Kuznetsov's trace formula and the Hecke eigenvalues of Maass forms

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Abstract

We give an adelic treatment of the Kuznetsov trace formula as a relative trace formula on GL(2) over **Q**. The result is a variant which incorporates a Hecke eigenvalue in addition to two Fourier coefficients on the spectral side. We include a proof of a Weil bound for the generalized twisted Kloosterman sums which arise on the geometric side. As an application, we show that the Hecke eigenvalues of Maass forms at a fixed prime, when weighted as in the Kuznetsov formula, become equidistributed relative to the Sato-Tate measure in the limit as the level goes to infinity.

Contents

1 Introduction

1.1 Some history

A *Fourier trace formula* for GL(2) is an identity between a product of two Fourier coefficients, averaged over a family of automorphic forms on GL(2), and a series involving Kloosterman sums and the Bessel J-function. The first example, arising from Petersson's computation of the Fourier coefficients of Poincaré series in 1932 [P1] and his introduction of the inner product in 1939 [P2], has the form

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}}\sum_{f\in\mathcal{F}_k(N)}\frac{a_m(f)\overline{a_n(f)}}{\|f\|^2}=\delta_{m,n}+2\pi i^k\sum_{c\in N\mathbf{Z}^+}\frac{S(m,n;c)}{c}J_{k-1}(\frac{4\pi\sqrt{mn}}{c}),$$

where $\mathcal{F}_k(N)$ is an orthogonal basis for the space of cusp forms $S_k(\Gamma_0(N))$, and

$$S(m,n;c)=\sum_{x\overline{x}\equiv 1 \bmod c} e^{2\pi i(mx+n\overline{x})/c}$$

is a Kloosterman sum. Because of the existence of the Weil bound

$$|S(m,n;c)|\le\tau(c)(m,n,c)^{1/2}c^{1/2} \tag{1.1}$$

where τ is the divisor function, and the bound

$$J_{k-1}(x)\ll\min(x^{k-1},x^{-1/2})$$

for the Bessel function, the Petersson formula is useful for approximating expressions involving Fourier coefficients of cusp forms. For example, Selberg used it in 1964 ([Sel3]) to obtain the nontrivial bound

$$a_n(f)=O(n^{(k-1)/2+1/4+\varepsilon}) \tag{1.2}$$

in the direction of the Ramanujan-Petersson conjecture $a_n(f)=O(n^{(k-1)/2+\varepsilon})$ subsequently proven by Deligne.

In his paper, Selberg mentioned the problem of extending his method to the case of Maass forms. This was begun in the late 1970's independently by Bruggeman and Kuznetsov ([Brug], [Ku]). The left-hand side of the above Petersson formula is now replaced by a sum of the form

$$\sum_{u_j\in\mathcal{F}}\frac{a_m(u_j)\overline{a_n(u_j)}}{\|u_j\|^2}\frac{h(t_j)}{\cosh(\pi t_j)}, \tag{1.3}$$

where $m,n>0$, $\mathcal{F}$ is an (orthogonal) basis of Maass cusp forms of weight $k=0$ and level $N=1$, t_j is the spectral parameter defined by $\Delta u_j=(\frac{1}{4}+t_j^2)u_j$ for the Laplacian Δ, and $h(t)$ is an even holomorphic function with sufficient decay.

There is a companion term coming from the weight 0 part of the continuous spectrum, describable in terms of the Eisenstein series

$$E(s,z) = \frac{1}{2} \sum_{\substack{c,d\in\mathbf{Z} \\ (c,d)=1}} \frac{y^{1/2+s}}{|cz+d|^{1+2s}} \qquad (\mathrm{Re}(s) > \tfrac{1}{2}, y>0, z=x+iy).$$

More accurately, it involves the analytic continuation to s on the imaginary line. This analytic continuation is provided by the Fourier expansion

$$\begin{aligned} E(s,z) = y^{1/2+s} + y^{1/2-s} \frac{\sqrt{\pi}\,\Gamma(s)\zeta(2s)}{\Gamma(1/2+s)\zeta(1+2s)} \\ + \frac{2y^{1/2}\pi^{1/2+s}}{\Gamma(1/2+s)\zeta(1+2s)} \sum_{m\neq 0} \sigma_{2s}(m)|m|^s K_s(2\pi|m|y)e^{2\pi i m x}. \end{aligned} \tag{1.4}$$

Here $\sigma_{2s}(m) = \sum_{0<d|m} d^{2s}$ is the divisor sum, and K_s is the K-Bessel function. The continuous contribution to the Kuznetsov/Bruggeman formula is the following integral of the product of two Fourier coefficients of $E(it,z)$ against the function $h(t)$:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(m/n)^{it}\sigma_{2it}(m)\overline{\sigma_{2it}(n)}}{|\zeta(1+2it)|^2} h(t)dt. \tag{1.5}$$

The Fourier trace formula is then the equality between the sum of (1.3) and (1.5) on the so-called spectral side, with the geometric side given by

$$\frac{\delta_{m,n}}{\pi^2} \int_{-\infty}^{\infty} h(t)\tanh(\pi t)\,t\,dt + \frac{2i}{\pi} \sum_{c\in\mathbf{Z}^+} \frac{S(m,n;c)}{c} \int_{-\infty}^{\infty} J_{2it}(\frac{4\pi\sqrt{mn}}{c}) \frac{h(t)\,t}{\cosh(\pi t)} dt. \tag{1.6}$$

Using this together with the Weil bound (9.2), Kuznetsov proved a mean-square estimate for the Fourier coefficients $a_n(u_j)$ ([Ku], Theorem 6), which immediately implies the bound

$$a_n(u_j) \ll_{j,\varepsilon} n^{1/4+\varepsilon}$$

in the direction of the (still open) Ramanujan conjecture $a_n(u_j) = O(n^{\varepsilon})$. (See also [Brug], §4.) This extended Selberg's result (1.2) to the case of Maass forms.

Kuznetsov also "inverted" the formula to give a variant in which a general test function appears on the geometric side in place of the Bessel integral. (Motohashi has given an interesting conceptual explanation of this, showing that the procedure is reversible, [Mo2].) This allows for important applications to bounding sums of Kloosterman sums. Namely, Kuznetsov proved that the estimate

$$\sum_{c\leq X} \frac{S(m,n;c)}{c} \ll_{m,n,\varepsilon} X^{\theta+\varepsilon} \tag{1.7}$$

holds with $\theta = \frac{1}{6}$ ([Ku], Theorem 3). The Weil bound alone yields only $\theta = \frac{1}{2}$, showing that Kuznetsov's method detects considerable cancellation among the Kloosterman sums due to the oscillations in their arguments. Linnik had conjectured in 1962 that (1.7) holds with $\theta = 0$, and Selberg remarked that this would imply the Ramanujan-Petersson conjecture for holomorphic cusp forms of level 1, ([Sel3]; see also §4 of [Mu]). By studying the Dirichlet series

$$Z(s,m,n) = \sum_c \frac{S(m,n;c)}{c^{2s}},$$

Selberg also codified a relationship between sums of Kloosterman sums and the smallest eigenvalue λ_1 of the Laplacian, leading him to conjecture that $\lambda_1 \geq \frac{1}{4}$ for congruence subgroups. He obtained the inequality $\lambda_1 \geq \frac{3}{16}$ using the Weil bound (9.2). This inequality is also a consequence of the generalized Kuznetsov formula given in 1982 by Deshouillers and Iwaniec ([DI]).

Fourier trace formulas have since become a staple tool in analytic number theory. We mention here a sampling of notable results in which they have played a role. Deshouillers and Iwaniec used the Kuznetsov formula to deduce bounds for very general weighted averages of Kloostermans sums, showing in particular that Linnik's conjecture holds on average ([DI], §1.4). They list some interesting consequences in §1.5 of their paper. For example, there are infinitely many primes p for which $p+1$ has a prime factor greater than $p^{21/32}$. They also give applications to the Brun-Titchmarsh theorem and to mean-value theorems for primes in arithmetic progressions (see also [Iw1], §12-13).

Suppose $f(x) \in \mathbf{Z}[x]$ is a quadratic polynomial with negative discriminant. If p is prime and ν is a root of f in $\mathbf{Z}/p\mathbf{Z}$, then the fractional part $\{\frac{\nu}{p}\} \in [0,1)$ is independent of the choice of representative for ν in $\mathbf{Z}$. Duke, Friedlander, and Iwaniec proved that for (p,ν) ranging over all such pairs, the set of these fractional parts is uniformly distributed in $[0,1]$, i.e. for any $0 \leq \alpha < \beta \leq 1$,

$$\frac{\#\{(p,\nu)|\, p \leq x, f(\nu) \equiv 0 \bmod p, \alpha \leq \{\frac{\nu}{p}\} < \beta\}}{\#\{p \leq x|\, p \text{ prime}\}} \sim (\beta - \alpha)$$

as $x \to \infty$ ([DFI]). Their proof uses the Kuznetsov formula to bound a certain related Poincaré series via its spectral expansion. See also Chapter 21 of [IK].

Applications of Fourier trace formulas to the theory of L-functions abound. Using the results of [DI], Conrey showed in 1989 that more than 40% of the zeros of the Riemann zeta function are on the critical line ([Con]).[1] Motohashi's book [Mo1] discusses other applications to $\zeta(s)$, including the asymptotic formula for its fourth moment. In his thesis, Venkatesh used a Fourier trace formula to carry out the first case of Langlands' *Beyond Endoscopy* program for GL(2) ([L], [V1], [V2]). This provided a new proof of the result of Labesse and Langlands characterizing as dihedral those forms for which the symmetric square L-function has a pole, as well as giving an asymptotic bound for the dimension of holomorphic cusp forms of weight 1, extending results of Duke. Fourier trace formulas have also been used by many authors in establishing subconvexity bounds for GL(1), GL(2) and Rankin-Selberg L-functions; see [MV] and its references, although this definitive paper does not actually use trace formulas. Subconvexity

[1]Conrey, Iwaniec and Soundararajan have recently proven that more than 56% of the zeros of the family of Dirichlet L-functions lie on the critical line, [CIS].

bounds have important arithmetic applications, notably to Hilbert's eleventh problem of determining the integers that are integrally represented by a given quadratic form over a number field ([IS1], [BH]). Other applications of Fourier trace formulas include nonvanishing of L-functions at the central point ([Du], [IS2], [KMV]) and the density of low-lying zeros of automorphic L-functions (starting with [ILS]).

1.2 Overview of the contents

Zagier is apparently the first one to observe that Kuznetsov's formula can be obtained by integrating each variable of an automorphic kernel function over the unipotent subgroup. His proof is detailed by Joyner in §1 of [Joy]. See also the description by Iwaniec on p. 258 of [Iw1], and the article [LiX] by X. Li, who also extended the formula to the setting of Maass forms for $\mathrm{SL}_n(\mathbf{Z})$, [Gld]. Related investigations have been carried out by others, notably in the context of base change by Jacquet and Ye (cf. [Ja] and its references).

Our primary purpose is to give a detailed account of this method over the adeles of $\mathbf{Q}$, for Maass cusp forms of arbitrary level N and nebentypus ω'. We obtain a variant of the Kuznetsov trace formula by using the kernel function attached to a Hecke operator $T_{\mathtt{n}}$. The final formula is given in Theorem 7.14 on page 86, and it differs from the usual version by the inclusion of eigenvalues of $T_{\mathtt{n}}$ on the spectral side. The cuspidal term thus has the form

$$\sum_{u_j\in\mathcal{F}(N)} \frac{\lambda_{\mathtt{n}}(u_j)\, a_{m_1}(u_j)\overline{a_{m_2}(u_j)}}{\|u_j\|^2}\frac{h(t_j)}{\cosh(\pi t_j)}. \tag{1.8}$$

This is a complement to the article [KL1], which dealt with Petersson's formula from the same viewpoint. As we pointed out there, the above variant can alternatively be derived from the classical version (see Section 7.7 below). It is also possible to invert the final formula to get a version with the test function appearing on the geometric side rather than the spectral side, although we will not pursue this. See Theorem 2 of [BKV] or [A], p. 135.

The incorporation of Hecke eigenvalues in (1.8) allows us to prove a result about their distribution (Theorem 10.2). To state a special case, assume for simplicity that the nebentypus is trivial, and that the basis $\mathcal{F}(N)$ is chosen so that $a_1(u_j) = 1$ for all j. Then for any prime $p \nmid N$, we prove that the multiset of Hecke eigenvalues $\lambda_p(u_j)$, when weighted by

$$w_j = \frac{1}{\|u_j\|^2}\frac{h(t_j)}{\cosh(\pi t_j)},$$

becomes equidistributed relative to the Sato-Tate measure in the limit as $N \to \infty$. This means that for any continuous function f on $\mathbf{R}$,

$$\lim_{N\to\infty}\frac{\sum_{u_j\in\mathcal{F}(N)} f(\lambda_p(u_j))w_j}{\sum_{u_j\in\mathcal{F}(N)} w_j} = \frac{1}{\pi}\int_{-2}^{2} f(x)\sqrt{1-\tfrac{x^2}{4}}\,dx.$$

This can be viewed as evidence for the Ramanujan conjecture, which asserts that $\lambda_p(u_j) \in [-2,2]$ for all j. The above result holds independently of both p and the choice of h from a large family of suitable functions. We discuss some of the history of this problem and its relation to the Sato-Tate conjecture in Section 10.

The material in the first six sections can be used as a basis for any number of investigations of Maass forms with the GL(2) trace formula. Sections 2-4 are chiefly expository. We begin with the goal of explaining the connection between the Laplace eigenvalue of a Maass form and the principal series representation of $\mathrm{GL}_2(\mathbf{R})$ determined by it. We then give a detailed account of the passage between a Maass form on the upper half-plane and its adelic counterpart, which is a cuspidal funcion on $\mathrm{GL}_2(\mathbf{A})$. We also describe the adelic Hecke operators of weight $\mathtt{k} = 0$ and level N corresponding to the classical ones $T_{\mathtt{n}}$.

Although similar in spirit with the derivation of Petersson's formula in [KL1], the analytic difficulties in the present case are considerably more subtle. Whereas in the holomorphic case the relevant Hecke operator is of finite rank, in the weight zero case it is not even Hilbert-Schmidt. The setting for the adelic trace formula is the Hilbert space

$$L^2(\omega) = \begin{cases} \phi : G(\mathbf{A}) \to \mathbf{C} \\ \phi(z\gamma g) = \omega(z)\phi(g) \quad (z \in Z(\mathbf{A}), \gamma \in G(\mathbf{Q})), \\ \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} |\phi|^2 < \infty, \end{cases}$$

where $G = \mathrm{GL}_2$, Z is the center, and ω is a finite order Hecke character. Relative to the right regular representation R of $G(\mathbf{A})$ on $L^2(\omega)$, there is a spectral decomposition $L^2(\omega) = L^2_{\text{disc}}(\omega) \oplus L^2_{\text{cont}}(\omega)$. The classical cusp forms correspond to certain elements in the discrete part, while the continuous part is essentially a direct integral of certain principal series representations $H(it)$ of $G(\mathbf{A})$. We begin Section 6 by describing this in detail, following Gelbart and Jacquet [GJ]. For a function $f \in L^1(\overline{\omega})$ attached to a classical Hecke operator, we then investigate the kernel

$$K(x,y) = \sum_{\gamma \in Z(\mathbf{Q})\backslash G(\mathbf{Q})} f(x^{-1}\gamma y) \tag{1.9}$$

of the operator $R(f)$. We assume that f_∞ is bi-invariant under SO(2), compactly supported in $\overline{G}(\mathbf{R})^+$, and sufficiently differentiable. Then letting ϕ range through an orthonormal basis for the subspace of vectors in $H(0)$ of weight 0 and level N, the main result of the section is a proof that the spectral expansion

$$\begin{aligned} K(x,y) = \; & \delta_{\omega,1} \frac{3}{\pi} \int_{\overline{G}(\mathbf{A})} f(g)dg + \sum_{\varphi \in \mathcal{F}(N)} \frac{R(f)\varphi(x)\overline{\varphi(y)}}{\|\varphi\|^2} \\ & + \frac{1}{4\pi} \sum_{\phi} \int_{-\infty}^{\infty} E(\pi_{it}(f)\phi_{it}, x)\overline{E(\phi_{it}, y)}dt, \end{aligned}$$

is absolutely convergent and valid for all x, y. These are, respectively, the residual, cuspidal, and continuous components of the kernel.

In Section 5, we discuss the Eisenstein series. We give an explicit description of the finite set of Eisenstein series $E(\phi_s, g)$ that contribute to the above expression for $K(x, y)$. Their Fourier coefficients involve generalized divisor sums and Dirichlet L-values on the right edge of the critical strip, directly generalizing (1.4). We derive bounds for these Fourier coefficients, which are useful for both the convergence and applications of the Kuznetsov formula. For this purpose we require lower bounds for Dirichlet L-functions on the right edge of the critical strip, reviewed in Section 2. (We note that more generally, in establishing absolute convergence of the spectral side of Jacquet's $\mathrm{GL}(n)$ relative trace formula, Lapid makes use of lower bounds for Rankin-Selberg L-functions on the right edge of the critical strip due to Brumley, [Lap], [Brum].)

In Section 7 we integrate each variable of $K(x, y)$ against a character over the unipotent group $N(\mathbf{Q})\backslash N(\mathbf{A})$. Using the geometric form (1.9) of the kernel, we obtain the geometric side of the Kuznetsov formula as a sum of orbital integrals whose finite parts evaluate to generalized twisted Kloosterman sums, defined by

$$S_{\omega'}(m_2, m_1; \mathtt{n}; c) = \sum_{dd' \equiv \mathtt{n} \bmod c} \overline{\omega'(d)} e^{2\pi i (dm_2 + d'm_1)/c} \qquad \text{(for } N|c\text{)},$$

where ω' is the Dirichlet character of modulus N attached to ω. These sums also arise in the generalized Petersson formula of [KL1]. After an extra averaging at the archimedean place, we obtain the J-Bessel integrals as in (1.6). Using the spectral form of the kernel we obtain the spectral side of the Kuznetsov formula, giving the main result, Theorem 7.14. The function $h(t)$ of (1.8) is the Selberg transform of the archimedean test function f_∞.

The hypothesis that f_∞ be smooth and compactly supported amounts to requiring that $h(iz)$ be an even Paley-Wiener function. This is very restrictive, ruling out well-behaved functions like the Gaussian $h(t) = e^{-t^2}$. In Section 8, we carefully study the various transforms involved under more relaxed hypotheses, and show that the Kuznetsov formula remains valid. We start with a function f on $G(\mathbf{A})$ which is C^m for m sufficiently large, and has polynomial decay rather than compact support. We then express f as a limit of compactly supported C^m functions (for which we have already established the Kuznetsov formula), and then show that the Kuznetsov formula is preserved in the limit. A key step is proving that $R(f)$ is a Hilbert-Schmidt operator on the cuspidal subspace (cf. Corollary 8.33).

In Section 9, we prove the Weil bound

$$|S_\chi(a, b; \mathtt{n}; c)| \le \tau(\mathtt{n})\tau(c)(a\mathtt{n}, b\mathtt{n}, c)^{1/2} c^{1/2} \mathfrak{c}_\chi^{1/2}, \tag{1.10}$$

where $\mathfrak{c}_\chi$ is the conductor of χ, and τ is the divisor function. Various identities relate the generalized sum to classical twisted Kloosterman sums $S_\chi(a, b; c) = S_\chi(a, b; 1; c)$. Therefore we reduce to proving a Weil bound for the latter sums. The latter is well-known, but seems to be a gap in the literature. Furthermore, it is sometimes erroneously asserted that $|S_\chi(a, b; c)| \le \tau(c)(a, b, c)^{1/2} c^{1/2}$. We give a counterexample on p. 125. For these reasons, we have included all of the details of the proof of (1.10).

For simplicity, in this paper we only treat forms of weight $\mathtt{k} = 0$ over $\mathbf{Q}$, and we deal only with positive integer Fourier coefficients for the cusp at infinity. There are many expositions of the Kuznetsov formula in the classical language which extend beyond this scope and give other applications. See especially [DI], [CPS], [Mo3] and [BM]. The latter incorporates general weights and cusps over a totally real field. We also recommend the text of Baker [B].

1.3 Acknowledgements

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2 Preliminaries

2.1 Notation and Haar measure

Notation and normalization of measures is the same as in [KL2], where full details are given. Let $G = \mathrm{GL}_2$, let $M = \{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)\} \subseteq G$ be the diagonal subgroup, and let $N = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\} \subseteq G$ be the upper triangular unipotent subgroup. The Borel subgroup of upper triangular matrices is denoted $B = MN = NM$. We write $\overline{G}$ for G/Z, where Z is the center of G, and generally for a subset $S \subseteq G$, $\overline{S}$ denotes the image of S in $\overline{G}$. Let

$$K_\infty = \{k_\theta := \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) | \, \theta \in \mathbf{R}\} \tag{2.1}$$

denote the compact subgroup SO(2) of $G(\mathbf{R})$.

