E (an infinite series involving confluent hypergeometric functions) satisfies

$$|E| \leq \frac{(4\pi rn)^{k-1}\varphi(D)\gcd(r,n)B(\sigma,k-\sigma)\prod_{p|N}(1-\frac{1}{p})}{N^{2\sigma}D^{\sigma-k+\frac{1}{2}}(k-2)!}2\cosh(\tfrac{\pi\tau}{2})\zeta(k-\sigma)\zeta(\sigma)$$

for the Beta function $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\leq 1$ and Euler's φ-function.

Remark: If $(r,N)>1$, then (4.1) vanishes. This is a consequence of the fact that the local representations at $p|N$ are supercuspidal, which implies that $a_p(u)=0$ (see e.g. the proof of Corollary 45 of [GHL10]) and hence $a_r(u)=0$. This is reflected on the geometric side in (3.5).

The L-functions in Theorem 4.1 are normalized so that the central point is $\frac{k}{2}$. In order to free s from dependence on k we shift the variable so that the critical strip becomes $[0,1]$ in the following.

Corollary 4.2. *Suppose for simplicity that $(n,r)=1$. Then for any point s in the critical strip $0<\mathrm{Re}(s)<1$, the sum*

$$\sum_{u\in\mathcal{F}_k^{new}(N^3,\omega')}\frac{\lambda_n(u)\overline{a_r(u)}\Lambda(s+\frac{k-1}{2},u,\chi)}{\psi(N^3)\|u\|^2}$$

is nonzero as long as $N+k$ is sufficiently large.

Proof. See the proof of Corollary 4.4 below. □

As another corollary, we can show that the central values $L(\frac{k}{2},h,\chi)$ satisfy the Lindelöf hypothesis on average as $k+N\to\infty$ when χ is real and ω' is trivial. (Under these conditions, the central value is a nonnegative real number, [Gu].)

Corollary 4.3. *Suppose that ω' is trivial and χ is real. Then for $k>2$,*

$$\sum_{u\in\mathcal{F}_k^{new}(N^3)} L(\tfrac{k}{2},u,\chi)\ll_D (kN^3)^{1+\varepsilon}.$$

Proof. The proof is identical to that of Corollary 1.3 of [JK]. □

Remark: This is the same bound we would get by assuming the Lindelöf hypothesis $L(\frac{k}{2},u,\chi)\ll(D^2kN^3)^{\varepsilon}$, in view of the fact that $|\mathcal{F}_k^{new}(N^3)|\sim\frac{k-1}{12}\psi^{new}(N^3)$, where $(N^3)^{1-\varepsilon}\ll\psi^{new}(N^3)\leq N^3$ ([Ser], (60) on p. 86).

Returning to the case of general ω' and χ, let $\langle\Lambda_n(s,\chi),a_r\rangle^{new}$ denote the sum in (4.1). We can regard this as an inner product of elements of the dual space of $S_k(N^3,\omega')^{new}$. One can also define $\langle\Lambda_n(s,\chi),a_r\rangle$ in the same way, but where the sum is taken over an orthogonal basis for the full space $S_k(N^3,\omega')$. It is interesting to compare the two.

Corollary 4.4. *Let s belong to the critical strip $\frac{k-1}{2}<\mathrm{Re}(s)<\frac{k+1}{2}$, and suppose that $(n,r)=1$. Then with notation as above, we have*

$$\frac{\langle\Lambda_n(s,\chi),a_r\rangle^{new}}{\langle\Lambda_n(s,\chi),a_r\rangle}\sim\prod_{p|N}(1-\frac{1}{p})$$

as $N+k\to\infty$.

Remark: From this we can observe two extremes in behavior. If $N = p$ is prime, and $p \to \infty$, the above tends to 1, so the contribution of oldforms begomes negligible. This agrees with a prediction of Ellenberg ([El], Remark 3.11). On the other hand, if we take N to be the product of the first ℓ primes not dividing D and let $\ell \to \infty$, the above goes to 0 and the contribution of the newforms becomes negligible.

Proof. In Theorem 1.1 of [JK], it is proven that

$$\langle \Lambda_n(s,\chi), a_r\rangle = F' + E',$$

where $F' = \dfrac{F}{\prod_{p|N}(1-\frac{1}{p})}$, and $|E'|$ satisfies a bound similar to the one given for $|E|$ in the above theorem, but without the factor of $\prod_{p|N}(1-\frac{1}{p})$, and with $N^{3\sigma}$ in the denominator rather than $N^{2\sigma}$. In the last line of the paper [JK], it is shown using Stirling's approximation that for $s = \frac{k}{2} + \delta + i\tau$ with $|\delta| < \frac{1}{2}$,

$$\left|\frac{E'}{F'}\right| \ll \frac{(4D\pi rne)^{k/2}}{(N^3)^{\frac{k-1}{2}} k^{\frac{k}{2}-1}},$$

where the implied constant depends on δ, D, R, n, τ. Clearly the above goes to 0 as $N + k \to \infty$. The same bound holds for $|\frac{E}{F}|$, but with N^2 in place of N^3, since the extra factor of $\prod(1-\frac{1}{p})$ in the numerator and denominator cancels out. Likewise

$$\left|\frac{E}{F'}\right| = \left|\frac{E}{F}\right| \prod_{p|N}(1-\frac{1}{p}) \le \left|\frac{E}{F}\right| \ll \frac{(4D\pi rne)^{k/2}}{(N^2)^{\frac{k-1}{2}} k^{\frac{k}{2}-1}}.$$

Now consider the quotient

$$\frac{\langle \Lambda_n(s,\chi), a_r\rangle^{new}}{\langle \Lambda_n(s,\chi), a_r\rangle} = \frac{F+E}{F'+E'} = \frac{\frac{F}{F'} + \frac{E}{F'}}{1 + \frac{E'}{F'}} = \frac{\prod_{p|N}(1-\frac{1}{p}) + O\left(\frac{(4D\pi rne)^{k/2}}{(N^2)^{\frac{k-1}{2}} k^{\frac{k}{2}-1}}\right)}{1 + O\left(\frac{(4D\pi rne)^{k/2}}{(N^3)^{\frac{k-1}{2}} k^{\frac{k}{2}-1}}\right)}$$

The corollary now follows easily. ☐

4.1. **Proof of Theorem 4.1.** In the holomorphic case, the spectral side (3.2) becomes

$$\frac{n^{1-k/2}}{e^{2\pi r}} \sum_{u \in \mathcal{F}_k^{new}(N^3,\omega')} \frac{\lambda_n(u)\overline{a_r(u)}}{\|u\|^2} \Lambda(s,u,\chi).$$

By the local calculation (3.7), the geometric side has the form

$$I_1(s) + \sum_{a \in \mathbf{Q}^*} I_{\delta_a}(s).$$

As is typical, the identity term $I_1(s)$ is the dominant term as $N + k \to \infty$. Multiplying the local results (3.11) together, when $k > 2$ (so $k' = k$) we obtain:

$$\frac{e^{2\pi r} n^{k/2-1}}{\psi(N^3)} I_1(s) = \frac{2^{k-1}(2\pi rn)^{k-s-1}}{(k-2)!}\Gamma(s) \prod_{p|N} \frac{p(p+1)(p-1)}{p^2(p+1)} \sum_{d|(r,n)} d^{2s-k+1}\overline{\omega'(\tfrac{n}{d})}\chi(\tfrac{rn}{d^2}).$$

This is the leading term of (4.1).

Theorem 4.1 now follows immediately from the following proposition involving the remaining orbital integrals.

Proposition 4.5. *For $\delta_a = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with $a \in \mathbf{Q}^*$, the orbital integral $I_{\delta_a}(s)$ is absolutely convergent on the strip $0 < \sigma < k$. It vanishes unless $a = \frac{N^2 b}{nD}$ for $b \in \mathbf{Z} - \{0\}$. When $s = \sigma + i\tau$ for $1 < \sigma < k-1$, the sum $\sum_{a\in\mathbf{Q}^*} I_{\delta_a}(s)$ is absolutely convergent, and*

$$\frac{e^{2\pi r} n^{k/2-1}}{\psi(N^3)} \sum_{a\in\mathbf{Q}^*} |I_{\delta_a}(s)|$$
$$\leq \frac{(4\pi r n)^{k-1}\varphi(D)\gcd(r,n)B(\sigma,k-\sigma)\prod_{p|N}(1-\frac{1}{p})}{N^{2\sigma}D^{\sigma-k+\frac{1}{2}}(k-2)!} 2\cosh(\tfrac{\pi\tau}{2})\zeta(k-\sigma)\zeta(\sigma).$$