Let $\mathbf{Z}^+$ denote the set of positive integers and let $\mathbf{R}^+$ denote the group of positive reals. If p is prime, we let $\mathbf{Q}_p$ and $\mathbf{Z}_p$ denote the p-adic numbers and p-adic integers, respectively. For any rational integer $x > 0$, we often use the notation

$$x_p = \mathrm{ord}_p(x),$$

so that $x = \prod_p p^{x_p}$, $N = \prod_p p^{N_p}$, etc.

Let $\mathbf{A}, \mathbf{A}_{\mathrm{fin}}$ be the adeles and finite adeles of $\mathbf{Q}$. Then $\widehat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$ is an open compact subgroup of $\mathbf{A}_{\mathrm{fin}}$. For an element $d \in \mathbf{Q}^*$, we let

$$d_N \in \mathbf{A}^* \tag{2.2}$$

be the idele which agrees with d at places $p|N$ and is 1 at all other places.

Let $K_p = G(\mathbf{Z}_p)$ and $K_{\mathrm{fin}} = G(\widehat{\mathbf{Z}})$ denote the standard maximal compact subgroups of $G(\mathbf{Q}_p)$ and $G(\mathbf{A}_{\mathrm{fin}})$ respectively. By the Iwasawa decomposition,

$$G(\mathbf{A}) = M(\mathbf{A})N(\mathbf{A})K,$$

where

$$K = K_\infty \times K_{\mathrm{fin}}.$$

For an integer $N \geq 1$, define the following nested congruence subgroups of K_{fin}:

$$K_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_{\mathrm{fin}} | \, c \equiv 0 \bmod N\widehat{\mathbf{Z}}\},$$

$$K_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(N) | \, d \equiv 1 \bmod N\widehat{\mathbf{Z}}\},$$

$$K(N) = \{k \in K_{\mathrm{fin}} | \, k \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod N\widehat{\mathbf{Z}}\}.$$

Each of these is open and compact in $G(\mathbf{A}_{\mathrm{fin}})$. By the strong approximation theorem, we have

$$G(\mathbf{A}) = G(\mathbf{Q})(G(\mathbf{R})^+ \times K_1(N)), \tag{2.3}$$

where as usual $G(\mathbf{Q})$ is embedded diagonally in $G(\mathbf{A})$, and $G(\mathbf{R})^+$ is the subgroup of $\mathrm{GL}_2(\mathbf{R})$ consisting of matrices with positive determinant. We will also use the local subgroups $K_0(N)_p = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_p |\, c \in N\mathbf{Z}_p\}$, and similarly for $K_1(N)_p$.

We take $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ to be the intersections of the above congruence subgroups with $\mathrm{SL}_2(\mathbf{Z})$ as usual. We set

$$\psi(N) = [K_{\mathrm{fin}} : K_0(N)] = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)] = N \prod_{\substack{p|N \\ p \text{ prime}}} (1 + \frac{1}{p}), \tag{2.4}$$

and locally $\psi(N) = \prod_p \psi_p(N)$, where

$$\psi_p(N) = [K_p : K_0(N)_p] = p^{N_p - 1}(p+1).$$

Haar measure will be normalized as follows. See §7 of [KL2] for more detail. On $\mathbf{R}$ we take Lebesgue measure dx, and on $\mathbf{R}^*$ we take $\frac{dy}{|y|}$. On $\mathbf{Q}_p$ we normalize by $\mathrm{meas}(\mathbf{Z}_p) = 1$, and on $\mathbf{Q}_p^*$ we take $\mathrm{meas}(\mathbf{Z}_p^*) = 1$. These choices determine measures on $\mathbf{A}$ and $\mathbf{A}^* \cong Z(\mathbf{A})$, with $\mathrm{meas}(\mathbf{Q}\backslash\mathbf{A}) = 1$. We normalize dk on K_∞ by $\mathrm{meas}(K_\infty) = 1$, and use the above measures on $\mathbf{R}$ and $\mathbf{R}^*$ to define measures on $N(\mathbf{R}) \cong \mathbf{R}$ and $M(\mathbf{R}) \cong \mathbf{R}^* \times \mathbf{R}^*$. These choices determine a measure on $G(\mathbf{R})$ by the Iwasawa decomposition: writing $g = mnk$, we take $dg = dm\, dn\, dk$. We normalize Haar measure on $G(\mathbf{Q}_p)$ so that $\mathrm{meas}(K_p) = 1$, and on $G(\mathbf{A}_{\mathrm{fin}})$ by taking $\mathrm{meas}(K_{\mathrm{fin}}) = 1$. We then adopt the product measure on $G(\mathbf{A}) = G(\mathbf{R}) \times G(\mathbf{A}_{\mathrm{fin}})$. Having fixed measures on $G(\mathbf{A})$ and $Z(\mathbf{A}) \cong \mathbf{A}^*$ as above, we give $\overline{G}(\mathbf{A}) = G(\mathbf{A})/Z(\mathbf{A})$ the associated quotient measure. It has the property that $\mathrm{meas}(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})) = \pi/3$. In the quotient measure on $\overline{G}(\mathbf{Q}_p)$, we have $\mathrm{meas}(\overline{K_p}) = 1$. We also take $\mathrm{meas}(\overline{K_\infty}) = 1$, which is *not* the quotient measure on $K_\infty/\{\pm 1\}$.

For any real number x, we denote

$$e(x) = e^{2\pi i x}.$$

We let $\theta : \mathbf{A} \longrightarrow \mathbf{C}^*$ denote the standard character of $\mathbf{A}$. It is defined by

$$\theta_p(x) = \begin{cases} e(-x) = e^{-2\pi i x} & \text{if } p = \infty \\ e(r_p(x)) = e^{2\pi i r_p(x)} & \text{if } p < \infty, \end{cases} \tag{2.5}$$

where $r_p(x) \in \mathbf{Q}$ is the p-principal part of x, a number with p-power denominator characterized (up to $\mathbf{Z}$) by $x \in r_p(x) + \mathbf{Z}_p$. Then θ is trivial on $\mathbf{Q}$, and $\theta_{\mathrm{fin}} = \prod_{p<\infty} \theta_p$ is trivial precisely on $\widehat{\mathbf{Z}}$. For $m \in \mathbf{Q}$, we define the character θ_m by

$$\theta_m(x) = \theta(-mx) = \overline{\theta(mx)}.$$

It is well-known that every character of $\mathbf{Q}\backslash\mathbf{A}$ arises in this way, i.e. $\mathbf{Q} \cong \widehat{\mathbf{Q}\backslash\mathbf{A}}$ by the the map $m \mapsto \theta_m$.

If V is a space of functions on a group G, then unless otherwise specified, we denote the right regular action of G on V by R. Thus for $\phi \in V$ and $g, x \in G$,

$$R(g)\phi(x) = \phi(xg).$$

2.2 Characters and Dirichlet L-functions

For a positive integer N, a **Dirichlet character** modulo N is a homomorphism

$$\widetilde{\chi} : (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*, \tag{2.6}$$

extended to a function on $\mathbf{Z}$ by taking $\widetilde{\chi}(n) = 0$ if $\gcd(n, N) > 1$. The simplest example is when (2.6) is the trivial homomorphism. In this case we say that $\widetilde{\chi}$ is the **principal character** modulo N.

If $d|N$ and χ' is a Dirichlet character modulo d, then it defines a Dirichlet character $\widetilde{\chi}$ modulo N by the composition

$$\widetilde{\chi} : (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow (\mathbf{Z}/d\mathbf{Z})^* \longrightarrow \mathbf{C}^*, \tag{2.7}$$

where the last arrow is χ'. We say that $\widetilde{\chi}$ is the character of modulus N **induced** from χ'. Conversely, if $\widetilde{\chi}$ is a Dirichlet character modulo N that factors through the projection to $(\mathbf{Z}/d\mathbf{Z})^*$ for some positive $d|N$ as above, then we say d is an **induced modulus** for $\widetilde{\chi}$. The **conductor** of $\widetilde{\chi}$ is the smallest induced modulus $\mathfrak{c}_{\widetilde{\chi}}$ for $\widetilde{\chi}$. Equivalently, $\mathfrak{c}_{\widetilde{\chi}}$ is the smallest positive divisor of N for which $\widetilde{\chi}(a) = 1$ whenever $\gcd(a, N) = 1$ and $a \equiv 1 \bmod \mathfrak{c}_{\widetilde{\chi}}$. If $\mathfrak{c}_{\widetilde{\chi}} = N$, then $\widetilde{\chi}$ is **primitive**.

Write $\mathbf{A}^* = \mathbf{Q}^*(\mathbf{R}^+ \times \widehat{\mathbf{Z}}^*)$. A **Hecke character** is a continuous homomorphism $\chi : \mathbf{A}^* \longrightarrow \mathbf{C}^*$, trivial on $\mathbf{Q}^*$. The restriction of χ to $\mathbf{R}^+$ is of the form $x \mapsto x^s$ for a unique complex number s. Therefore the Hecke character

$$\chi_0(a) = \chi(a)|a|^{-s}$$

is trivial on $\mathbf{Q}^*\mathbf{R}^+$, so it has finite order (cf. Lemma 12.1 of [KL2]; beware that in the bijection discussed after that lemma, *Dirichlet characters* should read *primitive Dirichlet characters*). Thus an arbitrary Hecke character is uniquely of the form $\chi_0 \otimes |\cdot|^s$, where χ_0 has finite order. The local components $\chi_p : \mathbf{Q}_p^* \longrightarrow \mathbf{C}^*$ ($p \leq \infty$) are given by

$$\chi_p(a) = \chi(1, \ldots, 1, \overset{p^{\text{th}}}{a}, 1, 1, \ldots),$$

so that $\chi = \prod_p \chi_p$.

For a finite order Hecke character χ, we let

$$\mathfrak{c}_\chi \in \mathbf{Z}^+$$

denote the conductor of χ. This is the smallest positive integer which has the property that $\chi(a) = 1$ for all $a \in (1 + \mathfrak{c}_\chi \widehat{\mathbf{Z}}) \cap \widehat{\mathbf{Z}}^*$. For any $N \in \mathfrak{c}_\chi \mathbf{Z}^+$ we can attach to χ a Dirichlet character $\chi' = \chi'_N$ of modulus N and conductor $\mathfrak{c}_\chi$, via

$$\chi : \mathbf{Q}^*(\mathbf{R}^+ \times \widehat{\mathbf{Z}}^*) \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*, \tag{2.8}$$

where the last arrow defines χ'. The case $N = \mathfrak{c}_\chi$ defines a bijection between the set of finite order Hecke characters of conductor N and the set of primitive Dirichlet characters modulo N. For any integer d prime to N, we have

$$\chi'(d) = \prod_{p|N} \chi_p(d) = \chi(d_N) \qquad (d, N) = 1, \tag{2.9}$$

with d_N as in (2.2).

Lemma 2.1 (Dirichlet vs. Hecke L-functions)**.** *In the above situation,*

$$L(s, \overline{\chi'}) = L_N(s, \chi), \tag{2.10}$$

where the partial L-function on the right is defined by the Euler product

$$L_N(s, \chi) = \prod_{p \nmid N} (1 - \chi_p(p) p^{-s})^{-1}. \tag{2.11}$$

Remark: If $N = \mathfrak{c}_\chi$, i.e. χ' is primitive, then $L_N(s, \chi) = L(s, \chi)$ by definition.

Proof. It is easy to show that $1 = \chi(p) = \chi_p(p)\chi'(p)$ for any $p \nmid N$ ([KL2], (12.7)). Therefore

$$L(s, \overline{\chi'}) = \sum_{n>0} \overline{\chi'(n)} n^{-s} = \prod_{p \nmid N} (1 - \overline{\chi'(p)} p^{-s})^{-1} = L_N(s, \chi). \qquad \square$$

We will need lower bounds for Dirichlet L-functions on the right edge of the critical strip, since such L-values arise in the denominators of the Fourier coefficients of Eisenstein series.

Theorem 2.2. *Let χ be a non-principal Dirichlet character modulo N. Write $s = \sigma + it$. There exists a constant $c > 0$ for which the following statements hold.*

1. *If χ is non-real, then for* $1 - \dfrac{c}{(\log(N(\lfloor |t| \rfloor + 2)))^9} < \sigma \le 2$,

$$L(s, \chi)^{-1} \ll \Big(\log\big(N(\lfloor |t| \rfloor + 2)\big)\Big)^7$$

   for an absolute implied constant.

2. *If χ is real, then in the region* $1 - \dfrac{c}{(\log(N(\lfloor |t| \rfloor + 2)))^9} < \sigma \le 2$, $|t| \ge 1$,

$$L(s, \chi)^{-1} \ll \Big(\log\big(N(\lfloor |t| \rfloor + 2)\big)\Big)^7$$

   for an absolute implied constant.

3. *If χ is real and $\varepsilon > 0$ is given such that $N^\varepsilon \ge \log N$, then in the region $1 - \frac{c}{N^{9\varepsilon}} < \sigma \le 2$, $\frac{1}{10N^\varepsilon} \le |t| \le 1$, we have*

$$L(s, \chi)^{-1} \ll N^{7\varepsilon}$$

   for an absolute implied constant.

4. *If χ is real and $\varepsilon > 0$ is given, then when N is sufficiently large (depending on ε), for $|s - 1| \le \frac{1}{N^{\varepsilon/2}}$ we have*

$$L(s, \chi)^{-1} \ll_\varepsilon N^{\varepsilon/2}$$

   for an ineffective implied constant depending on ε.

Proof. See equations (3), (4) and (5) on page 218 of Ramachandra's book [Ra]. The fourth case requires Siegel's Theorem, which is why the constant in that case is not effective. □

Corollary 2.3. *Fix $\varepsilon > 0$. For all Dirichlet characters χ of modulus N,*

$$L(1+it,\chi)^{-1} \ll_\varepsilon N^\varepsilon (\log(|t|+3))^7 \tag{2.12}$$

for an ineffective implied constant depending only on ε.

Proof. Note that since $\log 3 > 1$,

$$\log(N(\lfloor |t| \rfloor + 2)) \le \log N + \log(|t|+3) = \log(|t|+3)\left(\frac{\log N}{\log(|t|+3)} + 1\right)$$

$$\le \log(|t|+3)(\log N + 1) \ll_\varepsilon \log(|t|+3) N^{\varepsilon/7}.$$

Therefore by parts 1 and 2 of the theorem, (2.12) holds if χ is non-real, or if χ is a non-principal real character and $|t| \ge 1$.

Suppose χ is real and non-principal, and let $\varepsilon' = \varepsilon/7$. We need to establish (2.12) for $|t| \le 1$. Because $\frac{1}{10N^{\varepsilon'}} \le \frac{1}{N^{\varepsilon'/2}}$, we see that either $\frac{1}{10N^{\varepsilon'}} \le |t| \le 1$ or $|t| \le \frac{1}{N^{\varepsilon'/2}}$ must hold. Therefore as long as N is sufficiently large ($N \ge C(\varepsilon)$),

$$L(1+it,\chi)^{-1} \ll N^{7\varepsilon'} \ll N^\varepsilon (\log(3+|t|))^7,$$

as needed. We still have to treat the case $N < C(\varepsilon)$, $|t| \le 1$. We know that $L(1+it,\chi)^{-1}$ is continuous in t, and hence bounded on $|t| \le 1$. There are only finitely many characters χ with modulus $< C(\varepsilon)$, so their L-functions can be bounded uniformly on $|t| \le 1$. Thus $L(1+it,\chi) \ll 1$ on $|t| \le 1$ when $N < C(\varepsilon)$.

Lastly, suppose χ is the principal character modulo N. Recall the well-known estimate $\zeta(1+it)^{-1} \ll (\log(3+|t|))^7$ ([In], Theorem 10, p.28). Then

$$L(1+it,\chi)^{-1} = \zeta(1+it)^{-1} \prod_{p|N} (1-p^{-(1+it)})^{-1} \tag{2.13}$$

$$\ll (\log(3+|t|))^7 \prod_{p|N} (1-\frac{1}{p})^{-1} \ll (\log(3+|t|))^7 \prod_{p|N} 2$$

$$\ll_\varepsilon (\log(3+|t|))^7 N^\varepsilon.$$

□

3 Bi-K_∞-invariant functions on $\mathrm{GL}_2(\mathbf{R})$

Our objective is to study the cusp forms of weight 0, realized as certain right K_∞-invariant L^2-functions on $G(\mathbf{R}) \times G(\mathbf{A}_{\mathrm{fin}})$. In order to isolate the K_∞-invariant subspace of L^2, we will use an operator $R(f_\infty \times f_{\mathrm{fin}})$, where f_∞ is a bi-K_∞-invariant function on $G(\mathbf{R})$. In this section we review the properties of such functions which will be useful in what follows.

3.1 Several guises

Let m be a fixed nonnegative integer or ∞. Define $C_c^m(G^+//K)$ to be the space of m-times continuously differentiable functions f on

$$G(\mathbf{R})^+ = \{g \in G(\mathbf{R}) |\ \det(g) > 0\},$$

whose support is compact modulo $Z(\mathbf{R})$, and which satisfy

$$f(zkgk') = f(g) \tag{3.1}$$

for all $z \in Z(\mathbf{R})$ and $k, k' \in K_\infty$. In later sections, we will view these as functions on $G(\mathbf{R})$ by setting $f(g) = 0$ if $\det(g) < 0$. When $m = 0$, we sometimes denote the space by $C_c(G^+//K)$.

In terms of the Cartan decomposition

$$G(\mathbf{R})^+ = Z(\mathbf{R}) K_\infty \left\{ \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \right\} K_\infty, \tag{3.2}$$

an element $f \in C_c^m(G^+//K)$ depends only on the parameter y. As a function of y, it is invariant under $y \mapsto y^{-1}$, since $f(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) g \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)) = f(g)$. Thus we have the following isomorphism

$$C_c^m(G^+//K) \longrightarrow C_c^m(\mathbf{R}^+)^w,$$

where $C_c^m(\mathbf{R}^+)^w$ is the space of smooth compactly supported functions on $\mathbf{R}^+$ (the set of positive real numbers) that are invariant under $y \mapsto y^{-1}$. The value of such a function depends only on the unordered pair $\{y, y^{-1}\}$. The set of such pairs is in 1-1 correspondence with the real interval $[0, \infty)$ via $\{y, y^{-1}\} \leftrightarrow y + y^{-1} - 2$.

Proposition 3.1. *Suppose $m \geq 0$ and $0 \leq 3m' \leq m + 1$. Then for $y \in \mathbf{R}^+$, the substitution*

$$u = y + y^{-1} - 2 \tag{3.3}$$

defines a $\mathbf{C}$-linear injection $C_c^m(\mathbf{R}^+)^w \longrightarrow C_c^{m'}([0, \infty))$ whose image contains $C_c^m([0, \infty))$. In particular, this map is an isomorphism in the two cases $m = m' = 0$ and $m = m' = \infty$.

Proof. We first consider the case of smooth functions. Let $a(y) \in C_c^\infty(\mathbf{R}^+)^w$, and let $A(u) = a(y)$ be the associated function of $u \in [0, \infty)$. It is easy to see that A is C^∞ on $(0, \infty)$, however the smoothness at the endpoint $u = 0$ is not obvious because $\frac{dy}{du} = (1 - \frac{1}{y^2})^{-1}$ blows up at $y = 1$. It is helpful to write $y = e^x$, and define $h(x) = a(e^x) = A(u)$. Then:

- $h(-x) = h(x)$ is even
- $h \in C_c^\infty(\mathbf{R})$
- $u = e^x + e^{-x} - 2$, so $\frac{du}{dx} = e^x - e^{-x} = 2\sinh(x)$.

For $u > 0$, write

$$A^{(n)}(u) = \frac{p_n(x)}{2^n(\sinh x)^{2n-1}}.$$

When $n = 1$ this holds with $p_1(x) = h'(x)$, and by differentiating, we find in general that

$$p_{n+1}(x) = \sinh(x)p_n'(x) - (2n-1)\cosh(x)p_n(x). \tag{3.4}$$

We claim that $p_{n+1}(x)$ vanishes at least to order $2n+1$ at 0. We prove the claim by induction, the base case being $p_1(0) = h'(0) = 0$, h' being odd since h is even. It is clear that $p_{n+1}(x)$ vanishes to at least order $2n-1$ at 0, since this is true of both terms on the right-hand side of (3.4) by the inductive hypothesis. It remains to show that

$$p_{n+1}^{(2n-1)}(0) = p_{n+1}^{(2n)}(0) = 0.$$

By differentiating (3.4), we see that for $0 \leq j \leq 2n-1$,

$$p_{n+1}^{(j)}(x) = \sinh(x)p_n^{(j+1)}(x) - (2n-1-j)\cosh(x)p_n^{(j)}(x) + L_j(x),$$

where $L_0(x) = 0$ and $L_j(x)$, $j > 0$, is a linear combination of derivatives of p_n of order $< j$. In particular, $L_j(0) = 0$ for all $j \leq 2n-1$. Taking $j = 2n-1$ gives

$$p_{n+1}^{(2n-1)}(0) = \sinh(0)p_n^{(2n)}(0) + L_{2n-1}(0) = 0.$$

Furthermore, it follows inductively from (3.4) that $p_{n+1}(x)$ is an odd function, so that the even order derivative $p_{n+1}^{(2n)}(0)$ vanishes. This proves the claim.