Using the results of [JK] and (3.10) above, one can actually give a rather explicit formula for the sum of the $I_{\delta_a}(s)$ as an infinite series. However, this involves a lot of bookkeeping and seems of limited value, so we will just present the bound. First, by (3.10), we note that for $p|N$, $I_{\delta_a}(s)_p$ vanishes unless $a = \frac{N^2 b}{nD} \in N^2\mathbf{Z}_p$, and

$$|I_{\delta_a}(s)_p| \leq |N^2 b|_p^{2\sigma-k} p(p+1)(p-1) \leq |N^2|_p^{2\sigma-k} p(p+1)(p-1) \sum_{d_p=0}^{b_p} |p^{d_p}|_p^{2\sigma-k}. \tag{4.2}$$

Now suppose $p \nmid N\infty$. Then the value of $I_{\delta_a}(s)_p$ is not quite stated explicitly in [JK], but a closely related integral is given. Start with

$$I_{\delta_a}(s)_p = \int_{\mathbf{Q}_p^*}\int_{\mathbf{Q}_p} f_p(\begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix})\overline{\theta_p(rx)}dx\,\chi_p(y)|y|_p^{k/2-s}d^*y.$$

A matrix belongs to the support of f_p only if its determinant is of the form $(p^m)^2 nu$ for some $m \in \mathbf{Z}$ and $u \in \mathbf{Z}_p^*$. This can be seen from (2.24) (for $p|n$), from the expression for f_p^χ three lines above (2.17) if $p|D$, and from (2.14) if $p \nmid nDN\infty$. (In the latter two cases, n is a unit.) The determinant of the matrix in the above integral is y, so the integrand vanishes unless $y \in p^{-2d_p} n\mathbf{Z}_p^*$ for some $d_p \in \mathbf{Z}$. Write $a = \frac{N^2 b}{nD}$, where (for now) $b \in \mathbf{Q}^*$. It will be convenient to write $y = \frac{nu}{(N^2 d)^2}$ for $d = p^{d_p}$ and $u \in \mathbf{Z}_p^*$. Then the above becomes

$$\sum_{d_p\in\mathbf{Z}} \frac{\chi_p(\frac{n}{N^4 d^2})}{|\frac{n}{d^2}|_p^{s-k/2}} \int_{\mathbf{Z}_p^*}\int_{\mathbf{Q}_p} f_p(\left(\begin{smallmatrix} N^2 d & \\ & N^2 d \end{smallmatrix}\right)^{-1} \begin{pmatrix} \frac{nu}{N^2 d}\frac{N^2 b}{nD} & \frac{nu}{N^2 d}(\frac{xN^2 b}{nD} - 1) \\ N^2 d & dN^2 x \end{pmatrix})\overline{\theta_p(rx)}dx\,\chi_p(u)d^*u.$$

Since the second matrix has determinant $un \in n\mathbf{Z}_p^*$, all entries must lie in $\mathbf{Z}_p$ for the integrand to be nonzero. In particular, $d_p \geq 0$. We may substitute $x' = dN^2 x$, so that $dx' = |dN^2|_p\, dx = |d|_p\, dx$. Using the fact that the central character ω_p is unramified, we obtain

$$\sum_{d_p\geq 0} \frac{\chi_p(\frac{n}{N^4 d^2})\omega_p(d)}{|\frac{n}{d^2}|_p^{s-k/2}} |d|_p^{-1} \int_{\mathbf{Z}_p^*}\int_{\mathbf{Z}_p} f_p(\begin{pmatrix} \frac{ub}{dD} & \frac{ubx}{N^2 d^2 D} - \frac{nu}{N^2 d} \\ N^2 d & x \end{pmatrix})\overline{\theta_p(\tfrac{rx}{N^2 d})}dx\,\chi_p(u)d^*u.$$

The latter double integral coincides with (8.4) of [JK], but with N where we have N^2. It is computed explicitly in §8.1-8.2 of [JK]. In particular, it vanishes unless $0 \leq d_p \leq b_p$, proving the assertion in Proposition 4.5 that the global integral vanishes unless $a = \frac{N^2 b}{nD}$ for nonzero $b \in \mathbf{Z}$. Multiplying the coefficient by the double integral, whose value is given in (8.7), (8.8), and (8.12) of [JK], we find

$$|I_{\delta_a}(s)_p| \le \frac{\varphi(p^{D_p})}{|\tau(\overline{\chi})_p|} \sum_{d_p=0}^{b_p} |p^{d_p}|_p^{2\sigma-k} \qquad (p|D) \tag{4.3}$$

$$|I_{\delta_a}(s)_p| \le \frac{|n|_p^{k/2-\sigma}}{|\gcd(r,n)|_p} \sum_{d_p=0}^{b_p} |p^{d_p}|_p^{2\sigma-k} \qquad (p \nmid DN\infty). \tag{4.4}$$

(For the latter, we have used the fact that in [JK], $\gcd(\frac{b}{d}, Nd) | \gcd(r, Dn)$.) Multiplying the local bounds (4.2)-(4.4) together, we have

$$|I_{\delta_a}(s)_{\mathrm{fin}}| \le \frac{n^{\sigma-k/2}\varphi(D)\gcd(r,n)}{(N^2)^{2\sigma-k}|\tau(\overline{\chi})|}\Big(\prod_{p|N} p(p+1)(p-1)\Big)\sum_{d|b}\frac{1}{d^{2\sigma-k}}. \tag{4.5}$$

For the archimedean part, by (8.15) of [JK] we have

$$|I_{\delta_a}(s)_\infty| = \left|\frac{(4\pi r)^{k-1}(N^2)^{\sigma-k}b^{s-k}e^{i\pi s/2}}{(k-2)!(nD)^{\sigma-k}e^{2\pi r}}\,{}_1f_1(s;k;-\tfrac{2\pi irnD}{N^2 b})\right|, \tag{4.6}$$

where $b^s = e^{-i\pi s}|a|^s$ if $b<0$, and for $\mathrm{Re}(k) > \mathrm{Re}(s) > 0$,

$$ {}_1f_1(s,k;w) = B(s,k-s)\,{}_1F_1(s;k;w) = \int_0^1 e^{wx}x^{s-1}(1-x)^{k-s-1}dx$$

([Sl], §3.1). Noting that $|{}_1f_1(s;k;-\frac{2\pi irnD}{N^2b})| \le \int_0^1 x^{\sigma-1}(1-x)^{k-\sigma-1}dx = B(\sigma, k-\sigma)$, and that

$$\frac{\prod_{p|N} p(p+1)(p-1)}{\psi(N^3)} = \prod_{p|N}\frac{p(p+1)(p-1)}{p^2(p+1)} = \prod_{p|N}(1-\frac{1}{p}),$$

we multiply (4.5) by (4.6) to get

$$\frac{e^{2\pi r}n^{k/2-1}}{\psi(N^3)}|I_{\delta_a}(s)| \le \frac{(4\pi rn)^{k-1}\varphi(D)e^{-\pi\tau/2}}{N^{2\sigma}D^{\sigma-k+\frac12}(k-2)!}\big(\prod_{p|N}(1-\tfrac1p)\big)\big|b^{s-k}\big|B(\sigma,k-\sigma)\sum_{d|b}\frac{\gcd(r,n)}{d^{2\sigma-k}}.$$

Now we need to bound the sum over $b \in \mathbf{Z} - \{0\}$. Write $b = \pm cd$ for $c, d > 0$, and group the $c, -c$ terms together, so that

$$|c^{s-k}| + |(-c)^{s-k}| = c^{\sigma-k} + |(e^{-i\pi})^{s-k}c^{s-k}| = c^{\sigma-k}(1+e^{\pi\tau}).$$

Noting that $e^{-\pi\tau/2}(1+e^{\pi\tau}) = 2\cosh(\frac{\pi\tau}{2})$, we obtain

$$e^{-\pi\tau/2}\sum_{b\neq0}|b^{s-k}|\sum_{d|b}\frac{1}{d^{2\sigma-k}} = 2\cosh(\tfrac{\pi\tau}{2})\sum_{c,d>0}c^{\sigma-k}d^{-\sigma} = 2\cosh(\tfrac{\pi\tau}{2})\zeta(k-\sigma)\zeta(\sigma).$$

Proposition 4.5 now follows immediately.

5. The case of non-holomorphic cusp forms

5.1. **Integral transforms.** Here we define various integral transforms involving spherical functions. We refer to §3 of [KL3] for further detail.