Now we can prove by induction that $A^{(n)}(u)$ is defined and continuous at $u = 0$. This is clear when $n = 0$. Assuming it holds for some given $n \geq 0$, we note that by L'Hospital's rule,

$$A^{(n+1)}(0) = \lim_{u\to 0^+} \frac{A^{(n)}(u) - A^{(n)}(0)}{u} = \lim_{u\to 0^+} A^{(n+1)}(u) = \lim_{x\to 0} \frac{p_{n+1}(x)}{2^{n+1}(\sinh x)^{2n+1}},$$

provided the limit exists. The denominator vanishes exactly to order $2n+1$ at 0, and we just showed that $p_{n+1}(x)$ vanishes at least to order $2n+1$ at 0. Therefore we may apply L'Hospital's rule $2n+1$ times, at which point we get a nonvanishing denominator at 0, giving a finite limit as needed.

Now suppose $a(y)$ (hence $h(x)$) is only assumed to be m-times continuously differentiable. In order for the above argument to run with $m' = n+1$, we need $p_{n+1}(x)$ (hence $h^{(n+1)}(x)$) to be $(2n+1)$-times continuously differentiable, i.e. $m \geq 3n+2 = 3m'-1$. It is clear that the resulting map $C_c^m(\mathbf{R}^+)^w \to C_c^{m'}([0,\infty))$ is injective. On the other hand, given $A \in C_c^m([0,\infty))$ and defining $a(y) = A(y + y^{-1} - 2) = A(u)$, it follows from the fact that $\frac{du}{dy} = (1 - y^{-2})$ is smooth on $\mathbf{R}^+$ that $a \in C_c^m(\mathbf{R}^+)^w$, so A is in the image of the map. $\square$

We define, for $f \in C_c^m(G^+//K)$,

$$V(u) = V(y + y^{-1} - 2) = f(\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}). \tag{3.5}$$

By the preceding discussion, we have the following.

Proposition 3.2. *Suppose $0 \le 3m' \le m+1$. Then the assignment $f \mapsto V$ defines an injection $C_c^m(G^+//K) \to C_c^{m'}([0,\infty))$ whose image contains $C_c^m([0,\infty))$. In particular, it is an isomorphism if $m = m' = 0$ or $m = m' = \infty$.*

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbf{R})^+$, we can recover the parameter $y + y^{-1}$ as follows. Writing $g = (\sqrt{\det g})k_{\theta'}\left(\begin{smallmatrix} y^{1/2} & \\ & y^{-1/2} \end{smallmatrix}\right)k_\theta$, we see that ${}^t g\, g = (\det g)k_\theta^{-1}\left(\begin{smallmatrix} y & \\ & y^{-1} \end{smallmatrix}\right)k_\theta$, where ${}^t g$ denotes the transpose. Therefore

$$y + y^{-1} = \frac{\mathrm{tr}({}^t g\, g)}{\det g} = \frac{a^2+b^2+c^2+d^2}{ad-bc}. \tag{3.6}$$

Thus we can recover f from V via:

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = V(\frac{a^2+b^2+c^2+d^2}{ad-bc} - 2). \tag{3.7}$$

We can also identify f with a function of two variables on the complex upper half-plane $\mathbf{H}$. Recall the correspondence

$$G(\mathbf{R})^+/Z(\mathbf{R})K_\infty \longleftrightarrow \mathbf{H}$$

induced by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(i) = \frac{ai+b}{ci+d}$. By this, the following function is well-defined:

$$k(z_1, z_2) = f(g_1^{-1}g_2) \qquad (z_1, z_2 \in \mathbf{H}), \tag{3.8}$$

where $g_j(i) = z_j$ for $j = 1, 2$. Clearly

$$k(\gamma z_1, \gamma z_2) = k(z_1, z_2)$$

for all $\gamma \in G(\mathbf{R})^+$, and in particular for any real scalar $c > 0$,

$$k(cz_1, cz_2) = k(z_1, z_2). \tag{3.9}$$

Proposition 3.3. *With notation as above, for $g_1, g_2 \in G(\mathbf{R})^+$ we have*

$$k(z_1, z_2) = V\left(\frac{|z_1 - z_2|^2}{y_1 y_2}\right) = f(g_1^{-1}g_2).$$

Proof. Writing $g_1^{-1}g_2 = \left(\begin{smallmatrix} y_1 & x_1 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} y_2 & x_2 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{y_2}{y_1} & \frac{(x_2-x_1)}{y_1} \\ 0 & 1 \end{smallmatrix}\right)$, by (3.6) we have

$$u = \tfrac{a^2+b^2+c^2+d^2}{ad-bc} - 2 = \frac{y_2}{y_1} + \frac{(x_2-x_1)^2}{y_1y_2} + \frac{y_1}{y_2} - \frac{2y_1y_2}{y_1y_2} = \frac{(x_2-x_1)^2 + (y_2-y_1)^2}{y_1y_2}.$$

□

3.2 The Harish-Chandra transform

Given $f \in C_c^m(G^+//K)$, its Harish-Chandra transform is the function of $y \in \mathbf{R}^+$ defined by

$$(\mathcal{H}f)(y) = y^{-1/2} \int_{\mathbf{R}} f(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix})dx.$$

The absolute convergence follows easily from the compactness of the support of f. It is clear that $\mathcal{H}f$ is also compactly supported.

If we identify f with V, and let $u = y + y^{-1} - 2$ as before, the transform is traditionally denoted

$$Q(u) = \int_{\mathbf{R}} V(u + x^2)dx,$$

following Selberg. To see the equivalence, by (3.6) we have

$$\begin{aligned}(\mathcal{H}f)(y) = y^{-1/2} \int_{\mathbf{R}} f(\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix})dx &= \int_{\mathbf{R}} f(\begin{pmatrix} y^{1/2} & x \\ & y^{-1/2} \end{pmatrix})dx \\ = \int_{\mathbf{R}} V(y + y^{-1} + x^2 - 2)dx &= \int_{\mathbf{R}} V(u + x^2)dx. \end{aligned} \tag{3.10}$$

From this, we see that $\mathcal{H}f$ belongs to the space $C_c^m(\mathbf{R}^+)^w$.

Proposition 3.4. *Suppose $m, m' \geq 0$ with $3m' \leq m + 1$. Then the Harish-Chandra transform defines a commutative diagram*

$$\begin{array}{ccc} C_c^m(G^+//K) & \xrightarrow{f \mapsto \mathcal{H}f} & C_c^m(\mathbf{R}^+)^w \\ \downarrow & & \downarrow \\ C_c^{m'}([0,\infty)) & \xrightarrow{V \mapsto Q} & C_c^{m'}([0,\infty)), \end{array}$$

where all arrows are injective. The image of the bottom map $V \mapsto Q$ contains $C_c^m([0,\infty))$. When $m = m' = \infty$, all arrows are isomorphisms. Generally, if $m' > 0$ then for $u = y + y^{-1} - 2$ and

$$\mathcal{H}f(y) = Q(u) = \int_{\mathbf{R}} V(u + x^2)dx,$$

the inverse transform is given by

$$V(u) = -\frac{1}{\pi} \int_{\mathbf{R}} Q'(u + w^2)dw. \tag{3.11}$$

Remarks: For the smooth case, see also [Lang], §V.3, Theorem 3, p. 71. For more detail about the inverse transformation, see Propositions 8.16 and 8.17 below. For example, we will show that the image of the bottom map contains $C_c^{m'+1}([0,\infty))$. In fact, given Q in this space, if we *define* V by (3.11), then $V \in C_c^{m'}([0,\infty))$ and $Q(u) = \int_{\mathbf{R}} V(u + x^2)dx$.

Proof. The commutativity of the diagram follows from (3.10). The vertical maps are injective by Propositions 3.1 and 3.2 above. As described there, the image of the right-hand vertical map $\mathcal{H}f \mapsto Q$ contains $C_c^m([0,\infty))$, so by commutativity of the diagram, the same holds for the image of the bottom map $V \mapsto Q$. It follows that when $m' = \infty$, all arrows are surjective. The injectivity of the horizontal arrows is a consequence of the inversion formula (3.11), so it just remains to prove the latter. We can differentiate $Q(u)$ under the integral sign because $V \in C_c^{m'}([0,\infty))$ for $m' \geq 1$ (cf. Proposition 8.3). Thus

$$Q'(u) = \int_{\mathbf{R}} V'(u+x^2)dx.$$

Hence

$$-\frac{1}{\pi}\int_{\mathbf{R}} Q'(u+w^2)dw = -\frac{1}{\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} V'(u+w^2+x^2)dxdw$$

$$= -\frac{1}{\pi}\int_0^{2\pi}\int_0^\infty V'(u+r^2)rdrd\theta = -2\int_0^\infty V'(u+t)\frac{dt}{2} = V(u). \qquad \square$$

3.3 The Mellin transform

For $\Phi \in C_c^m(\mathbf{R}^+)$, its Mellin transform is the function of $\mathbf{C}$ defined by

$$(\mathcal{M}\Phi)(s) = \int_0^\infty \Phi(y)y^s\frac{dy}{y}. \tag{3.12}$$

This is a Fourier transform on the multiplicative group $\mathbf{R}^+$. We also denote the above by $\mathcal{M}_s\Phi$. It is easily shown to be an entire function of s. When $m \geq 2$, we have

$$\Phi(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma}(\mathcal{M}\Phi)(s)y^{-s}ds \tag{3.13}$$

for any $\sigma \in \mathbf{R}$. This is the Mellin inversion formula, which we will prove under somewhat more general hypotheses in Propositions 8.10 and 8.11.

We say that an entire function $\eta : \mathbf{C} \longrightarrow \mathbf{C}$ is **Paley-Wiener of order** m if there exists a real number $C \geq 1$ depending only on η such that

$$|\eta(\sigma+it)| \ll_{m,\eta} \frac{C^{|\sigma|}}{(1+|t|)^m}. \tag{3.14}$$

We let $PW^m(\mathbf{C})$ denote the space of such functions. If the above holds for all $m > 0$ with the same C, then η belongs to the **Paley-Wiener space** $PW^\infty(\mathbf{C}) = PW(\mathbf{C})$.

Proposition 3.5. *Suppose $m \geq 0$. Then the Mellin transform defines an injection*

$$\mathcal{M} : C_c^m(\mathbf{R}^+) \longrightarrow PW^m(\mathbf{C})$$

whose image contains $PW^{m+2}(\mathbf{C})$. On $PW^{m+2}(\mathbf{C})$, the inverse map is given by (3.13). In particular, if $m = \infty$ the transform is an isomorphism.

Proof. (See also [Lang], §V.3, Theorem 4, p. 76.) First we show that the image of the Mellin transform lies in $PW^m(\mathbf{C})$. When $m = 0$ this is obvious. Given $\Phi \in C_c^m(\mathbf{R}^+)$ for $m > 0$, we may apply integration by parts to (3.12) to get

$$\mathcal{M}\Phi(s) = \Phi(y)\frac{y^s}{s}\Big|_0^\infty - \frac{1}{s}\int_0^\infty \Phi'(y)y^{s+1}\frac{dy}{y}.$$

Since Φ is compactly supported in $\mathbf{R}^+$, the first term on the right vanishes. Continuing, we find

$$\mathcal{M}\Phi(s) = \frac{(-1)^m}{s(s+1)\cdots(s+m-1)}\int_0^\infty \Phi^{(m)}(y)y^{s+m}\frac{dy}{y}.$$

Using this, it is straightforward to see that $\mathcal{M}\Phi$ satisfies (3.14). The injectivity of the map is immediate from the inversion formula (3.13).

For $\eta \in PW^2(\mathbf{C})$, we can define $\Phi(y) = \frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma} \eta(s)y^{-s}ds$ as in (3.13), the convergence being absolute for any σ by (3.14) with $m = 2$. To see that Φ is compactly supported, consider

$$\Phi(y) = \frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma} \eta(s)y^{-s}ds \ll \int_{-\infty}^\infty \frac{C^\sigma |y|^{-\sigma}}{(1+|t|)^2}dt \ll (C|y|^{-1})^\sigma.$$

If $|y| > |C|$, then the right-hand side approaches 0 as $\sigma \to \infty$, so $\Phi(y) = 0$. Thus $\operatorname{Supp}\phi \subseteq [-C, C]$.

Lastly, if $\eta \in PW^{m+2}(\mathbf{C})$ and Φ is defined as above, then $\eta = \mathcal{M}\Phi$ (cf. Proposition 8.11), and it is not hard to show that $\Phi \in C_c^m(\mathbf{R}^+)$. The idea is that after differentiating under the integral sign m times, we still have an integrand with sufficient (quadratic) decay in $t = \operatorname{Im} s$ for convergence. See Proposition 8.13 below for details. □

3.4 The Selberg transform

If we restrict the Mellin transform to the space $C_c^m(\mathbf{R}^+)^w$ defined on p. 15, it gives an injection to the space $PW^m(\mathbf{C})^{\text{even}}$ of *even* functions that are Paley-Wiener of order m. The composition of the Harish-Chandra and Mellin transforms is called the **spherical transform**, which we denote by

$$(\mathcal{S}f)(s) = \mathcal{M}_s(\mathcal{H}f).$$

Because $\mathcal{M}_s$ and $\mathcal{H}$ are injective, we immediately see the following.

Proposition 3.6. *For $m \geq 0$, the spherical transform*

$$\mathcal{S} : C_c^m(G^+//K) \xrightarrow{f \mapsto \mathcal{S}f} PW^m(\mathbf{C})^{\text{even}}$$

is injective, and its image contains $PW^{m+2}(\mathbf{C})^{even}$. In particular, when $m = \infty$ it is an isomorphism.

The **Selberg transform** of $f \in C_c^m(G^+//K)$ is a variant of the above, defined by

$$h(t) = (\mathcal{S}f)(it) = \mathcal{M}_{it}\mathcal{H}f. \tag{3.15}$$

It is given explicitly by

$$h(t) = \int_0^\infty \int_{-\infty}^\infty f(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y^{1/2} & \\ & y^{-1/2} \end{smallmatrix}\right))y^{\frac{1}{2}+it}\frac{dx\,dy}{y^2} = \iint_{\mathbf{H}} k(i,z)y^{\frac{1}{2}+it}dz, \tag{3.16}$$

where $dz = \frac{dx\,dy}{y^2}$ is the $G(\mathbf{R})^+$-invariant measure on $\mathbf{H}$. Note that $s \mapsto h(-is)$ is Paley-Wiener of order m.

Proposition 3.7. *Suppose $m > 2$ and $h(-is) = (\mathcal{S}f)(s) \in PW^m(\mathbf{C})^{\mathrm{even}}$. Then the inverse of the Selberg transform is given by*

$$V(u) = \frac{1}{4\pi}\int_{-\infty}^\infty P_{-\frac{1}{2}+it}(1+\tfrac{u}{2})h(t)\tanh(\pi t)\,t\,dt,$$

for the Legendre function $P_s(z) = P_s^0(z)$. In particular, we have

$$f(1) = V(0) = \frac{1}{4\pi}\int_{-\infty}^\infty h(t)\tanh(\pi t)\,t\,dt. \tag{3.17}$$

Proof. (See also (2.24) of [Za] or (1.64′) of [Iw2].) Beginning with the fact that $\mathcal{M}_s(\mathcal{H}f) = h(-is)$, we apply Mellin inversion (3.13) to get, for $y > 0$,

$$(\mathcal{H}f)(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=0} h(-is)y^{-s}ds = \frac{1}{2\pi}\int_{\mathbf{R}} h(r)y^{-ir}dr = \frac{1}{2\pi}\int_{\mathbf{R}} h(r)y^{ir}dr,$$

since h is even. Write $y = e^v$ and $u = y + y^{-1} - 2 = e^v + e^{-v} - 2$, and define

$$g(v) = Q(u) = (\mathcal{H}f)(y).$$

Then $g(v) = \frac{1}{2\pi}\int_{\mathbf{R}} h(r)e^{irv}dr$, and differentiating (cf. Proposition 8.3),

$$g'(v) = \frac{i}{2\pi}\int_{\mathbf{R}} rh(r)e^{irv}dr = -\frac{1}{2\pi}\int_{\mathbf{R}} \sin(rv)r\,h(r)dr$$

since h is even. We have used the fact that $m > 2$, so in particular the above is absolutely convergent. Now we invert the Harish-Chandra transform via (3.11) to get

$$\begin{aligned} V(w) &= -\frac{1}{\pi}\int_{\mathbf{R}} Q'(w+x^2)dx = -\frac{2}{\pi}\int_0^\infty Q'(w+x^2)dx \\ &= -\frac{1}{\pi}\int_w^\infty \frac{Q'(u)du}{\sqrt{u-w}} = -\frac{1}{\pi}\int_{\cosh^{-1}(1+\frac{w}{2})}^\infty \frac{g'(v)dv}{\sqrt{e^v+e^{-v}-2-w}} \\ &= \frac{1}{2\pi^2}\int_{-\infty}^\infty \int_{\cosh^{-1}(1+\frac{w}{2})}^\infty \frac{\sin(rv)}{\sqrt{e^v+e^{-v}-2-w}}dv\,rh(r)dr. \end{aligned}$$

The interchange of the integrals is justified by the absolute convergence of the integral, which follows easily by the fact that $m > 2$. As observed by Zagier ([Za] (2.24)), using the identities 8.715.2 and 8.736.7 of [GR], it is straightforward to show that the above is

$$= \frac{1}{4\pi} \int_{\mathbf{R}} P_{-\frac{1}{2}+ir}(1 + \frac{w}{2}) \tanh(\pi r) r \, h(r) dr. \qquad \square$$

It is sometimes desirable to extend all of these transforms to functions with sufficient decay rather than just the case of compact support. We will discuss this in detail in Section 8, but we mention here that the following conditions are equivalent:

- $V(u) = O(u^{-\frac{1+A}{2}})$ as $u \to \infty$
- $Q(u) = O(u^{-A/2})$ as $u \to \infty$
- $h(t)$ is holomorphic in the horizontal strip $|\operatorname{Im}(t)| < A/2$.

See [Za], p. 320.

3.5 The principal series of $G(\mathbf{R})$

Here we recall the construction of the principal series of $G(\mathbf{R})$ and prove some well-known simple properties. Detailed background is given, e.g., in §11 of [KL2]. For $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ and $s_1, s_2 \in \mathbf{C}$, define a character $\chi = \chi(\varepsilon_1, \varepsilon_2, s_1, s_2)$ of $B(\mathbf{R})$ by

$$\chi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) = \operatorname{sgn}(a)^{\varepsilon_1} |a|^{s_1} \operatorname{sgn}(d)^{\varepsilon_2} |d|^{s_2}.$$

Every character of $B(\mathbf{R})$ has this form. We let $\pi_\chi = \pi(\varepsilon_1, \varepsilon_2, s_1, s_2)$ denote the representation of $G(\mathbf{R})$ unitarily induced from χ. The underlying representation space V_χ consists of measurable functions on $G(\mathbf{R})$ satisfying

$$\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \chi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) \left| \frac{a}{d} \right|^{1/2} \phi(g),$$

with inner product given by

$$\langle \phi_1, \phi_2 \rangle = \int_{K_\infty} \phi_1(k) \overline{\phi_2(k)} dk.$$

The action of $G(\mathbf{R})$ is given by right translation

$$\pi_\chi(g)\phi(x) = \phi(xg).$$

The representation π_χ is unitary when χ is unitary, i.e. when $s_1, s_2 \in i\mathbf{R}$. See §11.3 of [KL2] for details.

We say that a vector has **weight** $\mathtt{k}$ if it transforms by the scalar $e^{i\mathtt{k}\theta}$ under the action of $k_\theta \in K_\infty$. A natural basis for V_χ is $\{\phi_{\mathtt{k}} | \, \mathtt{k} \in \varepsilon_1 + \varepsilon_2 + 2\mathbf{Z}\}$, where $\phi_{\mathtt{k}}$ is characterized by

$$\phi_{\mathtt{k}}(k_\theta) = e^{i\mathtt{k}\theta}.$$

This function spans the one-dimensional space of weight $\mathtt{k}$ vectors in V_χ.

We define the **spectral parameter** of π_χ by

$$t = -\frac{i}{2}(s_1 - s_2). \tag{3.18}$$

The representation π_χ is reducible if and only if $t \neq 0$ and $2it + \varepsilon_1 + \varepsilon_2$ is an odd integer. Furthermore, the Casimir element Δ in the center of the universal enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$, whose right regular action on $C^\infty(G(\mathbf{R})^+)$ is given in the coordinates $z\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y^{1/2} & \\ & y^{-1/2} \end{smallmatrix}\right)k_\theta$ by

$$R(\Delta)\phi = -y^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + y\frac{\partial^2\phi}{\partial x \partial \theta}, \tag{3.19}$$

acts on the K_∞-finite vectors of V_χ by the scalar

$$\pi_\chi(\Delta) = \frac{1}{4} + t^2. \tag{3.20}$$

(See e.g. [KL2], pp. 169, 185.)

The only irreducible finite dimensional unitary representations of $G(\mathbf{R})$ are the unitary characters. For the infinite dimensional ones, we have the following.

Proposition 3.8. *Let π be an irreducible infinite dimensional unitary representation of $G(\mathbf{R})$. Then π contains a nonzero vector of weight 0 (resp. weight 1) if and only if $\pi \cong \pi(\varepsilon_1, \varepsilon_2, s_1, s_2)$ is an irreducible principal series representation with $\varepsilon_1 + \varepsilon_2$ even (resp. odd), and either:*

· *$s_1, s_2 \in i\mathbf{R}$ (unitary principal series) or*

· *$t \in i\mathbf{R}$, $0 < |t| < \frac{1}{2}$, and $s_1 + s_2 \in i\mathbf{R}$ (complementary series),*

for t as in (3.18). The vector is unique up to scalar multiples.

Proof. Any irreducible unitary representation π is infinitesimally equivalent with a subrepresentation of a principal series representation of $G(\mathbf{R})$. A proper subrepresentation containing a vector of weight 0 or 1 is necessarily finite dimensional (see e.g. [KL2], p. 164). Therefore $\pi \cong \pi(\varepsilon_1, \varepsilon_2, s_1, s_2)$ is itself a principal series representation. Since π is unitary, one of the two given scenarios must hold. □

Generally, if ϕ is any continuous function on $G(\mathbf{R})$, we extend the right regular action of $G(\mathbf{R})$ to an action of $f \in C_c^m(G^+//K)$ by defining

$$R(f)\phi(g') = \int_{\overline{G}(\mathbf{R})} f(g)R(g)\phi(g')dg = \int_{\overline{G}(\mathbf{R})} f(g)\phi(g'g)dg.$$

If ϕ is right $Z(\mathbf{R})K_\infty$-invariant, it can be viewed as a function on $\mathbf{H}$, and after replacing g by $g'^{-1}g$ in the above, we find

$$R(f)\phi(g') = \int_{\mathbf{H}} k(z',z)\phi(z)\frac{dx\,dy}{y^2}, \tag{3.21}$$

where $k(z',z)$ is the function attached to f in (3.8).

As shown by Selberg [Sel2], if $\phi(z)$ is an eigenfunction of the Laplacian with eigenvalue $\frac{1}{4}+t^2$, then in the sense of (3.21), $R(f)\phi = h(t)\phi$ for the Selberg transform h of f (see also Theorem 1.16 of [Iw2]). We prove this here in the special case of interest to us. We use the setting of weight $\mathtt{k}$ functions as an example of how the results of this section immediately generalize from $\mathtt{k}=0$.

Proposition 3.9. *Let $\pi=\pi_\chi$ be as above. Let f_∞ be a continuous function whose support lies in $G(\mathbf{R})^+$ and is compact modulo $Z(\mathbf{R})$, satisfying*

$$f_\infty(zk_{\theta_1}^{-1}gk_{\theta_2}) = \chi(z)^{-1}e^{-i\mathtt{k}(\theta_2-\theta_1)}f_\infty(g) \qquad (z\in Z(\mathbf{R}),\ k_{\theta_j}\in K_\infty).$$

Then the operator $\pi(f_\infty)$ preserves the one-dimensional subspace $V_{\mathtt{k}}$ of weight $\mathtt{k}$ vectors in V_χ, and vanishes on its orthogonal complement. If this subspace is nonzero (i.e. $\mathtt{k}\equiv\varepsilon_1+\varepsilon_2$ mod 2), then

$$\pi(f_\infty)\phi_{\mathtt{k}} = h(t)\phi_{\mathtt{k}}, \tag{3.22}$$

where t is the spectral parameter (3.18) *of π, and h is the Selberg transform of f_∞, defined in* (3.15).

Proof. We will prove the first claim in a more general context in Lemma 3.10 below, so we grant it for now. Hence if $\phi=\phi_{\mathtt{k}}\in V_\chi$, ϕ is an eigenvector of $\pi(f_\infty)$ since $\dim V_{\mathtt{k}}=1$. The eigenvalue λ is given by

$$\lambda = \pi(f_\infty)\phi(1) = \int_{\overline{G}(\mathbf{R})} f_\infty(g)\phi(g)dg = \int_{\mathrm{SL}_2(\mathbf{R})} f_\infty(g)\phi(g)dg,$$

by our normalization of Haar measure (cf. (7.27) on page 95 of [KL2]). Here we have used the fact that f_∞ is supported on $G(\mathbf{R})^+$. Now since f_∞ and ϕ have opposite weights on the right, the integrand is right K_∞-invariant. Therefore we have

$$\begin{aligned}
\lambda &= \int_0^\infty\int_{-\infty}^\infty f_\infty\left(\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}y^{1/2}&\\&y^{-1/2}\end{smallmatrix}\right)\right)\phi\left(\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}y^{1/2}&\\&y^{-1/2}\end{smallmatrix}\right)\right)\frac{dx\,dy}{y^2}\\
&= \int_0^\infty\int_{-\infty}^\infty f_\infty\left(\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}y^{1/2}&\\&y^{-1/2}\end{smallmatrix}\right)\right)y^{(s_1-s_2)/2}y^{1/2}y^{-2}dx\,dy\\
&= \int_0^\infty\left[y^{-1/2}\int_{-\infty}^\infty f_\infty\left(\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}y^{1/2}&\\&y^{-1/2}\end{smallmatrix}\right)\right)dx\right]y^{it}\frac{dy}{y} = \mathcal{M}_{it}\mathcal{H}(f_\infty) = h(t),
\end{aligned}$$

as required. ☐

Lemma 3.10. *Let G be a locally compact group, let $K \subseteq G$ be a closed subgroup, and let π be a unitary representation of G on a Hilbert space V with central character χ. Then for any bi-K-invariant function $f \in L^1(G, \chi^{-1})$ (i.e. integrable mod center, with central character χ^{-1}), the operator $\pi(f)$ on V given by*

$$\pi(f)v = \int_{\overline{G}} f(g)\pi(g)v \, dg$$

has its image in the K-fixed subspace V^K, and annihilates the orthogonal complement of this subspace.

Remark: If the bi-K-invariance of f is replaced by the property

$$f(k^{-1}gk') = \tau(k)\tau(k')^{-1}f(g)$$

for a character τ of K, then the above holds with $V_\tau = \{v \in V \,|\, \pi(k)v = \tau(k)v\}$ in place of V^K, as is easily seen by adjusting the proof below.

Proof. See page 140 of [KL2] for a detailed discussion of $\pi(f)$. In particular, the vector $\pi(f)v$ is characterized by the property that

$$\langle \pi(f)v, w\rangle = \int_{\overline{G}} f(g) \langle \pi(g)v, w\rangle \, dg$$

for all $w \in V$. Since π is unitary, for any $k \in K$ we have

$$\langle \pi(k)\pi(f)v, w\rangle = \left\langle \pi(f)v, \pi(k^{-1})w\right\rangle = \int_{\overline{G}} f(g) \left\langle \pi(g)v, \pi(k^{-1})w\right\rangle dg$$

$$= \int_{\overline{G}} f(g) \langle \pi(kg)v, w\rangle \, dg = \int_{\overline{G}} f(k^{-1}g) \langle \pi(g)v, w\rangle \, dg = \langle \pi(f)v, w\rangle$$

by the left K-invariance of f. Thus $\pi(k)\pi(f)v = \pi(f)v \in V^K$ as claimed.

The adjoint of $\pi(f)$ is the operator $\pi(f^*)$, where

$$f^*(g) = \overline{f(g^{-1})} \in L^1(G, \chi^{-1}).$$

The right K-invariance of f means that f^* is left K-invariant, so the operator $\pi(f)^* = \pi(f^*)$ also has its image in V^K. If $w \in (V^K)^\perp$, then for any $v \in V^K$ we have

$$\langle \pi(f)w, v\rangle = \langle w, \pi(f^*)v\rangle = 0.$$

Hence $\pi(f)w \in V^K \cap (V^K)^\perp = \{0\}$, as needed. □

4 Maass cusp forms

Here we review some well-known properties of Maass cusp forms, and spell out their connection with the representation theory of the adele group $\mathrm{GL}_2(\mathbf{A})$.

4.1 Cusp forms of weight 0

More detail on the material below can be found e.g. in Iwaniec [Iw2]. Fix a level $N \in \mathbf{Z}^+$, and let ω' be a Dirichlet character whose conductor divides N and which satisfies

$$\omega'(-1) = 1. \tag{4.1}$$

We view ω' as a character of $\Gamma_0(N)$ via $\omega'(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \omega'(d)$. Note that $\omega'(\gamma) = 1$ if $\gamma \in \Gamma_1(N)$. Let $L^2(N, \omega')$ denote the space of measurable functions $u : \mathbf{H} \longrightarrow \mathbf{C}$ (modulo functions that are 0 a.e.) such that

$$u(\gamma z) = \overline{\omega'(\gamma)} u(z) \tag{4.2}$$

for all $\gamma \in \Gamma_0(N)$, and whose Petersson norm

$$\|u\|^2 = \frac{1}{\psi(N)} \int_{\Gamma_0(N)\backslash\mathbf{H}} |u(x+iy)|^2 \frac{dx\,dy}{y^2} \tag{4.3}$$

is finite. Taking $\gamma = \left(\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right)$ in (4.2) gives $\omega'(-1) = 1$ if $u(z) \neq 0$, which is why we imposed (4.1).

Let $\delta \in G(\mathbf{Q})^+$, and write

$$\delta^{-1}\Gamma_1(N)\delta \cap N(\mathbf{Q}) = \{\left(\begin{smallmatrix} 1 & tM_\delta \\ 0 & 1 \end{smallmatrix}\right) |\, t \in \mathbf{Z}\},$$

where $M_\delta > 0$ (see Lemma 3.7 of [KL2]). If u is any continuous function satisfying (4.2), we set

$$u_\delta(z) = u(\delta(z)).$$

Then $u_\delta(z + M_\delta) = u_\delta(z)$, so for all $y > 0$, it has a Fourier expansion about the cusp $q = \delta(\infty)$ of the form

$$u_\delta(z) = \sum_{m=-\infty}^{\infty} a_{m,\delta}(u, y)\, e(nx/M_\delta).$$

We drop δ and just write $a_m(u, y)$ when $q = \infty$. An element $u \in L^2(N, \omega')$ is **cuspidal** if its constant terms vanish:

$$a_{0,\delta}(u, y) = \frac{1}{M_\delta} \int_0^{M_\delta} u(\delta(x+iy))dx = 0 \tag{4.4}$$

for all $\delta \in G(\mathbf{Q})^+$ and a.e. $y > 0$. The subspace of cuspidal functions is denoted $L^2_0(N, \omega')$.

The hyperbolic Laplacian is defined as an operator on $C^\infty(\mathbf{H})$ by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \tag{4.5}$$

This operator commutes with the action of $G(\mathbf{R})^+$:

$$(\Delta u)(gz) = \Delta(u(gz)).$$

By this invariance, Δ descends to an operator on $C_c^\infty(\Gamma_1(N)\backslash\mathbf{H})$, which is dense in $L^2(N, \omega')$. One can show that relative to the Petersson inner product, this operator is symmetric and positive:

$$\langle \Delta\phi, \psi \rangle = \langle \phi, \Delta\psi \rangle, \tag{4.6}$$

$$\langle \Delta\phi, \phi \rangle = \tfrac{1}{[\Gamma(1):\Gamma_1(N)]} \int_{\Gamma_1(N)\backslash\mathbf{H}} \|\nabla\phi(x+iy)\|^2 dx\, dy \geq 0 \tag{4.7}$$

([Lang], §XIV.4). The operator Δ extends to an elliptic operator on the distribution space $\mathcal{D}'(\Gamma_1(N)\backslash\mathbf{H})$ of continuous linear functionals on $C_c^\infty(\Gamma_1(N)\backslash\mathbf{H})$. See [F1], p. 284. Identifying $\phi \in L^2(N, \omega')$ with the functional $f \mapsto \langle f, \phi \rangle$ realizes $L^2(N, \omega')$ as a subspace of $\mathcal{D}'(\Gamma_1(N)\backslash\mathbf{H})$, although this subspace is not stable under the extended operator Δ.

A **Maass cusp form** of level N and nebentypus ω' is an eigenfunction u of Δ in the subspace $L_0^2(N, \omega')$ ([Ma]). By the elliptic regularity theorem, such an eigenfunction is necessarily smooth, i.e. $u \in C^\infty(\mathbf{H})$ (cf. [F2] p. 214, or [Lang] p. 407). We write $\Delta u = (\frac{1}{4} + t^2)u$ for the Laplace eigenvalue and call t the **spectral parameter** of u. It is also customary to use $s(1-s)$ for the eigenvalue, where the relationship is given by $s = \frac{1}{2} + it$. We will not use this notation, preferring instead to use $s = it$.

Theorem 4.1. *The cuspidal subspace $L_0^2(N, \omega')$ has an orthogonal basis consisting of Maass cusp forms. Each cuspidal eigenspace of Δ is finite dimensional, and the eigenvalues are positive real numbers $\lambda_1 \leq \lambda_2 \leq \cdots$ with no finite limit point.*

Remarks: (1) A famous conjecture of Selberg asserts that $\lambda_1 \geq \frac{1}{4}$, or equivalently, that all of the spectral parameters t are real, [Sar1]. (It is not hard to show that the set of $t \notin \mathbf{R}$ is finite; see Corollary 7.3 on page 72.) Selberg proved that $\lambda_1 \geq \frac{3}{16}$. See §6.2 of [DI], where this is proven as a consequence of the Kuznetsov formula. The best bound to date is $\lambda_1 \geq \frac{1}{4} - (\frac{7}{64})^2 \approx 0.238037...$, due to Kim and Sarnak [KS].

(2) In the case of level $N = 1$, Cartier conjectured that the eigenvalues occur with multiplicity one ([Car]). Until very recently, it was widely believed that the eigenvalues of Δ on the *newforms* of level N should occur with multiplicity one. However, Strömberg has discovered counterexamples on $\Gamma_0(9)$ which, despite coming from newforms, nevertheless arise out of the spectrum of a congruence subgroup of lower level ([St]). Some of his examples were found independently by Farmer and Lemurell.

Proof. (Sketch. See also [Iw2], §4.3 and [IK], §15.5.) The existence of the basis is a consequence of the complete reducibility of $L^2_0(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}),\omega)$ (see Proposition 4.8 below). The discreteness of the set of eigenvalues and the finite dimensionality of the eigenspaces both follow from (7.3) on page 71. The fact that there are infinitely many linearly independent cusp forms can be seen from Weyl's Law (see (7.4)). By (4.7) the eigenvalues of Δ are nonnegative. If $\Delta u = 0$ for $u \in L^2_0(N,\omega')$, then u is a harmonic function on $\Gamma_1(N)\backslash\mathbf{H}$. By the maximum principle ([F2], p. 72), the supremum of $|u(z)|$ occurs on the boundary, i.e. at a cusp, where u vanishes. Hence $u = 0$. This shows that the λ_j are strictly positive. □

If u is a Maass cusp form with Δ-eigenvalue $\frac{1}{4}+t^2$, its Fourier expansion at ∞ has the well-known form

$$u(x+iy) = \sum_{m\in\mathbf{Z}-\{0\}} a_m(u)\, y^{1/2} K_{it}(2\pi|m|y) e^{2\pi i m x} \tag{4.8}$$

for constants $a_m(u)$ called the **Fourier coefficients** of u (see e.g. [Bu], §1.9). The K-Bessel function can be defined by

$$K_s(z) = \frac{1}{2}\int_0^\infty e^{-z(w+w^{-1})/2}\, w^s\, \frac{dw}{w}, \tag{4.9}$$

for $s \in \mathbf{C}$ and $\mathrm{Re}(z) > 0$.

4.2 Hecke operators

For $u \in L^2(N,\omega')$ and an integer $\mathtt{n} > 0$, the Hecke operator $T_{\mathtt{n}}$ is given by

$$T_{\mathtt{n}}u(z) = \mathtt{n}^{-1/2} \sum_{\substack{ad=\mathtt{n}\\ d>0}} \sum_{r=0}^{d-1} \overline{\omega'(a)}\, u(\frac{az+r}{d}).$$

One shows in the usual way that $T_{\mathtt{n}}u \in L^2(N,\omega')$.

We also define $T_{-1}u(x+iy) = u(-x+iy)$. A Maass cusp form is *even* (resp. *odd*) if $T_{-1}u = u$ (resp. $T_{-1}u = -u$). If u is even, then in (4.8) we have $a_{-n} = a_n$, while if u is odd, $a_n = -a_{-n}$.

Proposition 4.2. *The Hecke operators for* $\gcd(\mathtt{n}, N) = 1$ *are normal operators on* $L^2(N,\omega')$. *They commute with each other and with* Δ. *Hence the family of operators* $\{\Delta, T_{-1}, T_{\mathtt{n}} |\ \gcd(\mathtt{n},N) = 1\}$ *is simultaneously diagonalizable on* $L^2_0(N,\omega')$.

Proof. We can compute the adjoint of $T_{\mathtt{n}}$ as in §3.9 of [KL2]. The proof of diagonalizability in the holomorphic case relies crucially on the finite dimensionality of $S_{\mathtt{k}}(N,\omega')$. In order to get the diagonalizability of $T_{\mathtt{n}}$ on $L^2_0(N,\omega')$ we can use the fact that each $T_{\mathtt{n}}$ preserves the Δ-eigenspace $L^2_0(N,\omega',\frac{1}{4}+t^2)$, which is finite-dimensional. These subspaces exhaust the cuspidal spectrum by Theorem 4.1. See also Proposition 4.8 below. □

A **Maass eigenform** is a cusp form u which is a simultaneous eigenvector of the operators $T_{\mathtt{n}}$ for $\mathtt{n} \geq 1$, $(\mathtt{n}, N) = 1$. We write $T_{\mathtt{n}} u = \lambda_{\mathtt{n}}(u) u$ for the Hecke eigenvalue. In this situation,

$$a_{\mathtt{n}}(u) = a_1(u) \lambda_{\mathtt{n}}(u)$$

whenever $\gcd(N, \mathtt{n}) = 1$. This is a consequence of the fact that for any cusp form u,

$$a_m(T_{\mathtt{n}} u) = \sum_{\ell \mid \gcd(\mathtt{n}, m)} \overline{\omega'(\ell)}\, a_{\frac{\mathtt{n}m}{\ell^2}}(u), \tag{4.10}$$

which is proven in the same way as for holomorphic cusp forms.

We now define a function which serves as the adelic counterpart to $T_{\mathtt{n}}$ (see Lemma 4.6 below). Fix integers $N, \mathtt{n} \in \mathbf{Z}^+$ with $\gcd(\mathtt{n}, N) = 1$, and let ω be a Hecke character of conductor dividing N. Define $f^{\mathtt{n}} : G(\mathbf{A}_{\text{fin}}) \to \mathbf{C}$ as follows. Let

$$M_1(\mathtt{n}, N) = \{ g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\widehat{\mathbf{Z}}) \mid \det g \in \mathtt{n} \widehat{\mathbf{Z}}^* \text{ and } c, (d-1) \in N \widehat{\mathbf{Z}} \}.$$

Let $M_1(\mathtt{n}, N)_p$ be the local component of this set in $G(\mathbf{Q}_p)$. Note that if $p \nmid \mathtt{n}$, then $M_1(\mathtt{n}, N)_p = K_1(N)_p = K_1(N) \cap K_p$. The function $f^{\mathtt{n}}$ is supported on $Z(\mathbf{A}_{\text{fin}}) M_1(\mathtt{n}, N)$ and given by

$$f^{\mathtt{n}}(zm) = \frac{\overline{\omega(z)}}{\text{meas}(\overline{K_1(N)})} = \frac{\psi(N)}{\omega(z)}. \tag{4.11}$$

It is clear that $f^{\mathtt{n}}$ is well-defined and bi-$K_1(N)$-invariant. For any finite prime p, define a local function $f_p^{\mathtt{n}}$ on $G(\mathbf{Q}_p)$, supported on $Z(\mathbf{Q}_p) M_1(\mathtt{n}, N)_p$, by

$$f_p^{\mathtt{n}}(zm) = \frac{\overline{\omega_p(z)}}{\text{meas}(\overline{K_1(N)}_p)}. \tag{4.12}$$

Then $f^{\mathtt{n}}(g) = \prod_p f_p^{\mathtt{n}}(g_p)$.