Let $f_\infty \in C_c^\infty(G(\mathbf{R})^+//K_\infty)$ as in (2.15). The **Harish-Chandra transform** of f_∞ is the function on $\mathbf{R}^+$ defined by

$$\mathcal{H}f_\infty(y) = y^{-1/2}\int_{-\infty}^{\infty} f_\infty(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix})dx.$$

We will also encounter the twisted variant

$$\mathcal{H}_r f_\infty(y) = y^{-1/2} \int_{-\infty}^{\infty} f_\infty(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix}) e^{-2\pi i r x} dx \tag{5.1}$$

for $r \in \mathbf{R}$, and a twisted variant in the big Bruhat cell

$$\mathcal{H}_{r,\alpha} f_\infty(y) = y^{-1/2} \int_{-\infty}^{\infty} f_\infty(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix}) e^{-2\pi i r x} dx \tag{5.2}$$

for $\alpha \in \mathbf{R}$. Each of the above functions is smooth with compact support in $\mathbf{R}^+$.

For $\phi \in C_c^\infty(\mathbf{R}^+)$, the Mellin transform is denoted

$$\mathcal{M}_s \phi = \int_0^\infty \phi(y) y^s d^*y.$$

Composing with the Harish-Chandra transform, we obtain the **spherical transform**

$$\mathcal{S} f_\infty(s) = \mathcal{M}_s \mathcal{H} f_\infty.$$

The **Selberg transform** of f_∞ is defined by

$$h(t) = \mathcal{S} f_\infty(it) = \mathcal{M}_{it} \mathcal{H} f_\infty. \tag{5.3}$$

Then $h(it)$ is an even Paley-Wiener function. This means that it is holomorphic and there exists a real number $C \geq 1$ depending only on h such that for any integer $M > 0$, we have

$$h(a + ib) \ll_{M,h} \frac{C^{|b|}}{(1 + |a|)^M}. \tag{5.4}$$

Using (5.1) we also define a **twisted spherical transform** of f_∞ by

$$h_r(s) = \mathcal{M}_s \mathcal{H}_r f_\infty, \tag{5.5}$$

and a twisted variant in the big Bruhat cell

$$h_{r,\alpha}(s) = \mathcal{M}_s \mathcal{H}_{r,\alpha} f_\infty \tag{5.6}$$

for $\alpha \in \mathbf{R}$, as in (5.2). These functions likewise are holomorphic and satisfy (5.4), though they are not even in general. Note that $h_0(s) = h(s)$.

5.2. **Nonholomorphic case: spectral side.** When $k = 0$, the spectral side (3.2) of the relative trace formula becomes

$$I = \frac{\sqrt{n}}{2} \sum_{u_j \in \mathcal{F}_+^{new}(N^3, \omega')} \frac{\lambda_n(u_j) \overline{a_r(u_j)} \Lambda(s, u_j, \chi)}{\|u_j\|^2} h(t_j) K_{it_j}(2\pi |r|). \tag{5.7}$$

Proposition 5.1. *Let $k = 0$ and $r \in \mathbf{Q}$. If $\sigma = \mathrm{Re}(s)$ is sufficiently large, then the integral*

$$\int_{\mathbf{Q}^* \backslash \mathbf{A}^*} \int_{\mathbf{Q} \backslash \mathbf{A}} \sum_{u \in \mathcal{F}_0^{new}(N^3, \omega')} \Big| \frac{\lambda_f(u) R(f^\chi) \phi_u((\begin{smallmatrix} y & \\ & 1 \end{smallmatrix})) \overline{\phi_u((\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}))}}{\|u\|^2} |y|^{s-1/2} \Big| dx\, d^*y$$

is absolutely convergent. Hence as in Proposition 3.1, the integral (3.1) *is equal to* (5.7) *for such s. The sum* (5.7) *converges absolutely for all $s \in \mathbf{C}$, and defines an entire function.*