We now recall the definition of the **unramified principal series** of $G(\mathbf{Q}_p)$. Suppose $p \nmid N$, so ω_p is an unramified unitary character of $\mathbf{Q}_p^*$. For $\nu \in \mathbf{C}$, let

$$\chi(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)) = \chi_1(a) \chi_2(d) \left| \frac{a}{d} \right|_p^{\nu} \tag{4.13}$$

be an unramified quasicharacter of the Borel subgroup $B(\mathbf{Q}_p)$. Here we take χ_1 and χ_2 to be finite order unramified characters of $\mathbf{Q}_p^*$ with

$$\chi_1(z) \chi_2(z) = \omega_p(z).$$

Let V_χ be the space of functions $\phi : G(\mathbf{Q}_p) \longrightarrow \mathbf{C}$ with the following properties:

(i) For all $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B(\mathbf{Q}_p)$ and all $g \in G(\mathbf{Q}_p)$,

$$\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \chi_1(a) \chi_2(d) \left| \frac{a}{d} \right|_p^{\nu + 1/2} \phi(g).$$

(ii) There exists an open compact subgroup $J \subseteq G(\mathbf{Q}_p)$ such that $\phi(gk) = \phi(g)$ for all $k \in J$ and all $g \in G(\mathbf{Q}_p)$.

We let π_χ denote the representation of $G(\mathbf{Q}_p)$ on V_χ by right translation. It is unitary when χ is unitary, i.e. when $\nu \in i\mathbf{R}$. The space V_χ has a one-dimensional subspace of K_p-fixed vectors, spanned by the function

$$\phi_0(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)k) = \chi_1(a)\chi_2(d)\left|\frac{a}{d}\right|_p^{\nu+1/2} \qquad (k \in K_p). \tag{4.14}$$

Proposition 4.3. *The representation (π_χ, V_χ) defined above is irreducible. Every irreducible admissible unramified representation of $G(\mathbf{Q}_p)$ with central character ω_p is either one-dimensional or of the form π_χ for some χ as above. If π_χ is unitary then either:*
· $\nu \in i\mathbf{R}$ *(unitary principal series), or*
· $0 < |\operatorname{Re}\nu| < \frac{1}{2}$ *(complementary series).*

Proof. Refer, e.g., to Theorems 4.5.1, 4.6.4, and 4.6.7 of [Bu]. □

The local component $f_p^{\mathtt{n}}$ of $f^{\mathtt{n}}$ acts on the unramified vector ϕ_0 in the following way.

Proposition 4.4. *Assume $p \nmid N$, and let $\mathtt{n}_p = \operatorname{ord}_p(\mathtt{n}) \geq 0$. With $f_p^{\mathtt{n}}$ as above, the function ϕ_0 of* (4.14) *is an eigenvector of the local Hecke operator $\pi_\chi(f_p^{\mathtt{n}})$ with eigenvalue*

$$p^{\mathtt{n}_p/2}\lambda_{p^{\mathtt{n}_p}}(\chi_1, \chi_2, \nu),$$

where

$$\lambda_{p^{\mathtt{n}_p}}(\chi_1, \chi_2, \nu) = \sum_{j=0}^{\mathtt{n}_p}\left(\frac{p^{\mathtt{n}_p}}{p^{2j}}\right)^{\nu}\chi_1(p)^j\chi_2(p)^{\mathtt{n}_p-j}. \tag{4.15}$$

Proof. The fact that ϕ_0 is an eigenvector is due to Proposition 3.10, together with the fact that the space of K_p-fixed vectors is one-dimensional. Thus the eigenvalue is equal to $\pi_\chi(f_p^{\mathtt{n}})\phi_0(1)$, which can be computed using the decomposition

$$M_1(\mathtt{n}, N)_p = \bigcup_{j=0}^{\mathtt{n}_p}\bigcup_{a\in\mathbf{Z}/p^j\mathbf{Z}}\begin{pmatrix} p^j & a \\ & p^{\mathtt{n}_p-j}\end{pmatrix}K_p \tag{4.16}$$

([KL2], Lemma 13.4) as follows:

$$\int_{\overline{G}(\mathbf{Q}_p)} f_p^{\mathtt{n}}(g)\phi_0(g)dg = \sum_{j=0}^{\mathtt{n}_p} p^j\left|\frac{p^j}{p^{\mathtt{n}_p-j}}\right|_p^{\nu+\frac{1}{2}}\chi_1(p^j)\chi_2(p^{\mathtt{n}_p-j}) = p^{\mathtt{n}_p/2}\lambda_{p^{\mathtt{n}_p}}(\chi_1, \chi_2, \nu).$$

□

4.3 Adelic Maass forms

Let ω be the Hecke character attached to ω' as in (2.8). Using (2.9) and (4.1), we have

$$\omega_\infty(-1) = \omega_\infty(-1)\omega'(-1) = \omega_\infty(-1)\prod_{p|N}\omega_p(-1) = \omega(-1) = 1.$$

Since ω_∞ is trivial on $\mathbf{R}^+$, this implies that for all $x \in \mathbf{R}^*$,

$$\omega_\infty(x) = 1. \tag{4.17}$$

Let $L^2(\omega) = L^2(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}), \omega)$ be the space of measurable $\mathbf{C}$-valued functions ψ on $G(\mathbf{A})$ (modulo functions that are 0 a.e.) satisfying $\psi(z\gamma g) = \omega(z)\psi(g)$ for all $\gamma \in G(\mathbf{Q})$ and $z \in Z(\mathbf{A}) \cong \mathbf{A}^*$, and which are square integrable over $\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})$. A function $\psi \in L^2(\omega)$ is **cuspidal** if its constant term ψ_N vanishes for a.e. $g \in G(\mathbf{A})$:

$$\psi_N(g) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \psi(ng)dn = 0.$$

Let $L_0^2(\omega) \subseteq L^2(\omega)$ denote the subspace of cuspidal functions. We let R denote the right regular representation of $G(\mathbf{A})$ on $L^2(\omega)$, and let R_0 denote its restriction to $L_0^2(\omega)$, which is easily seen to be an invariant subspace.

Let $L^1(\overline{\omega})$ denote the space of measurable functions $f : G(\mathbf{A}) \longrightarrow \mathbf{C}$ satisfying $f(zg) = \overline{\omega(z)}f(g)$ for all $z \in Z(\mathbf{A})$ and $g \in G(\mathbf{A})$, and which are absolutely integrable over $\overline{G}(\mathbf{A})$. Such a function defines an operator $R(f)$ on $L^2(\omega)$ via

$$R(f)\phi(x) = \int_{\overline{G}(\mathbf{A})} f(y)\phi(xy)dy,$$

the integral converging absolutely. Recall in fact that $\|R(f)\phi\|_{L^2} \leq \|f\|_{L^1}\|\phi\|_{L^2}$ (see e.g. [KL2], p. 140). The restriction of $R(f)$ to $L_0^2(\omega)$ is denoted $R_0(f)$. For $f, h \in L^1(\overline{\omega})$, the convolution

$$f * h(x) = \int_{\overline{G}(\mathbf{A})} f(y)h(y^{-1}x)dy$$

also belongs to $L^1(\overline{\omega})$, and by a straightforward computation we have

$$R(f * h) = R(f)R(h).$$

To each $u \in L^2(N, \omega')$ we associate a function φ_u on $G(\mathbf{A})$ using strong approximation (2.3) by setting

$$\varphi_u(\gamma(g_\infty \times k)) = u(g_\infty(i)) \tag{4.18}$$

for $\gamma \in G(\mathbf{Q})$, $g_\infty \in G(\mathbf{R})^+$, and $k \in K_1(N)$. Using the modularity of u, it is easy to check that φ_u is well-defined.

Proposition 4.5. *The map $u \mapsto \varphi_u$ defines surjective linear isometries*

$$L^2(N,\omega') \longrightarrow L^2(\omega)^{K_\infty \times K_1(N)}$$

and

$$L^2_0(N,\omega') \longrightarrow L^2_0(\omega)^{K_\infty \times K_1(N)},$$

where the spaces on the right denote those functions satisfying $\varphi(gk) = \varphi(g)$ for all $k \in K_\infty \times K_1(N)$.

Proof. First we check that $\varphi_u(zg) = \omega(z)\varphi_u(g)$ for all $z \in Z(\mathbf{A})$. By strong approximation, we can assume that $g = g_\infty \times k \in G(\mathbf{R})^+ \times K_1(N)$. Write $z = z_{\mathbf{Q}}(z_\infty \times z_{\mathrm{fin}})$ for $z_\infty \in \mathbf{R}^+$ and $z_{\mathrm{fin}} \in \widehat{\mathbf{Z}}^*$. We have $z_{\mathrm{fin}} \equiv a \bmod N\widehat{\mathbf{Z}}$ for some integer a relatively prime to N. Then $\omega(z) = \omega'(a)$ as in (2.8). Choose b, c, d such that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then $\gamma z_{\mathrm{fin}} \in K_1(N)$, and

$$\begin{aligned}\varphi_u(zg) &= \varphi_u(\gamma z g) = \varphi_u(\gamma g_\infty \times \gamma z_{\mathrm{fin}} k) = u(\gamma g_\infty(i)) \\ &= \overline{\omega'(d)} u(g_\infty(i)) = \omega'(a)\varphi_u(g) = \omega(z)\varphi_u(g)\end{aligned}$$

as needed.

For the square integrability, let D_N be a fundamental domain in $\mathbf{H}$ for $\Gamma_0(N)\backslash\mathbf{H}$. We identify D_N with a subset of $G(\mathbf{R})^+$ via $x + iy \leftrightarrow \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$. Then by Proposition 7.43 of [KL2],

$$\begin{aligned}\int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} |\varphi_u(g)|^2 dg &= \int_{D_N K_\infty \times K_0(N)} |\varphi_u(g)|^2 dg \\ &= \mathrm{meas}(K_0(N)) \iint_{D_N} |u(x+iy)|^2 \frac{dx\,dy}{y^2} = \|u\|^2.\end{aligned}$$

This proves that the map $u \mapsto \varphi_u$ is an isometry of $L^2(N,\omega')$ into $L^2(\omega)^{K_\infty \times K_1(N)}$, since it is clear from the definition (4.18) that φ_u is invariant under $K_\infty \times K_1(N)$. For the surjectivity, we note that the inverse map is given by

$$u(z) := \varphi(g_\infty),$$

where $g_\infty \in G(\mathbf{R})^+$ is any element satisfying $g_\infty(i) = z$. The function u is well-defined since φ is K_∞-invariant. The fact that $u(z)$ satisfies (4.2) can be seen as follows. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we can write $\gamma^{-1} = \left(\begin{smallmatrix} a_N & \\ & a_N \end{smallmatrix}\right) k$ for $k \in K_1(N)$ (and a_N as in (2.2)). Thus

$$\begin{aligned}u(\gamma z) &= \varphi(\gamma_\infty g_\infty) = \varphi(g_\infty \times \gamma_{\mathrm{fin}}^{-1}) = \varphi(g_\infty \times \left(\begin{smallmatrix} a_N & \\ & a_N \end{smallmatrix}\right) k) \\ &= \omega(a_N)\varphi(g_\infty) = \omega'(a) u(z) = \overline{\omega'(d)} u(z).\end{aligned}$$

Lastly, for any $g \in G(\mathbf{R})^+ \times K_1(N)$, there exists $\delta \in G(\mathbf{Q})^+$ determined by g_{fin} such that

$$(\varphi_u)_N(g) = a_{0,\delta}(u, y)$$

for $y = \mathrm{Im}\, g_\infty(i)$. This is proven just as in the holomorphic case, making the obvious adjustments. See [KL2], pp. 200-201. Therefore u is cuspidal if and only if φ_u is cuspidal. $\square$

Next, we describe some properties of the correspondence $u \mapsto \varphi_u$.

Lemma 4.6. *The correspondence is equivariant for both Δ and the Hecke operators, in the following sense: For all $u \in L^2(N, \omega')$,*

$$R(f^{\mathtt{n}})\varphi_u = \int_{\overline{G}(\mathbf{A}_{\mathrm{fin}})} f^{\mathtt{n}}(g)R(g)\varphi_u \, dg = \sqrt{\mathtt{n}}\,\varphi_{T_{\mathtt{n}}u} \tag{4.19}$$

for $f^{\mathtt{n}}$ defined in (4.11)*, and if u is smooth,*

$$R(\Delta)\varphi_u = \varphi_{\Delta u}. \tag{4.20}$$

Proof. In both cases it suffices by strong approximation (2.3) to show that the two functions agree on elements of the form $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in G(\mathbf{R}^+)$ when u is smooth. In (4.20), the symbol Δ is used in two different ways. On the right, Δ is the Laplace operator (4.5), and on the left it is the Casimir element whose effect on $C^\infty(G(\mathbf{R}))$ is given by (3.19). But because ϕ_u is K_∞-invariant, $\frac{\partial}{\partial\theta}\phi_u = 0$, so we can drop the second term of (3.19) to conclude

$$\begin{aligned} R(\Delta)\varphi_u(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)) &= -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi_u(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)) \\ &= \Delta u(x+iy) = \varphi_{\Delta u}(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)), \end{aligned} \tag{4.21}$$

as needed.

The proof of (4.19) is the same as that of the version for holomorphic cusp forms given in [KL2], Proposition 13.6. □

By a theorem of Gelfand and Piatetski-Shapiro, the right regular representation R_0 of $G(\mathbf{A})$ on $L^2_0(\omega)$ decomposes into a direct sum of irreducible unitary representations π. Each such cuspidal representation π is a restricted tensor product of local representations: $\pi = \pi_\infty \otimes \pi_{\mathrm{fin}} = \pi_\infty \otimes \bigotimes_p \pi_p$ (cf. [Bu], §3.4).

Proposition 4.7. *We have the following decomposition:*

$$L^2_0(\omega)^{K_\infty \times K_1(N)} = \bigoplus_\pi \mathbf{C}v_\infty \otimes \pi_{\mathrm{fin}}^{K_1(N)}, \tag{4.22}$$

where π runs through the irreducible cuspidal representations with infinity type of the form $\pi_\infty = \pi(\varepsilon, \varepsilon, s, -s)$, where either $s \in i\mathbf{R}$ or $-\frac{1}{2} < s < \frac{1}{2}$, and v_∞ is a nonzero vector of weight 0 *(unique up to multiples). Equivalently, π runs through the constituents of $L^2_0(\omega)$ which contain a nonzero $K_\infty \times K_1(N)$-fixed vector.*

Remarks: (1) Selberg's conjecture asserts that the complementary series of Proposition 3.8 do not actually show up here, i.e. that $s \in i\mathbf{R}$. Likewise, according to the Ramanujan conjecture, the unramified local factors of π_{fin} are unitary principal series rather than complementary series (cf. Proposition 4.3).

(2) Caution about notation: In §11 of [KL2], when discussing $\pi(\varepsilon_1, \varepsilon_2, s_1, s_2)$ we used the notation $s = s_1 - s_2$. In the present document, we take $s = 2(s_1 - s_2)$.

Proof. For any irreducible cuspidal representation π, the orthogonal projection map $L^2_0(\omega) \to \pi$ commutes with the right regular action $R(g)$. As an easy consequence, we have

$$L^2_0(\omega)^{K_\infty \times K_1(N)} = \bigoplus_\pi \pi^{K_\infty \times K_1(N)}.$$

The Casimir element Δ acts on the smooth vectors of an irreducible finite dimensional representation of $G(\mathbf{R})$ by a scalar which is ≤ 0 (cf. Theorem 11.15 and Proposition 11.22 of [KL2]). We conclude from the fact that $R_0(\Delta)$ is positive definite that π_∞ is infinite dimensional. The proposition now follows immediately from Proposition 3.8. Note that ω_∞ is the trivial character, so $s_1 + s_2 = 0$ in the notation of that proposition, and here we have set $s_1 = -s_2 = s$. □

An **adelic Hecke operator** of weight 0 is a function on $G(\mathbf{A})$ of the form

$$f = f_\infty \times f^{\mathtt{n}} \in L^1(G(\mathbf{A}), \overline{\omega}) \tag{4.23}$$

with $f_\infty \in C_c(G^+//K_\infty)$ and $f^{\mathtt{n}}$ as in (4.11). We now show that for such f, the operator $R_0(f)$ on $L^2_0(\omega)$ is diagonalizable. By Lemma 3.10, it suffices to consider its restriction to $L^2_0(\omega)^{K_\infty \times K_1(N)}$:

Proposition 4.8. *For each cuspidal $\pi = \pi(\varepsilon, \varepsilon, s, -s) \otimes \pi_{\mathrm{fin}}$ contributing to (4.22), choose an orthogonal basis $\mathcal{F}_\pi$ for the finite dimensional subspace $\mathbf{C}v_\infty \otimes \pi_{\mathrm{fin}}^{K_1(N)}$. Let $\mathcal{F}_{\mathbf{A}} = \bigcup_\pi \mathcal{F}_\pi$ be the resulting orthogonal basis for $L^2_0(\omega)^{K_\infty \times K_1(N)}$. Then each $\varphi \in \mathcal{F}_{\mathbf{A}}$ is an eigenfunction of $R_0(f)$ with eigenvalue of the form*

$$h(t)\sqrt{\mathtt{n}}\,\lambda_{\mathtt{n}}(\varphi), \tag{4.24}$$

where $t = -is$ is the spectral parameter of π_∞, h is the Selberg transform of f_∞, and $\lambda_{\mathtt{n}}(\varphi) = \prod_{p|\mathtt{n}} \lambda_{p^{\mathtt{n}_p}}$ is determined from π_p for $p|\mathtt{n}$ by (4.15). Furthermore, if $u \in L^2_0(N, \omega')$ is the function on $\mathbf{H}$ corresponding to φ, then u is a Maass eigenform with Δ-eigenvalue $\frac{1}{4} + t^2$, and Hecke eigenvalue $\lambda_{\mathtt{n}}(u) = \lambda_{\mathtt{n}}(\varphi)$.

Proof. Let π be one of the given cuspidal representations. When $p|\mathtt{n}$, $\pi_p^{K_1(N)_p} = \pi_p^{K_p}$ is nonzero, so π_p is an unramified unitary principal series representation. Write $\pi_p^{K_p} = \mathbf{C}v_p$. By Proposition 4.4, we have

$$\pi_p(f_p^{\mathtt{n}})v_p = p^{\mathtt{n}_p/2}\lambda_{p^{\mathtt{n}_p}}\, v_p.$$

At the archimedean place, Proposition 3.9 gives

$$\pi_\infty(f_\infty)v_\infty = h(t)v_\infty.$$

Consider any $v \in \mathbf{C}v_\infty \otimes \pi_{\mathrm{fin}}^{K_1(N)}$. For any object defined as a product of local objects, let us for the moment use a prime $'$ to denote the product over just the finite primes $p \nmid \mathtt{n}$. Then

$$v = v_\infty \otimes v' \otimes \bigotimes_{p|\mathtt{n}} v_p$$

for v_p as above and

$$v' \in (\pi')^{K_1(N)'} = \bigotimes_{p \nmid \mathtt{n}} \pi_p^{K_1(N)_p}.$$

By the definition (4.11) of $f^{\mathtt{n}}$, we see that

$$\pi'(f^{\mathtt{n}\prime})v' = \text{meas}(\overline{K_1(N)'})^{-1} \int_{\overline{K_1(N)'}} \pi'(k)v'dk = v'.$$

Letting $\varphi \in L_0^2(\omega)$ denote the function corresponding to v, it follows (e.g. by Proposition 13.17 of [KL2]) that

$$R(f)\varphi = \pi_\infty(f_\infty)v_\infty \otimes \pi'(f^{\mathtt{n}\prime})v' \otimes \bigotimes_{p|\mathtt{n}} \pi_p(f_p^{\mathtt{n}})v_p = \sqrt{\mathtt{n}}\lambda_{\mathtt{n}}(\varphi)h(t)\varphi.$$

Now let u be the element of $L_0^2(N,\omega')$ attached to v. We need to show that $\Delta u = (\frac{1}{4}+t^2)u$. For any $X \in \mathfrak{g}_{\mathbf{R}}$, we have

$$\pi(X)v \stackrel{\text{def}}{=} \left.\frac{d}{dt}\right|_{t=0} \pi(\exp(tX) \times 1_{\text{fin}})v = \left.\frac{d}{dt}\right|_{t=0} \pi_\infty(\exp(tX))v_\infty \otimes v_{\text{fin}}.$$

Therefore

$$\pi(\Delta)v = \pi_\infty(\Delta)v_\infty \otimes v_{\text{fin}} = (\tfrac{1}{4}+t^2)v$$

by (3.20). Equivalently, $R(\Delta)\varphi = (\frac{1}{4}+t^2)\varphi$, so by (4.20), $\Delta u = (\frac{1}{4}+t^2)u$. Lastly, by Lemma 4.6 we also have $\lambda_{\mathtt{n}}(u) = \lambda_{\mathtt{n}}(\varphi)$. $\square$

With $\mathcal{F}_{\mathbf{A}}$ as in the above proposition, we let

$$\mathcal{F} \subseteq L_0^2(N,\omega') \tag{4.25}$$

be the corresponding orthogonal basis. It consists of Maass eigenforms as shown above. Using Proposition 4.8, we can arrange further for each $u \in \mathcal{F}$ to be an eigenvector of T_{-1}.

5 Eisenstein series

The continuous part of $L^2(\omega)$ is explicitly describable in terms of Eisenstein series (see Sec. 6.1). Because we are interested in automorphic forms of weight $\mathtt{k} = 0$ and level N, we will concentrate on $K_\infty \times K_1(N)$-invariant Eisenstein series.

5.1 Induced representations of $G(\mathbf{A})$

We begin by constructing certain principal series representations of $G(\mathbf{A})$. These are representations induced from characters of the Borel subgroup $B(\mathbf{A}) = M(\mathbf{A})N(\mathbf{A})$. Any character of $B(\mathbf{A})$ is trivial on the commutator subgroup $N(\mathbf{A})$, and hence is really defined on the diagonal group $M(\mathbf{A}) \cong \mathbf{A}^* \times \mathbf{A}^*$. We are only interested in $G(\mathbf{Q})$-invariant functions, so we want a character of $B(\mathbf{Q})\backslash B(\mathbf{A})$, which by the above is nothing more than a pair of Hecke characters, say $\chi_1 \otimes |\cdot|^{s_1}$ and $\chi_2 \otimes |\cdot|^{s_2}$, where χ_1, χ_2 have finite order. Furthermore, we need the product of these two characters to equal our fixed central character ω, which has finite order. This means in particular that $s_2 = -s_1$.

Thus for finite order Hecke characters χ_1 and χ_2 with $\chi_1\chi_2 = \omega$, and $s \in \mathbf{C}$, we consider the character of $B(\mathbf{A})$ defined by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d) \left|\frac{a}{d}\right|^s.$$

We let $(\pi_s, H(\chi_1, \chi_2, s))$ denote the representation of $G(\mathbf{A})$ unitarily induced from this character. This Hilbert space has a dense subspace spanned by the continuous functions $\phi : G(\mathbf{A}) \longrightarrow \mathbf{C}$ satisfying

$$\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \chi_1(a)\chi_2(d) \left|\frac{a}{d}\right|^{s+1/2} \phi(g). \tag{5.1}$$

The inner product is defined by

$$\langle \phi, \psi \rangle = \int_K \phi(k)\overline{\psi(k)}dk.$$

This is nondegenerate since, by the decomposition $G = BK$, any $\phi \in H(\chi_1, \chi_2, s)$ is determined by its restriction to K. The right regular representation $\pi_s = \pi_s(\chi_1, \chi_2)$ of $G(\mathbf{A})$ on $H(\chi_1, \chi_2, s)$ is unitary if $s \in i\mathbf{R}$ (for the idea, see e.g. [KL2] Proposition 11.8). As explained in §4B-4C of [GJ], we have

$$H(\chi_1, \chi_2, s) \cong H(\chi_2, \chi_1, -s) \tag{5.2}$$

as representations of $G(\mathbf{A})$.

Restriction to K identifies $H(\chi_1, \chi_2, s)$ with the subspace of $L^2(K)$ consisting of functions satisfying

$$f(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)k) = \chi_1(a)\chi_2(d)f(k) \qquad (\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B(\mathbf{A}) \cap K).$$

In this way, the spaces $H(\chi_1, \chi_2, s)$ form a trivial vector bundle over the above subspace of $L^2(K)$. Given $\phi \in H(\chi_1, \chi_2, 0)$ and $s \in \mathbf{C}$, we define $\phi_s \in H(\chi_1, \chi_2, s)$ by

$$\phi_s(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k) = \chi_1(a)\chi_2(d) \left|\frac{a}{d}\right|^{s+1/2} \phi(k).$$

Equivalently, if $H : G(\mathbf{A}) \to \mathbf{R}^+$ is the **height function** defined by

$$H(g) = H(\begin{pmatrix} a & \\ & d \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k) = \log \left|\frac{a}{d}\right|, \tag{5.3}$$

then

$$\phi_s(g) = e^{sH(g)}\phi(g).$$

The map from $H(\chi_1, \chi_2, 0)$ to $H(\chi_1, \chi_2, s)$ taking $\phi \mapsto \phi_s$ is an isomorphism of Hilbert spaces. We set

$$H(\chi_1, \chi_2) \stackrel{\text{def}}{=} H(\chi_1, \chi_2, 0).$$

Lemma 5.1. *Suppose $\phi \in H(\chi_1, \chi_2, s)$ is a right $K_\infty \times K_1(N)$-invariant function, i.e. $\phi \in H(\chi_1, \chi_2, s)^{K_\infty \times K_1(N)}$. Define $\phi_\infty : G(\mathbf{R})^+ \longrightarrow \mathbf{C}$ by*

$$\phi_\infty(g_\infty) = \operatorname{Im}(z)^{s+1/2} \qquad (z = g_\infty(i) \in \mathbf{H}), \tag{5.4}$$

and for $p \nmid N$, define $\phi_p : G(\mathbf{Q}_p) \longrightarrow \mathbf{C}$ by

$$\phi_p(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k) = \chi_{1p}(a)\chi_{2p}(d) \left|\frac{a}{d}\right|_p^{s+1/2}.$$

Also set $\phi' = \prod_{p \nmid N} \phi_p$, and define $\phi_N : \prod_{p|N} G(\mathbf{Q}_p) \longrightarrow \mathbf{C}$ by

$$\phi_N(g_N) = \phi(1_\infty \times g_N \times 1').$$

(When $N = 1$, the above is just the constant $\phi(1)$.) Then ϕ is factorizable as

$$\phi(g_\infty \times g_N \times g') = \phi_\infty(g_\infty)\phi_N(g_N)\phi'(g').$$

Proof. Write

$$g_\infty = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k_\infty.$$

Then since ϕ is K_∞-invariant,

$$\begin{aligned}\phi(g_\infty \times g_{\text{fin}}) &= \omega_\infty(u)\chi_{1\infty}(y^{1/2})\chi_{2\infty}(y^{-1/2})y^{s+\frac{1}{2}}\phi(1_\infty \times g_{\text{fin}}) \\ &= y^{s+\frac{1}{2}}\phi(1_\infty \times g_{\text{fin}}) = \phi_\infty(g_\infty)\phi(1_\infty \times g_{\text{fin}})\end{aligned}$$

since ω_∞ is trivial by (4.17), and $\chi_{1\infty}, \chi_{2\infty}$ are trivial on $\mathbf{R}^+$. Now according to the Iwasawa decomposition, write $g' = b'k'$ for $k' \in K' = \prod_{p \nmid N} K_p$ and $b \in \prod_{p \nmid N} B(\mathbf{Q}_p)$. Then by the same argument, using the fact that ϕ is right invariant under K', we have

$$\phi(1_\infty \times g_N \times g') = \phi'(g')\phi(1_\infty \times g_N \times 1') = \phi'(g')\phi_N(g_N).$$

The lemma follows. □

Proposition 5.2. *If $f = f_\infty \times f^{\mathtt{n}}$ with $f_\infty \in C_c(G^+//K_\infty)$, the operator $\pi_s(f)$ acts by the scalar*

$$h(t)\sqrt{\mathtt{n}}\,\lambda_{\mathtt{n}}(\chi_1,\chi_2,it)$$

on $H(\chi_1,\chi_2,s)^{K_\infty\times K_1(N)}$, and vanishes on the orthogonal complement of this finite dimensional subspace. Here, $t=-is$, h is the Selberg transform of f_∞, and

$$\lambda_{\mathtt{n}}(\chi_1,\chi_2,s) = \prod_{p|\mathtt{n}} \lambda_{p^{\mathtt{n}_p}}(\chi_{1p},\chi_{2p},s) = \mathtt{n}^s \sum_{d|\mathtt{n}} \frac{\overline{\chi_1(d_N)\chi_2((\frac{\mathtt{n}}{d})_N)}}{d^{2s}}, \tag{5.5}$$

for $\lambda_{p^{\mathtt{n}_p}}(\chi_{1p},\chi_{2p},s)$ as in Proposition 4.4.

Proof. The dimension of $H(\chi_1,\chi_2,s)^{K_\infty\times K_1(N)}$ is computed in Section 5.4 below. By Lemma 3.10, $\pi_s(f)$ vanishes on its orthogonal complement. The second equality in (5.5) comes from the fact that for any finite order Hecke character χ of conductor dividing N, and any positive integer $d|\mathtt{n}$,

$$1 = \chi(d) = \prod_{p|\mathtt{n}} \chi_p(d) \prod_{p|N} \chi_p(d),$$

so

$$\prod_{p|\mathtt{n}} \chi_p(p^{d_p}) = \prod_{p|\mathtt{n}} \chi_p(d) = \prod_{p|N} \overline{\chi_p(d)} = \overline{\chi(d_N)}.$$

For $\phi \in H(\chi_1,\chi_2,s)^{K_\infty\times K_1(N)}$, write

$$\phi = \phi_\infty \otimes \phi_N \otimes \bigotimes_{p\nmid N} \phi_p$$

as in the above lemma. Note that ϕ_∞ is a weight 0 vector in an induced representation $\pi_\infty = \pi(\varepsilon_1,\varepsilon_2,s,-s)$ with spectral parameter $t=-is$. Likewise if $p\nmid N$, then ϕ_p is the K_p-fixed vector in the unramified representation π_p of $G(\mathbf{Q}_p)$ induced from the character (χ_{1p},χ_{2p}) of $B(\mathbf{Q}_p)$ (as in (4.13), taking $\nu=s$). Setting $f_N = \prod_{p|N} f_p$, we have

$$\pi_s(f)\phi = \pi_\infty(f_\infty)\phi_\infty \otimes R(f_N)\phi_N \otimes \bigotimes_{p\nmid N} \pi_p(f_p^{\mathtt{n}})\phi_p.$$

Here

$$\pi_\infty(f_\infty)\phi_\infty = h(t)\phi_\infty$$

by Proposition 3.9, and if $p\nmid N$,

$$\pi_p(f_p^{\mathtt{n}})\phi_p = p^{\mathtt{n}_p/2}\lambda_{p^{\mathtt{n}_p}}(\chi_{1p},\chi_{2p},s)\phi_p$$

by Proposition 4.4, while by (4.12)

$$R(f_N)\phi_N(x) = \frac{1}{\mathrm{meas}(\overline{K_1(N)})}\int_{\prod_{p|N}\overline{K_1(N)}_p} \phi_N(xk)dk = \phi_N(x).$$

Therefore $\pi_s(f)\phi = h(t)\sqrt{\mathtt{n}}\,\lambda_{\mathtt{n}}(\chi_1,\chi_2,s)\phi$ as claimed. □

5.2 Definition of Eisenstein series

The elements of $H(\chi_1, \chi_2, s)$ are $B(\mathbf{Q})$-invariant by construction. We use them to define $G(\mathbf{Q})$-invariant functions (automorphic forms) on $G(\mathbf{A})$ by averaging:

$$E(\phi, s, g) = E(\phi_s, g) = \sum_{\gamma \in B(\mathbf{Q}) \backslash G(\mathbf{Q})} \phi_s(\gamma g) \qquad (\phi \in H(\chi_1, \chi_2)).$$

This sum converges absolutely when $\mathrm{Re}(s) > 1/2$ (see Proposition 5.6 below). For now we will assume that s belongs to this domain. However, the Eisenstein series E has a meromorphic continuation to the complex plane. We will prove this well-known result below in the case where ϕ is $K_\infty \times K_1(N)$-invariant by writing down the Fourier expansion of E, which is seen to be meromorphic on $\mathbf{C}$ (cf. Theorem 5.16).

For a fixed level N, we will only be interested in the case where ϕ is a nonzero (right) $K_\infty \times K_1(N)$-invariant vector. Such ϕ exists if and only if the product $\mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}$ of the conductors divides N (see Corollary 5.11 below). In this case, $E(\phi, s, g)$ is left $G(\mathbf{Q})$-invariant and right $K_1(N)$-invariant. By strong approximation we have $G(\mathbf{A}) = G(\mathbf{Q})(G(\mathbf{R})^+ \times K_1(N))$. Therefore it suffices to investigate $E(\phi, s, g_\infty \times 1_{\mathrm{fin}})$, where $g_\infty \in G(\mathbf{R})^+ = Z(\mathbf{R})B(\mathbf{R})K_\infty$. Furthermore, because

1. $E(\phi, s, g_\infty)$ is right K_∞-invariant
2. the central character $\omega = \chi_1\chi_2$ is trivial on $Z(\mathbf{R})$ (see (4.17)),

the value of $E(\phi, s, g_\infty)$ depends only on $z = g_\infty(i) \in \mathbf{H}$. So for $z = x + iy \in \mathbf{H}$, we define

$$E_\phi(s, z) = E(\phi, s, g_\infty \times 1_{\mathrm{fin}}),$$

for any $g_\infty \in Z(\mathbf{R})\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y^{1/2} & \\ & y^{-1/2} \end{smallmatrix}\right)K_\infty$. Thus

$$E_\phi(s, z) = \sum_{\gamma \in B(\mathbf{Q}) \backslash G(\mathbf{Q})} \phi_s(\gamma g_\infty) = \sum_{\gamma \in B(\mathbf{Q}) \backslash G(\mathbf{Q})} e^{sH(\gamma_\infty g_\infty \times \gamma_{\mathrm{fin}})} \phi(\gamma_\infty g_\infty \times \gamma_{\mathrm{fin}}),$$

where H is the height function defined in (5.3).

Lemma 5.3. *A set of representatives for $B(\mathbf{Q}) \backslash G(\mathbf{Q})$ is given by $\pm N(\mathbf{Z}) \backslash \mathrm{SL}_2(\mathbf{Z})$. The latter set is in one-to-one correspondence with ordered pairs (c, d) of relatively prime integers with $c > 0$, together with $(0, 1)$, via $\pm N(\mathbf{Z})\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \leftrightarrow (c, d)$.*

Proof. The first assertion follows from the decomposition

$$G(\mathbf{Q}) = B(\mathbf{Q})\,\mathrm{SL}_2(\mathbf{Z}) \tag{5.6}$$

since $B \backslash B\Gamma \cong (B \cap \Gamma) \backslash \Gamma$. The decomposition (5.6) can easily be proven directly as follows. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbf{Q})$. Write $(c \;\; d) = t(c' \;\; d')$, where $t \in \mathbf{Q}$, $c', d' \in \mathbf{Z}$ and $\gcd(c', d') = 1$. There exist integers x and y such that $c'x - d'y = 1$. Then $\left(\begin{smallmatrix} d' & x \\ -c' & -y \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} d' & x \\ -c' & -y \end{smallmatrix}\right) \in B(\mathbf{Q})$.

For the representatives, view $\mathbf{Z}\times\mathbf{Z}$ as a set of row vectors, and consider the right action of $\mathrm{SL}_2(\mathbf{Z})$ on this set. The stabilizer of $(0\ \ 1)$ is $N(\mathbf{Z})$. Therefore $N(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{Z})$ is in one-to-one correspondence with the orbit of $(0\ \ 1)$. It is easy to see that this orbit is the set of ordered pairs of relatively prime integers:

$$(0\ \ 1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (c\ \ d).$$

Considering instead the set $\pm\backslash(\mathbf{Z}\times\mathbf{Z})$, the stabilizer of $\pm(0\ \ 1)$ is $\pm N(\mathbf{Z})$ and we can take $c \geq 0$, obtaining the given set of pairs (c, d). □

By the above and Lemma 5.1, we have

$$E_\phi(s,z) = \sum_{\gamma\in\pm N(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{Z})} e^{sH(\gamma_\infty g_\infty\times 1_{\mathrm{fin}})}\phi_\infty(\gamma g_\infty)\phi_{\mathrm{fin}}(\gamma).$$

This holds since $\gamma_{\mathrm{fin}}\in \mathrm{SL}_2(\mathbf{Z})\subseteq K_{\mathrm{fin}}$ and the height function is K_{fin}-invariant. Now using

$$e^{H(\gamma_\infty g_\infty\times 1_{\mathrm{fin}})} = |\operatorname{Im}(\gamma z)| = \frac{y}{|cz+d|^2} \qquad (\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)),$$

together with (5.4) (with $s=0$), we have

$$E_\phi(s,z) = y^{1/2+s}\phi_{\mathrm{fin}}(0,1) + \sum_{c>0}\sum_{\substack{d\in\mathbf{Z}\\ \gcd(c,d)=1}} \frac{y^{1/2+s}}{|cz+d|^{1+2s}}\phi_{\mathrm{fin}}(c,d). \tag{5.7}$$

Here we have written $\phi_{\mathrm{fin}}(c,d)$ to denote $\phi_{\mathrm{fin}}(\gamma)$. This notation is apt because ϕ_{fin} is left $B(\mathbf{Q})$-invariant, so that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbf{Z})$, $\phi_{\mathrm{fin}}(\gamma)$ depends only on (c,d) by Lemma 5.3.

5.3 The finite part of ϕ

We eventually need to compute the Fourier coefficients of $E_\phi(s,z)$ for ϕ in an orthonormal basis for $H(\chi_1,\chi_2)^{K_\infty\times K_1(N)}$. Since we can take ϕ_∞ to be the function determined in Lemma 5.1, in order to find such a basis we just need to write down the possibilities for ϕ_{fin}.

Lemma 5.4. $K_{\mathrm{fin}} = \mathrm{SL}_2(\mathbf{Z})K_1(N)$.

Proof. Let $S = \mathrm{SL}_2(\widehat{\mathbf{Z}})\cap K_1(N)$ denote the set of determinant 1 elements of $K_1(N)$. Note that S is an open subgroup of $\mathrm{SL}_2(\widehat{\mathbf{Z}})$. Hence

$$\mathrm{SL}_2(\widehat{\mathbf{Z}}) = \mathrm{SL}_2(\mathbf{Z})\cdot S$$

since $\mathrm{SL}_2(\mathbf{Z})$ is dense in $\mathrm{SL}_2(\widehat{\mathbf{Z}})$ (see e.g. Proposition 6.6 of [KL2]). From this we obtain the following decomposition:

$$K_{\mathrm{fin}} = \mathrm{SL}_2(\widehat{\mathbf{Z}})\begin{pmatrix} \widehat{\mathbf{Z}}^* & \\ & 1\end{pmatrix} = \mathrm{SL}_2(\mathbf{Z})\left[S\begin{pmatrix} \widehat{\mathbf{Z}}^* & \\ & 1\end{pmatrix}\right].$$

The lemma follows since the expression in the brackets is exactly $K_1(N)$. □

Lemma 5.5. *For a nonzero vector $\phi \in H(\chi_1, \chi_2)^{K_\infty \times K_1(N)}$, with ϕ_∞ given as in Lemma 5.1, the finite part ϕ_{fin} is determined by its restriction to* $\mathrm{SL}_2(\mathbf{Z})$. *For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, $\phi_{\mathrm{fin}}(\gamma)$ depends only on (c, d) mod N.*

Proof. By strong approximation for $B(\mathbf{A})$ ([KL2], Prop. 6.5),

$$G(\mathbf{A}) = B(\mathbf{A})K = B(\mathbf{Q})(B(\mathbf{R})^+ K_\infty \times K_{\mathrm{fin}}).$$

Hence by $B(\mathbf{Q})$-invariance, ϕ is determined by its restriction to $G(\mathbf{R})^+ \times K_{\mathrm{fin}}$. We assume that ϕ_∞ is the function given in Lemma 5.1. By $K_1(N)$-invariance and Lemma 5.4, ϕ_{fin} is determined by its values on $\mathrm{SL}_2(\mathbf{Z})$. Lastly, because $K(N) \subseteq K_1(N)$, ϕ_{fin} determines a function on $K_{\mathrm{fin}}/K(N) \cong G(\mathbf{Z}/N\mathbf{Z})$. Therefore $\phi_{\mathrm{fin}}(\gamma)$ depends only on the entries modulo N. □

Proposition 5.6. *For any $\phi \in H(\chi_1, \chi_2)^{K_\infty \times K_1(N)}$ and all $z \in \mathbf{H}$, the Eisenstein series $E_\phi(s, z)$ is absolutely convergent on* $\mathrm{Re}(s) > 1/2$. *For any $\delta > 0$ and compact set $C \subseteq \mathbf{H}$, the convergence is uniform on the set* $\mathrm{Re}(s) \geq \frac{1}{2} + \delta$ *and $z \in C$.*

Remark: A similar proof applies to the case of arbitrary $\phi \in H(\chi_1, \chi_2)$ (cf. [Bu], Proposition 3.7.2).