Proof. As in the proof of Proposition 3.1, by the fact there are at most finitely many u with exceptional spectral parameters, it suffices to sum over the set $\mathcal{F}'$ of newforms for which t is real. Thus we need to show that

$$\sum_{u\in\mathcal{F}'}\frac{|\lambda_f(u)|}{\|u\|^2}\int_{\mathbf{Q}^*\backslash\mathbf{A}^*}|\phi_{u_\chi}(\begin{pmatrix}y&\\&1\end{pmatrix})|\,y^{\sigma-1/2}d^*y\int_{\mathbf{Q}\backslash\mathbf{A}}|\phi_u(\begin{pmatrix}1&x\\0&1\end{pmatrix})|dx \tag{5.8}$$

is finite. The second integral is bounded by an absolute constant:

$$\int_{\mathbf{Q}\backslash\mathbf{A}}|\phi_u(\begin{pmatrix}1&x\\0&1\end{pmatrix})|dx=\int_0^1|u(i+x)|dx\le\int_0^1\sum_{m\neq0}\Big|a_m(u)K_{it}(2\pi|m|)e^{2\pi imx}\Big|dx$$
$$=\sum_{m\neq0}\Big|a_m(u)K_{it}(2\pi|m|)\Big|\ll\sum_{m\neq0}|m|^{1/2+\varepsilon}K_0(2\pi|m|)<\infty.$$

Here, we have used the fact that since t is real,

$$|K_{it}(y)|=\left|\tfrac12\int_0^\infty e^{-y(w+w^{-1})/2}w^{it}d^*w\right|\le\tfrac12\int_0^\infty e^{-y(w+w^{-1})/2}d^*w=K_0(y), \tag{5.9}$$

and also the bound $|a_m(u)|\ll|m|^{1/2+\varepsilon}$ ([IK], (5.92)). The strongest known bound of this nature is that of Kim and Sarnak [KS]:

$$|a_m(u)|\le\tau(|m|)|m|^{7/64}, \tag{5.10}$$

where τ is the divisor function. This, together with (2.26), gives $|\lambda_f(u)|\ll_n|h(t)|$. By the above observations, and following the proof of Lemma 2.2, we see that (5.8) is

$$\ll\sum_{u\in\mathcal{F}'}\frac{|h(t)|}{\|u\|^2}\int_0^\infty|u_\chi(iy)|y^{\sigma-1/2}d^*y. \tag{5.11}$$

Using the Fourier expansion of u_χ, the above integral is bounded by

$$\int_0^\infty\sum_{m\neq0}\Big|\chi(m)a_m(u)K_{it}(2\pi|m|y)\Big|y^\sigma d^*y\le(2\pi)^{-\sigma}\sum_{m\neq0}\frac{|a_m(u)|}{m^\sigma}\int_0^\infty|K_{it}(y)|y^\sigma d^*y.$$

Once again invoking (5.9), we have

$$\int_0^\infty|K_{it}(y)|y^\sigma d^*y\le\int_0^\infty K_0(y)y^\sigma d^*y=2^{\sigma-2}\Gamma(\sigma/2)^2$$

by a well-known identity for $\sigma>0$ (cf. [GRy], 6.561.16). Using (5.10), we see that the sum over m is bounded by an absolute constant when $\sigma\ge3$. This shows that the integral of u_χ in (5.11) is bounded by a constant independent of u and depending continuously on $\sigma\ge3$. As shown by Goldfeld, Hoffstein and Liemann,

$$\frac{1}{\|u\|^2}\ll_\varepsilon N^\varepsilon(1+|t|)^\varepsilon \tag{5.12}$$

for an absolute (ineffective) implied constant, [GHL94]. Thus we reduce to proving that

$$\sum_{u_j\in\mathcal{F}'}|h(t_j)|(1+|t_j|)^\varepsilon<\infty. \tag{5.13}$$

This follows from (5.4) and the fact that $|t_j|\to\infty$ (for details, see the end of the proof of Proposition 7.5 of [KL3]).

Now we prove that the sum (5.7) is absolutely convergent for all $s \in \mathbf{C}$. Once again, it suffices to sum over $u \in \mathcal{F}'$. Thus (using (5.10) to bound $a_r(u)$ and $\lambda_n(u)$), we need to show

$$\sum_{u_j \in \mathcal{F}'} \frac{\left|\Gamma(\frac{s+it_j}{2})\Gamma(\frac{s-it_j}{2})L(s,u_j,\chi)h(t_j)K_{it_j}(2\pi|r|)\right|}{\|u_j\|^2} < \infty. \tag{5.14}$$

By Stirling's formula ([AS], 6.1.39), for real $t \neq 0$ (taking $\arg(it) = \pm\frac{\pi}{2}$) we have

$$\left|\Gamma(\frac{s+it}{2})\Gamma(\frac{s-it}{2})\right| \sim 2\pi \left|\Big(\frac{it}{2}\Big)^{\frac{s+it-1}{2}}\Big(-\frac{it}{2}\Big)^{\frac{s-it-1}{2}}\right| = 2\pi\Big(\frac{|t|}{2}\Big)^{\sigma-1}e^{-\pi|t|/2} \tag{5.15}$$

as $|t| \to \infty$. Similarly, because t is real, we have

$$K_{it}(2\pi|r|) \ll e^{-\pi|t|/2} \tag{5.16}$$

as $|t| \to \infty$ (cf. eq. (19) on p. 88 of [Er]).

To bound the L-functions, by the functional equation we can assume without losing generality that $\sigma \geq 1/2$. For such s, we have the the uniform convexity bound

$$L(s,u_j,\chi) \ll_\varepsilon (D^2N^3)^{1/4+\varepsilon}(|s|+3)^{1/2+\varepsilon}(|t_j|+3)^{1/2+\varepsilon}$$

([IK], Theorem 5.41 and (5.8)). Here the implied constant is independent of u_j.

Using (5.12), we now find that the left-hand side of (5.14) is

$$\ll (|s|+3)^{1/2+\varepsilon}\sum_{u_j}(|t_j|+3)^{1/2+\varepsilon}(\tfrac{|t_j|}{2})^{\sigma-1}|h(t_j)|e^{-\pi|t_j|}$$

The finiteness of the above sum follows as for (5.13). It is clear as well that the convergence is uniform for s in compact sets, giving an entire function. □

5.3. **Non-holomorphic case: geometric side.** By Proposition 3.2 and (3.7), the geometric side is equal to

$$I_1(s) + \sum_{a \in \mathbf{Q}^*} I_{\delta_a}(s),$$

for $\delta_a = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The only changes from the holomorphic case discussed earlier are archimedean. For example, it remains true here that $I_{\delta_a}(s) \neq 0$ only if $a = \frac{N^2b}{nD}$ for some nonzero integer b. The local orbital integrals at ∞ are now given as general integral transforms of $f_\infty \in C_c^\infty(G(\mathbf{R})^+//K_\infty)$. Using the fact that f_∞ has compact support modulo Z_∞, we will see that all but finitely many of the geometric terms vanish, and indeed if N is sufficiently large, the only nonzero term is the main term.

For the main term we have, upon replacing y by y^{-1} in (3.3),

$$I_1(s)_\infty = \int_{\mathbf{R}^*}\int_{\mathbf{R}} f_\infty(\begin{pmatrix} y & xy \\ 0 & 1\end{pmatrix})e^{2\pi irx}dx\,\chi_\infty(y)|y|^{1/2-s}d^*y.$$

Since f_∞ is supported on $G(\mathbf{R})^+$, the first integral can be taken over $\mathbf{R}^+$, where χ_∞ is trivial. Furthermore, since f_∞ is bi-invariant under $Z_\infty K_\infty$, it follows easily (using the Cartan decomposition [KL3, §3.1]) that

$$f_\infty(g) = f_\infty(g^{-1}). \tag{5.17}$$

Therefore

$$I_1(s)_\infty = \int_0^\infty \int_{-\infty}^\infty f_\infty(\begin{pmatrix} y^{-1} & -x \\ 0 & 1\end{pmatrix})e^{2\pi i r x} dx\, y^{1/2-s} d^*y \tag{5.18}$$

$$= \int_0^\infty \left[y^{-1/2} \int_{-\infty}^\infty f_\infty(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}) e^{-2\pi i r x} dx \right] y^s d^*y = \mathcal{M}_s \mathcal{H}_r f_\infty = h_r(s)$$

as in (5.5).

Multiplying the above by the local non-archimedean values given in (3.11) (taking $k' = 1$), we obtain the following.

Proposition 5.2. *The global integral $I_1(s)$ is nonzero only if $\gcd(r, N) = 1$. In this case,*

$$I_1(s) = \frac{h_r(s)}{n^{s-1/2}} N \prod_{p|N} (p^2 - 1) \sum_{d| \gcd(n,r)} d^{2s} \overline{\omega'(\tfrac{n}{d})} \chi(\tfrac{rn}{d^2}).$$

For $\delta_a = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$, the archimedean orbital integral is

$$I_{\delta_a}(s)_\infty = \int_{\mathbf{R}^*} \int_{\mathbf{R}} f_\infty(\begin{pmatrix} y & \\ & 1 \end{pmatrix} \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) e^{2\pi i r x} dx\, \chi_\infty(y) |y|^{1/2-s} d^*y$$

$$= \int_0^\infty \int_{\mathbf{R}} f_\infty(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix}) e^{2\pi i r x} dx\, y^{1/2-s} d^*y$$

$$= \int_0^\infty \int_{\mathbf{R}} f_\infty(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix}) e^{-2\pi i r x} dx\, y^{s-1/2} d^*y, \tag{5.19}$$

$$= \mathcal{M}_s(\mathcal{H}_{r,a} f_\infty) = h_{r,a}(s)$$

as in (5.6).