Proof. It follows from Lemma 5.5 that ϕ_{fin} is a bounded function. So up to a constant multiple, $E_\phi(s, z)$ is majorized by the classical series

$$E(\mathrm{Re}(s), z) = \sum_{(c,d) \neq (0,0)} \frac{y^{\mathrm{Re}(s)+1/2}}{|cz + d|^{2\,\mathrm{Re}(s)+1}},$$

which is easily seen to converge when $\mathrm{Re}(s) > 1/2$. □

As indicated in the proof of Lemma 5.5, ϕ_{fin} can be viewed as a function on $G(\mathbf{Z}/N\mathbf{Z})$. Let $D(\chi_1, \chi_2, N)$ denote the space of all functions ϕ on $G(\mathbf{Z}/N\mathbf{Z})$ satisfying

$$\phi(\begin{pmatrix} a & b \\ & d \end{pmatrix} k \begin{pmatrix} a' & b' \\ & 1 \end{pmatrix}) = \chi_1(a)\chi_2(d)\phi(k)$$

for all $k \in G(\mathbf{Z}/N\mathbf{Z})$ and $a, d, a' \in (\mathbf{Z}/N\mathbf{Z})^*$ and $b, b' \in \mathbf{Z}/N\mathbf{Z}$, or equivalently,

$$\phi(\begin{pmatrix} a & b \\ & d \end{pmatrix} k \begin{pmatrix} a' & b' \\ & d' \end{pmatrix}) = \chi_1(a)\chi_2(d)\omega(d')\phi(k).$$

Here $d' \in (\mathbf{Z}/N\mathbf{Z})^*$, and we view χ_1 and χ_2 as characters of $(\mathbf{Z}/N\mathbf{Z})^*$ as in (2.8), i.e. $\chi_j(a) = \chi_j(a_N)$. We make $D(\chi_1, \chi_2, N)$ into a finite dimensional Hilbert space by defining

$$\langle \phi_1, \phi_2 \rangle = |G(\mathbf{Z}/N\mathbf{Z})|^{-1} \sum_{k \in G(\mathbf{Z}/N\mathbf{Z})} \phi_1(k)\overline{\phi_2(k)}. \tag{5.8}$$

Notice that if $\phi, \psi \in H(\chi_1, \chi_2)^{K_\infty \times K_1(N)}$, then with notation as in Lemma 5.1, it is easy to see that

$$\langle \phi, \psi \rangle = \int_{K_\infty} \phi_\infty(k)\overline{\psi_\infty(k)}dk \int_{K_N} \phi_N(k)\overline{\psi_N(k)}dk \int_{K'} \phi'(k)\overline{\psi'(k)}dk$$
$$= \int_{K_N} \phi_N(k)\overline{\psi_N(k)}dk, \tag{5.9}$$

where $K_N = \prod_{p|N} K_p$. Letting $K_N(N) = \{k \in K_N | \, k \equiv 1 \bmod N\}$, the $K_1(N)$-invariance then gives

$$\langle \phi, \psi \rangle = [K_N : K_N(N)]^{-1} \sum_{k \in K_N/K_N(N)} \phi_N(k)\overline{\psi_N(k)} \tag{5.10}$$
$$= |G(\mathbf{Z}/N\mathbf{Z})|^{-1} \sum_{k \in G(\mathbf{Z}/N\mathbf{Z})} \phi_N(k)\overline{\psi_N(k)}.$$

(When $N = 1$, this is just $\phi(1)\overline{\psi(1)}$.) In view of (5.8), this proves the following.

Lemma 5.7. *The identification of ϕ_{fin} with a function on $G(\mathbf{Z}/N\mathbf{Z})$ induces an isometry of $H(\chi_1, \chi_2)^{K_\infty \times K_1(N)}$ with $D(\chi_1, \chi_2, N)$.*

The space $D(\chi_1, \chi_2, N)$ can be analyzed locally since

$$G(\mathbf{Z}/N\mathbf{Z}) \cong \prod_{p|N} G(\mathbf{Z}_p/N\mathbf{Z}_p).$$

We let $D_p(\chi_1, \chi_2, N)$ denote the space of functions on $G(\mathbf{Z}_p/N\mathbf{Z}_p)$ satisfying[2]

$$\phi(\begin{pmatrix} a & b \\ & d \end{pmatrix} k \begin{pmatrix} a' & b' \\ & d' \end{pmatrix}) = \chi_1(a)\chi_2(d)\chi_1(d')\chi_2(d')\phi(k) \tag{5.11}$$

for $k \in G(\mathbf{Z}_p/N\mathbf{Z}_p)$, $a, d, a', d' \in (\mathbf{Z}_p/N\mathbf{Z}_p)^*$, and $b, b' \in \mathbf{Z}_p/N\mathbf{Z}_p$. This is a Hilbert space with inner product given by the local analog of (5.8):

$$\langle \phi_1, \phi_2 \rangle = [K_p : K_p(p^{N_p})]^{-1} \sum_{k \in G(\mathbf{Z}_p/N\mathbf{Z}_p)} \phi_1(k)\overline{\phi_2(k)}, \tag{5.12}$$

and we have isometries

$$H(\chi_1, \chi_2)^{K_\infty \times K_1(N)} \cong D(\chi_1, \chi_2, N) \cong \bigotimes_{p|N} D_p(\chi_1, \chi_2, N). \tag{5.13}$$

When $N = 1$, the empty tensor product on the right is to be interpreted as $\mathbf{C}$.

[2]Here and henceforth, for $a \in \mathbf{Q}_p^*$ we evaluate $\chi_1(a)$ by embedding a as an idele which is 1 outside p. This is equivalent to $\chi_{1p}(a)$ but we sometimes wish to avoid the extra subscript when the context is completely local.

5.4 An orthogonal basis for $H(\chi_1, \chi_2)^{K_\infty \times K_1(N)}$

Most of the material in this section is drawn from pages 305-306 of Casselman's article [Cas].

In order to construct an orthogonal basis for $H(\chi_1, \chi_2)^{K_\infty \times K_1(N)}$, we see from (5.13) that it suffices to do so for $D_p(\chi_1, \chi_2, N)$. Define

$$B(\mathbf{Z}_p/N\mathbf{Z}_p) = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a, d \in (\mathbf{Z}_p/N\mathbf{Z}_p)^*, b \in \mathbf{Z}_p/N\mathbf{Z}_p\}.$$

Proposition 5.8. *For a prime $p|N$, we have the following disjoint union:*

$$G(\mathbf{Z}_p/N\mathbf{Z}_p) = \bigcup_{i=0}^{N_p} B(\mathbf{Z}_p/N\mathbf{Z}_p) \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix} B(\mathbf{Z}_p/N\mathbf{Z}_p).$$

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbf{Z}_p/N\mathbf{Z}_p)$. Setting $i = \min(\mathrm{ord}_p(c), N_p)$, it is elementary to show that g belongs only to the double coset of $\left(\begin{smallmatrix} 1 & 0 \\ p^i & 1 \end{smallmatrix}\right)$. For future reference, we give the decomposition explicitly. There are three cases. If $c = 0$, then $p^{N_p} \equiv 0 \bmod N\mathbf{Z}_p$ and

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(\mathbf{Z}_p/N\mathbf{Z}_p) = B(\mathbf{Z}_p/N\mathbf{Z}_p) \begin{pmatrix} 1 & 0 \\ p^{N_p} & 1 \end{pmatrix} B(\mathbf{Z}_p/N\mathbf{Z}_p). \tag{5.14}$$

Second, suppose that $0 < i < N_p$. Then a is a unit mod p and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{p^i a}{c} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix} \begin{pmatrix} \frac{c}{p^i} & \frac{bc}{ap^i} \\ & \frac{ad-bc}{a} \end{pmatrix} \qquad (i = \mathrm{ord}_p(c)). \tag{5.15}$$

If $i = 0$, then c is a unit, and we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} - 1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c & d - \frac{ad-bc}{c} \\ & \frac{ad-bc}{c} \end{pmatrix}. \tag{5.16}$$

□

By equation (5.11) and the above proposition, we see that a function $\phi \in D_p(\chi_1, \chi_2, N)$ is determined by its values on the matrices $\left(\begin{smallmatrix} 1 & 0 \\ p^i & 1 \end{smallmatrix}\right)$, for $i = 0, \ldots, N_p$. Therefore if $D_p(\chi_1, \chi_2, N)$ is nonzero, it is spanned by functions $\phi_{p,i,N_p}^{\chi_{1p},\chi_{2p}}$ satisfying

$$\phi_{p,i,N_p}^{\chi_{1p},\chi_{2p}}(\begin{pmatrix} 1 & 0 \\ p^j & 1 \end{pmatrix}) = \delta_{ij}. \tag{5.17}$$

Often we denote the above by ϕ_i when the other parameters are clear from the context. Because the decomposition of g into the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ p^i & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$ is not unique, for some values of i it may not be possible to start with (5.17) and extend to $G(\mathbf{Z}_p/N\mathbf{Z}_p)$ via (5.11). We give here the conditions on i under which such a function exists:

Proposition 5.9. *The function* $\phi_i = \phi_{p,i,N_p}^{\chi_{1p},\chi_{2p}}$ *is well-defined on* $G(\mathbf{Z}_p/N\mathbf{Z}_p)$ *if and only if*

$$\mathrm{ord}_p(\mathfrak{c}_{\chi_2}) \le i \le N_p - \mathrm{ord}_p(\mathfrak{c}_{\chi_1}), \tag{5.18}$$

where $N_p = \mathrm{ord}_p(N)$.

Proof. First, we suppose that (5.18) holds, and we check that ϕ_i is well-defined. It suffices to show that if

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix} \begin{pmatrix} r & s \\ 0 & t \end{pmatrix}, \tag{5.19}$$

then the two values produced by ϕ_i using (5.11) coincide:

$$\chi_1(a)\chi_2(d) = \chi_1(t)\chi_2(t). \tag{5.20}$$

The equality (5.19) gives

$$\begin{pmatrix} r & s \\ 0 & t \end{pmatrix} = \begin{pmatrix} a + bp^i & b \\ dp^i - ap^i - bp^{2i} & d - bp^i \end{pmatrix}.$$

From the lower left corner, we see that $(d - bp^i)p^i \equiv ap^i \bmod p^{N_p}$, so

$$d - bp^i \equiv a \bmod p^{N_p - i}. \tag{5.21}$$

Because $\mathrm{ord}_p(\mathfrak{c}_{\chi_1}) \le N_p - i$, this implies

$$\chi_1(a) = \chi_1(d - bp^i) = \chi_1(t).$$

Similarly, because $\mathrm{ord}_p(\mathfrak{c}_{\chi_2}) \le i$, we have

$$\chi_2(d) = \chi_2(d - bp^i) = \chi_2(t).$$

This proves (5.20), so ϕ_i is well-defined.

Conversely, assume ϕ_i is well-defined. Thus we suppose that whenever (5.21) holds, we have the equality

$$\chi_1(a)\chi_2(d) = \chi_1(d - bp^i)\chi_2(d - bp^i).$$

Using this we must deduce (5.18). Set $a = 1$, $b = 0$, and $d = 1 + up^{N_p - i}$ for any $u \in \mathbf{Z}_p$. Then (5.21) holds, so by our hypothesis we get

$$\chi_2(1 + up^{N_p - i}) = \chi_1(1 + up^{N_p - i})\chi_2(1 + up^{N_p - i}).$$

This implies $\chi_1(1 + up^{N_p - i}) = 1$, so $\mathrm{ord}_p(\mathfrak{c}_{\chi_1}) \le N_p - i$ as needed. Now set $a = 1$ and $d = 1 + bp^i$ for any $b \in \mathbf{Z}_p$. Then (5.21) holds, so we have

$$\chi_2(1 + bp^i) = \chi_1(1)\chi_2(1) = 1.$$

Thus $\mathrm{ord}_p(\mathfrak{c}_{\chi_2}) \le i$ as needed. □

Corollary 5.10. *Given that* $(\chi_1\chi_2)_p = \omega_p$*, the space* $D_p(\chi_1, \chi_2, N)$ *is nonzero if and only if*

$$\mathrm{ord}_p(\mathfrak{c}_{\chi_1}) + \mathrm{ord}_p(\mathfrak{c}_{\chi_2}) \leq N_p,$$

i.e. if and only if $N\mathbf{Z}_p \subseteq \mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}\mathbf{Z}_p$*. If nonzero, its dimension is equal to* $1 + \mathrm{ord}_p(\frac{N}{\mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}})$*, with an orthogonal basis given by*

$$\mathcal{B}_p = \mathcal{B}_p(\chi_1, \chi_2) = \{\phi_i |\ \mathrm{ord}_p(\mathfrak{c}_{\chi_2}) \leq i \leq N_p - \mathrm{ord}_p(\mathfrak{c}_{\chi_1})\}.$$

Proof. The only point remaining is the orthogonality of $\{\phi_i\}$, which follows immediately from the definition of the inner product (5.12) since these functions have disjoint support. □

Tensoring the local spaces together, we have:

Corollary 5.11. *Given that* $\chi_1\chi_2 = \omega$*, the space* $H(\chi_1, \chi_2)^{K_\infty \times K_1(N)}$ *is nonzero if and only if* $\mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}|N$*. If nonzero, its dimension is* $\tau(\frac{N}{\mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}})$ *for the divisor function* τ*, with an orthogonal basis given by*

$$\mathcal{B} = \mathcal{B}(\chi_1, \chi_2) = \{\phi_{(i_p)} = \prod_{p|N} \phi_{i_p} |\ \phi_{i_p} \in \mathcal{B}_p\}.$$

Here we implicitly use the natural identification (5.13)*. The norm of* $\phi_{(i_p)} \in \mathcal{B}$ *is given by*

$$\|\phi_{(i_p)}\|^2 = \prod_{\substack{p|N \\ i_p=0}} \frac{p}{(p+1)} \prod_{\substack{p|N \\ 0<i_p<N_p}} \frac{p-1}{p^{i_p}(p+1)} \prod_{\substack{p|N \\ i_p=N_p}} \frac{1}{p^{N_p-1}(p+1)}. \tag{5.22}$$

Proof. The claim about the dimension follows from the fact that by (5.18) the number of tuples (i_p) is

$$\prod_{p|N}(N_p - \mathrm{ord}_p(\mathfrak{c}_{\chi_1}) - \mathrm{ord}_p(\mathfrak{c}_{\chi_2}) + 1) = \tau(\frac{N}{\mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}}).$$

For the norm, by (5.9) we have

$$\|\phi_{(i_p)}\|^2 = \prod_{p|N} \int_{K_p} |\phi_{i_p}(k)|^2 dk = \prod_{p|N} \mathrm{meas}\left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_p |\ \min(\mathrm{ord}_p(c), N_p) = i_p\right\}.$$

When $i_p = N_p$, the corresponding set is just $K_0(N)_p$, which has measure

$$\frac{1}{\psi_p(N)} = \frac{1}{p^{N_p-1}(p+1)}$$

(cf. [KL2], pp. 206-207). When $0 \leq i_p < N_p$, the corresponding set is equal to $K_0(p^{i_p})_p - K_0(p^{i_p+1})_p$, which has measure $\frac{1}{\psi_p(p^{i_p})} - \frac{1}{\psi_p(p^{i_p+1})}$. This works out to $\frac{p-1}{p^{i_p}(p+1)}$ if $0 < i_p < N_p$ and, using $\psi_p(1) = 1$, $\frac{p}{p+1}$ if $i_p = 0$. □

5.5 Evaluation of the basis elements

Given a basis element $\phi = \phi_{(i_p)}$ of $H(\chi_1, \chi_2)^{K_\infty \times K_1(N)}$, we will need to compute the Fourier expansion of the associated Eisenstein series. From our expression (5.7) for $E_\phi(s, z)$, we see that we need to be able to evaluate $\phi_{\text{fin}}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \text{SL}_2(\mathbf{Z})$.

Proposition 5.12. *Let $p|N$. For a local element $\phi_i \in \mathcal{B}_p$, and $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_p$, $\phi_i(k) = 0$ unless $i = \min(\text{ord}_p(c), N_p)$. If this condition is met, then*

$$\phi_i(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{cases} \chi_2(d) & i = N_p, \\ \chi_1(ad - bc)\overline{\chi_1(\frac{c}{p^i})}\,\chi_2(d) & 0 < i < N_p, \\ \chi_1(ad - bc)\overline{\chi_1(c)} & i = 0. \end{cases}$$

Proof. This follows from the definition (5.11) of $D_p(\chi_1, \chi_2, N)$ and the decompositions (5.14)-(5.16). If $i = N_p$, then by (5.18) $\text{ord}_p(\mathfrak{c}_{\chi_1}) \leq N_p - i = 0$, so χ_1 is unramified at p. Since $N_p > 0$, we see that a must be a unit, and by (5.11),

$$\phi_i(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \chi_1(a)\chi_2(d) = \chi_2(d).$$

If $0 < i < N_p$, then a is a unit and by (5.15) we have

$$\phi_i(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \chi_1(\frac{p^i a}{c})\chi_1(\frac{ad - bc}{a})\chi_2(\frac{ad - bc}{a})$$

$$= \overline{\chi_1(\tfrac{c}{p^i})}\chi_1(ad - bc)\chi_2(d - \tfrac{bc}{a}) = \overline{\chi_1(\tfrac{c}{p^i})}\chi_1(ad - bc)\chi_2(d)$$

since $\frac{bc}{a} \in c\mathbf{Z}_p = p^i\mathbf{Z}_p \subseteq \mathfrak{c}_{\chi_2}\mathbf{Z}_p$. When $i = 0$, we have $\text{ord}_p(\mathfrak{c}_{\chi_2}) = 0$, so χ_2 is unramified at p. Then (5.16) gives

$$\phi_i(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \omega(\frac{ad - bc}{c}) = \chi_1(ad - bc)\overline{\chi_1(c)}. \qquad \square$$

Multiplying these local results together, we have, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \text{SL}_2(\widehat{\mathbf{Z}})$,

$$\phi_{(i_p)}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_{\text{fin}}) = \prod_{\substack{p|N, \\ i_p = 0}} \overline{\chi_{1p}(c)} \prod_{\substack{p|N, \\ 0 < i_p < N_p}} \overline{\chi_{1p}(\tfrac{c}{p^{i_p}})}\chi_{2p}(d) \prod_{\substack{p|N, \\ i_p = N_p}} \chi_{2p}(d) \tag{5.23}$$

under the assumption that $\min(\text{ord}_p(c), N_p) = i_p$ for all p (otherwise the value is 0). We can express this as a product of two Dirichlet characters as follows. Let

$$N_1 \stackrel{\text{def}}{=} \prod_{\substack{p|N, \\ i_p < N_p}} p^{N_p}.$$

Note that $\mathfrak{c}_{\chi_1}|N_1$. Attach to χ_1 a Dirichlet character modulo N_1 by

$$\chi_1'(x) \stackrel{\text{def}}{=} \prod_{p|N_1} \chi_{1p}(x) \qquad (x, N_1) = 1$$

as in (2.8)-(2.9) with N_1 in place of N. We extend χ_1' to $\mathbf{Z}$ in the usual way by taking it to be 0 if $(x, N_1) > 1$. For convenience later, we also set $\chi_1'(x) = 0$ if x is not an integer. Let

$$M \stackrel{\text{def}}{=} \prod_{p|N} p^{i_p}.$$

Then assuming $\phi_{(i_p)}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \neq 0$, we have $M|c$ since $i_p \leq \mathrm{ord}_p(c)$ for all p, and

$$\chi_1'(\frac{c}{M}) = \prod_{p|N_1} \chi_{1p}(\frac{c}{p^{i_p}\frac{M}{p^{i_p}}}) = \prod_{p|N_1} \chi_{1p}(\tfrac{c}{p^{i_p}})\,\overline{\chi_{1p}(\tfrac{M}{p^{i_p}})}.$$

Therefore defining the constant

$$C_{(i_p)} = \prod_{p|N_1} \overline{\chi_{1p}(\tfrac{M}{p^{i_p}})},$$

we have

$$\prod_{p|N_1} \overline{\chi_{1p}(\tfrac{c}{p^{i_p}})} = C_{(i_p)}\overline{\chi_1'(\tfrac{c}{M})}.$$

Similarly, we set

$$N_2 \stackrel{\text{def}}{=} \prod_{\substack{p|N, \\ i_p > 0}} p^{N_p}$$

(the lexical ambiguity between the above definition and $N_2 = \mathrm{ord}_2(N)$ should not cause confusion). Observing that $\mathfrak{c}_{\chi_2}|\,M|\,N_2$, we define a Dirichlet character modulo N_2 by

$$\chi_2'(x) \stackrel{\text{def}}{=} \prod_{p|N_2} \chi_{2p}(x) \qquad (x, N_2) = 1,$$

extending to all of $\mathbf{Z}$ by $\chi_2'(x) = 0$ if $(x, N_2) > 1$. Note that because M and N_2 have the same set of prime divisors and $\mathfrak{c}_{\chi_2}|M$, we have

$$\chi_2'(d + Mx) = \chi_2'(d) \tag{5.24}$$

for all $x \in \mathbf{Z}$. With the above notation, (5.23) becomes

$$\phi_{(i_p)}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{\text{fin}}) = C_{(i_p)}\,\overline{\chi_1'(\tfrac{c}{M})}\,\chi_2'(d). \tag{5.25}$$

In the preceding discussion, (5.25) was established under the assumption that $\min(\mathrm{ord}_p(c), N_p) = i_p$ for all p. However, it actually holds in general:

Proposition 5.13. *Equation* (5.25) *is valid for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$.