Proposition 5.3. *For any choice of $f_\infty \in C_c^\infty(G(\mathbf{R})^+//K_\infty)$, $I_\delta(s) = 0$ for all but finitely many δ. Indeed, there exists a constant C, depending on f_∞ and n, such that*

$$\sum_\delta I_\delta(s) = I_1(s) = \frac{h_r(s)}{n^{s-1/2}} N \prod_{p|N} (p^2 - 1) \sum_{d| \gcd(n,r)} d^{2s} \overline{\omega'(\tfrac{n}{d})} \chi(\tfrac{rn}{d^2}) \tag{5.20}$$

whenever $N > C$.

Proof. The function $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \mapsto \frac{\alpha\gamma}{\alpha\delta - \beta\gamma}$ is well-defined in $\overline{G}(\mathbf{R})$. Hence it is bounded on the compact set $\mathrm{Supp}(f_\infty)/Z_\infty$. Taking

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} y & \\ & 1 \end{pmatrix} \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix},$$

we have $\frac{\alpha\gamma}{\alpha\delta-\beta\gamma} = a$. This shows that if $|a|$ is sufficiently large, the above matrix lies outside the support of f_∞ for all x, y, and hence $I_{\delta_a}(s) = 0$. Furthermore, because

$$|a| = \tfrac{N^2}{n}|b| \geq \tfrac{N^2}{n} \to \infty$$

as $N \to \infty$, when N is sufficiently large the only nonzero term is $I_1(s)$. □

Putting everything together, we now arrive at the main result for Maass forms.

Theorem 5.4. *Let r, n, N be positive integers with N square-free and $(rn, N) = 1$. Let χ be a primitive Dirichlet character of modulus D, where $(D, rnN) = 1$. Let $h(iz)$ be an even Paley-Wiener function. When the square-free integer N is sufficiently large, we have for all $s \in \mathbf{C}$,*

$$\sum_{u_j \in \mathcal{F}_+^{new}(N^3,\omega')} \frac{\lambda_n(u_j)\,\overline{a_r(u_j)}\,\Lambda(s,u_j,\chi)}{\psi(N^3)\|u_j\|^2} h(t_j)\,K_{it_j}(2\pi|r|) = \frac{2}{n^s} h_r(s) \prod_{p|N}(1-\frac{1}{p}) \sum_{d|\gcd(n,r)} d^{2s}\,\overline{\omega'(\tfrac{n}{d})}\,\chi(\tfrac{rn}{d^2}) \tag{5.21}$$

for $h_r(s)$ as in (5.5).

Remarks: (1) An immediate corollary (at least when $\gcd(n,r) = 1$) is the existence of a Maass newform of level N^3 for which $\lambda_n(u)$, $a_r(u)$, and $\Lambda(s, u, \chi)$ are simultaneously nonzero.

(2) When $\gcd(r, n) = 1$, the sum on RHS becomes $\overline{\omega'(n)}\chi(rn)$. If $r = n = 1$, then the RHS is independent of χ. This is the case stated as Theorem 1.1.

(3) Both sides vanish when $(r, N) > 1$. See the remark after Theorem 4.1.

(4) One can weaken the hypotheses somewhat. It is sufficient for $h(iz)$ to be Paley-Wiener of order $m \geq 8$ (cf. Corollary 6.12 and (3.14) of [KL3]).

Proof. As a consequence of Proposition 3.2 and the above discussion, the equality between the spectral side (5.7) and the geometric side (5.20) has been established in some right half-plane $\mathrm{Re}(s) \geq \alpha$ for α sufficiently large. Multiplying both sides of this relative trace formula by $\frac{2}{\sqrt{n}\psi(N^3)}$ we obtain (5.21) for such s. On the other hand, by Proposition 5.1, each side of (5.21) is an entire function of s. Hence the equality is valid for all complex s. □

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