Proof. When $\min(\mathrm{ord}_p(c), N_p) \neq i_p$ for some p, the left-hand side of (5.25) is equal to 0. Thus it suffices to show that the same is true of $\chi_1'(\frac{c}{M})$. By definition, $\chi_1'(\frac{c}{M})$ is nonzero if and only if $M|c$ and $\gcd(\frac{c}{M}, N_1) = 1$. This is equivalent to $\mathrm{ord}_p(c) \geq i_p$ for all p and $\mathrm{ord}_p(c) = i_p$ when $i_p < N_p$. These conditions occur precisely when $i_p = \min(\mathrm{ord}_p(c), N_p)$. □

5.6 Fourier expansion of Eisenstein series

For any $\phi \in H(\chi_1, \chi_2)^{K_\infty \times K_1(N)}$, the Eisenstein series $E_\phi(s, z)$ has period one as a function of $z \in \mathbf{H}$. Indeed, writing $z = g_\infty(i)$,

$$E_\phi(s, z+1) = E(\phi, s, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) g_\infty \times 1_{\mathrm{fin}}) = E(\phi, s, g_\infty \times \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)_{\mathrm{fin}}) = E_\phi(s, z),$$

the second equality holding by the left $G(\mathbf{Q})$-invariance of $E(\phi, s, g)$, and the third equality holding by the right $K_1(N)$-invariance of ϕ. It follows that $E_\phi(s, z)$ has a Fourier expansion

$$E_\phi(s, z) = \sum_{m \in \mathbf{Z}} a_m(s, y) e(mx), \tag{5.26}$$

valid when $\mathrm{Re}(s) > 1/2$ by Proposition 5.6. It turns out that the right-hand side also converges for other s, to a meromorphic function continuing $E_\phi(s, z)$. This will be described in the next section. Here we will compute the Fourier coefficients when $\phi = \phi_{(i_p)}$.

Henceforth we fix the tuple $(i_p)_{p|N}$, setting $\phi = \phi_{(i_p)}$ and $M = \prod_{p|N} p^{i_p}$ as before. Assuming $\mathrm{Re}(s) > 1/2$, by (5.7) and (5.25) we have

$$E_\phi(s, z) = y^{1/2+s} C_{(i_p)} \chi_1'(0) + y^{1/2+s} C_{(i_p)} \sum_{c>0} \sum_{\substack{d \in \mathbf{Z} \\ (d,c)=1}} \frac{\overline{\chi_1'(\frac{c}{M})} \chi_2'(d)}{|cz+d|^{1+2s}}.$$

Recall that

$$\chi_1'(0) = \begin{cases} 1 & \text{if } N_1 = 1, \text{ i.e. } i_p = N_p \text{ for all } p|N, \\ 0 & \text{if } N_1 > 1, \text{ i.e. } i_p < N_p \text{ for some } p|N, \end{cases}$$

and $\overline{\chi_1'(c/M)} = 0$ unless $M|c$.

It will be convenient to sum over all $d \in \mathbf{Z}$ rather than the restricted set $(d, c) = 1$. We need the following lemma.

Lemma 5.14. *Suppose* $\gcd(c, d) = n$. *Write* $c = nc'$ *and* $d = nd'$ *for integers* c', d'. *Then*

$$\overline{\chi_1'(c/M)} \chi_2'(d) = \overline{\chi_1'(n)} \chi_2'(n) \overline{\chi_1'(c'/M)} \chi_2'(d'). \tag{5.27}$$

Proof. If $\gcd(n, N_2) > 1$, then $\chi_2'(d) = 0 = \chi_2'(n)$. So equation (5.27) is valid in this case. On the other hand, suppose $(n, N_2) = 1$. Then $(n, M) = 1$ because $M|N_2$. If $M \nmid c$, then both sides of (5.27) vanish. If $M|c$, then $M|c'$, and (5.27) follows by the multiplicativity of Dirichlet characters. □

Using the above lemma, we have

$$\sum_{c>0}\sum_{d\in\mathbf{Z}}\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d)}{|cz+d|^{1+2s}}=\sum_{n>0}\sum_{c\in n\mathbf{Z}^+}\sum_{\substack{d\in\mathbf{Z}\\(d,c)=n}}\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d)}{|cz+d|^{1+2s}}$$

$$=\sum_{n>0}\frac{\overline{\chi_1'(n)}\chi_2'(n)}{n^{1+2s}}\sum_{c>0}\sum_{\substack{d\in\mathbf{Z}\\(d,c)=1}}\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d)}{|cz+d|^{1+2s}}$$

$$=L_N(1+2s,\chi_1\overline{\chi_2})\sum_{c>0}\sum_{\substack{d\in\mathbf{Z}\\(d,c)=1}}\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d)}{|cz+d|^{1+2s}}.$$

Here we have applied (2.10), using the fact that $\overline{\chi_1'}\chi_2'$ has modulus $\mathrm{lcm}(N_1,N_2)=N$. The above has period one as a function of z. This can be seen from the the fact that the Eisenstein series has period one, or it can be seen directly as follows:

$$\sum_{c>0}\sum_{d\in\mathbf{Z}}\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d)}{|cz+c+d|^{1+2s}}=\sum_{c>0}\sum_{d\in\mathbf{Z}}\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d-c)}{|cz+d|^{1+2s}}.$$

The summand vanishes unless $M|c$. Therefore by (5.24), $\chi_2'(d-c)=\chi_2'(d)$ in all nonzero terms, as needed. By this periodicity, the double sum has a Fourier expansion

$$\sum_{c>0}\sum_{d\in\mathbf{Z}}\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d)}{|cz+d|^{1+2s}}=\sum_{m\in\mathbf{Z}}b_m(s,y)e(mx).$$

The coefficient $b_m(s,y)$ is related to $a_m(s,y)$ of (5.26) since

$$E_\phi(s,z)=y^{1/2+s}C_{(i_p)}\chi_1'(0)+\frac{y^{1/2+s}C_{(i_p)}}{L_N(1+2s,\chi_1\overline{\chi_2})}\sum_{c>0}\sum_{d\in\mathbf{Z}}\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d)}{|cz+d|^{1+2s}}.\quad(5.28)$$

Explicitly, for $\mathrm{Re}(s)>1/2$ we have

$$a_m(s,y)=\begin{cases}y^{1/2+s}C_{(i_p)}\chi_1'(0)+y^{1/2+s}C_{(i_p)}L_N(1+2s,\chi_1\overline{\chi_2})^{-1}b_0(s,y) & \text{if } m=0,\\ y^{1/2+s}C_{(i_p)}L_N(1+2s,\chi_1\overline{\chi_2})^{-1}b_m(s,y) & \text{if } m\neq 0.\end{cases}$$

We now compute the coefficients $b_m(s,y)$. We have

$$b_m(s,y)=\sum_{c>0}\sum_{d\in\mathbf{Z}}\int_0^1\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d)}{|cz+d|^{1+2s}}e(-mx)dx$$

$$=\sum_{c>0}\sum_{d\in\mathbf{Z}/c\mathbf{Z}}\int_0^1\sum_{t\in\mathbf{Z}}\frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d+ct)}{|cz+d+ct|^{1+2s}}e(-mx)dx.$$

As before, the integrand is nonzero only if $M|c$, and under this assumption $\chi_2'(d+ct) = \chi_2'(d)$ by (5.24). Therefore the above is

$$= \sum_{c>0} \sum_{d \in \mathbf{Z}/c\mathbf{Z}} \int_{-\infty}^{\infty} \frac{\overline{\chi_1'(\frac{c}{M})}\chi_2'(d)}{|cz+d|^{1+2s}} e(-mx)dx$$

$$= \sum_{c>0} \frac{\overline{\chi_1'(\frac{c}{M})} \sum_{d \in \mathbf{Z}/c\mathbf{Z}} \chi_2'(d)}{c^{1+2s}} \int_{-\infty}^{\infty} \frac{e(-mx)}{|z+\frac{d}{c}|^{1+2s}} dx$$

$$= \sum_{c>0} \frac{\overline{\chi_1'(\frac{c}{M})} \sum_{d \in \mathbf{Z}/c\mathbf{Z}} \chi_2'(d)}{c^{1+2s}} \int_{-\infty}^{\infty} \frac{e(-m(x-\frac{d}{c}))}{(x^2+y^2)^{1/2+s}} dx$$

$$= \sum_{c \in M\mathbf{Z}^+} \frac{\overline{\chi_1'(\frac{c}{M})}}{c^{1+2s}} \left(\sum_{d \in \mathbf{Z}/c\mathbf{Z}} \chi_2'(d) e(\frac{dm}{c}) \right) \int_{-\infty}^{\infty} \frac{e(-mx)}{(x^2+y^2)^{1/2+s}} dx. \tag{5.29}$$

Now apply the well-known formula:

$$\int_{-\infty}^{\infty} \frac{e(-mx)}{(x^2+y^2)^{1/2+s}} dx = \begin{cases} \dfrac{2\pi^{1/2+s}|y|^{-s}|m|^s}{\Gamma(\frac{1}{2}+s)} K_s(2\pi|m||y|) & m \neq 0, \\ \\ \dfrac{\sqrt{\pi} y^{-2s}\Gamma(s)}{\Gamma(\frac{1}{2}+s)} & m = 0 \end{cases}$$

([Bu], p. 67). By (5.24), the character sum S in parentheses in (5.29) satisfies

$$S = \sum_{d \in \mathbf{Z}/c\mathbf{Z}} \chi_2'(d-M) e(\frac{dm}{c}) = \sum_{d \in \mathbf{Z}/c\mathbf{Z}} \chi_2'(d)\, e\Big(\frac{(d+M)m}{c}\Big) = e(\frac{mM}{c}) S. \tag{5.30}$$

Hence if $e(\frac{mM}{c}) \neq 1$ (or equivalently $c \nmid mM$), then $S = 0$. Therefore if $\mathrm{Re}(s) > 1/2$, the Fourier coefficient is given by

$$b_m(s,y) = \begin{cases} \dfrac{2\pi^{1/2+s} y^{-s}|m|^s}{\Gamma(\frac{1}{2}+s)} \sigma_s(\chi_1', \chi_2', m) K_s(2\pi|m|y) & m \neq 0, \\ \\ \dfrac{\sqrt{\pi} y^{-2s}\Gamma(s)}{\Gamma(\frac{1}{2}+s)} \sigma_s(\chi_1', \chi_2', 0) & m = 0, \end{cases}$$

for the sum (see also §5.8)

$$\sigma_s(\chi_1', \chi_2', m) = \sum_{\substack{c \in M\mathbf{Z}^+ \\ c|mM}} \frac{\overline{\chi_1'(\frac{c}{M})}}{c^{1+2s}} \sum_{d \in \mathbf{Z}/c\mathbf{Z}} \chi_2'(d)\, e(\frac{dm}{c}) \tag{5.31}$$

$$= \sum_{c|m} \frac{\overline{\chi_1'(c)}}{(Mc)^{1+2s}} \sum_{d \in \mathbf{Z}/Mc\mathbf{Z}} \chi_2'(d) e(\frac{dm}{Mc}).$$

In the second sum, each summand is defined for $d \bmod M\mathbf{Z}$, since M is a modulus for χ_2' and $e(\frac{(d+M)m}{Mc}) = e(\frac{dm}{Mc})$ since $c|m$. Thus

$$\sigma_s(\chi_1', \chi_2', m) = \frac{1}{M^{1+2s}} \sum_{c|m} \frac{\overline{\chi_1'(c)}}{c^{2s}} \sum_{d \in \mathbf{Z}/M\mathbf{Z}} \chi_2'(d)\, e(\frac{dm}{Mc}). \tag{5.32}$$

We emphasize that even though $m < 0$ is allowed, the sum is extended only over the *positive* divisors c of m. When $m \neq 0$, the sum is finite. However, when $m = 0$, the sum is extended over all $c \in \mathbf{Z}^+$, and only converges absolutely for $\mathrm{Re}(s) > 1/2$. Indeed we have the following.

Proposition 5.15. *When $m = 0$,*

$$\sigma_s(\chi_1', \chi_2', 0) = \begin{cases} \frac{\varphi(M)}{M^{1+2s}} L_{N_1}(2s, \omega) & \text{if } \chi_2 \text{ is trivial,} \\ 0 & \text{otherwise,} \end{cases}$$

where φ is the Euler φ-function.

Proof. By (5.32),

$$\sigma_s(\chi_1', \chi_2', 0) = \frac{1}{M^{1+2s}} \sum_{c=1}^{\infty} \frac{\overline{\chi_1'(c)}}{c^{2s}} \sum_{d \in \mathbf{Z}/M\mathbf{Z}} \chi_2'(d).$$

The sum over d vanishes unless χ_2' is the principal character modulo M. Indeed,

$$\sum_{d=1}^{M} \chi_2'(d) = \sum_{d \in (\mathbf{Z}/M\mathbf{Z})^*} \chi_2'(d) = \begin{cases} \varphi(M) & \text{if } \chi_2' \text{ is principal} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore if χ_2' is principal (in which case χ_2 is trivial by (2.8) with N_2 in place of N), we find

$$\sigma_s(\chi_1', \chi_2', 0) = \frac{\varphi(M)}{M^{2s+1}} L(2s, \overline{\chi_1'}).$$

Applying (2.10), the proposition follows since $\chi_1 = \omega$ in this case. □

5.7 Meromorphic continuation

To summarize the previous section, for the scaled basis element

$$\phi = \tfrac{1}{C_{(i_p)}} \phi_{(i_p)} \in H(\chi_1, \chi_2)^{K_\infty \times K_1(N)},$$

we have, for $\mathrm{Re}(s) > 1/2$,

$$\begin{aligned} E_\phi(s, z) &= y^{1/2+s} \chi_1'(0) + y^{1/2+s} \sum_{c>0} \sum_{(d,c)=1} \frac{\overline{\chi_1'(\frac{c}{M})} \chi_2'(d)}{|cz+d|^{1+2s}} \\ &= y^{1/2+s} \chi_1'(0) + y^{1/2-s} \delta_{\chi_2} \frac{\varphi(M) \sqrt{\pi}\, \Gamma(s)\, L_{N_1}(2s, \omega)}{M^{1+2s}\, \Gamma(\frac{1}{2}+s) L_N(1+2s, \omega)} \end{aligned} \tag{5.33}$$

$$+\frac{2y^{1/2}\pi^{1/2+s}}{\Gamma(\frac{1}{2}+s)L_N(1+2s,\chi_1\overline{\chi_2})}\sum_{m\neq 0}|m|^s\sigma_s(\chi_1',\chi_2',m)K_s(2\pi|m|y)e(mx). \quad (5.34)$$

Here $\delta_{\chi_2}\in\{0,1\}$ is nonzero if and only if χ_2 is the trivial character.

Theorem 5.16. *The Fourier expansion* (5.33)-(5.34) *defines a meromorphic function on* $\mathbf{C}$ *which continues* $E_\phi(s,z)$*. It is holomorphic in the half-plane* $\mathrm{Re}(s)\geq 0$*, except possibly for a simple pole at* $s=1/2$ *which occurs precisely when* χ_1 *and* χ_2 *are both trivial. In the event of a pole, its residue is*

$$\frac{3\varphi(M)}{\pi M^2}\prod_{\substack{p|N\\ i_p=N_p}}(1-p^{-2})^{-1}\prod_{\substack{p|N\\ i_p<N_p}}(1+p^{-1})^{-1}$$

for the Euler φ*-function.*

Proof. From the meromorphic continuation of Dirichlet L-functions, we see that the constant term (5.33) is meromorphic. Since $\omega'(-1)=1$, the completed L-function of ω has the form

$$\Lambda(2s,\omega)=\pi^{-s}\,\Gamma(s)L(2s,\omega)$$

and is entire unless $\omega=triv$ is the trivial character, in which case it has simple poles at $s=0,\frac{1}{2}$ ([Bu] Theorem 1.1.1). Therefore

$$\Gamma(s)L_{N_1}(2s,\omega)=\pi^s\Lambda(2s,\omega)\prod_{\substack{p|N_1\\ p\nmid c_\omega}}(1-\omega_p(p)p^{-2s})$$

is entire unless $\omega=triv$, in which case it has a simple pole at $s=1/2$ and possibly (if $N_1=1$) a simple pole at $s=0$. This possible pole at $s=0$ is cancelled by the simple pole of $L_N(1+2s,triv)$ at $s=0$ occurring in the denominator when ω is trivial. Recall also that in general $\Gamma(\frac{1}{2}+s)L_N(1+2s,\omega)$ is nonzero when $\mathrm{Re}(s)\geq 0$. This shows that (5.33) has the desired properties, as does the first factor of (5.34).

It remains to consider the sum in (5.34). From (5.32), for $m\neq 0$ we have

$$|\sigma_s(\chi_1',\chi_2',m)|\leq\frac{1}{M^{1+2\,\mathrm{Re}(s)}}\sum_{c|m}\frac{1}{c^{2\,\mathrm{Re}(s)}}\Big(\sum_{d\in\mathbf{Z}/M\mathbf{Z}}1\Big)=\frac{1}{M^{2\,\mathrm{Re}(s)}}\sum_{c|m}\frac{1}{c^{2\,\mathrm{Re}(s)}}.$$

When $\mathrm{Re}(s)\geq 0$, this is

$$\leq M^{-2\,\mathrm{Re}(s)}\tau(m)\ll|m|^\varepsilon,$$

while if $\mathrm{Re}(s)\leq 0$ it is

$$\leq(|m|M)^{-2\,\mathrm{Re}(s)}\tau(m)\ll|m|^{2|\mathrm{Re}(s)|+\varepsilon}.$$

Here as usual $\tau(d)$ denotes the number of positive divisors of d, and is well known to be $\ll |d|^\varepsilon$. Furthermore, the Bessel function decays exponentially. In fact, for real $x > 1 + |s|^2$,

$$K_s(x) = \sqrt{\frac{\pi}{2x}} e^{-x}(1 + O\left(\frac{|s|^2+1}{x}\right))$$

for an absolute implied constant ([Wa], p. 219, [Iw2], p. 204). Now suppose s and y are restricted to fixed compact subsets of $\mathbf{C}$ and $\mathbf{R}^+$ respectively. Then by the above, there exists a constant C, depending only on the two compact sets, such that $|K_s(2\pi|m|y)| \leq Ce^{-2\pi|m|y}$ for all m. It follows that the sum in (5.34) converges uniformly on compact sets, so the sum is entire.

In the event of a pole at $s = 1/2$, the singular part of the Eisenstein series is the term

$$y^{1/2-s}\frac{\varphi(M)}{M^{1+2s}}\frac{\Lambda(2s, triv)\prod_{p|N_1}(1-p^{-2s})}{\Lambda(1+2s, triv)\prod_{p|N}(1-p^{-1-2s})}.$$

The formula for the residue follows since $\Lambda(2s, triv)$ has residue $\frac{1}{2}$ at $s = 1/2$, while in the denominator $\Lambda(2, triv) = \pi^{-1}\zeta(2) = \frac{\pi}{6}$. □

5.8 Character sums

In order to prove Theorem 10.2, we will need good bounds for the Fourier coefficients of normalized Eisenstein series. For this purpose we now examine more closely the character sums occurring there.

Let χ be a Dirichlet character mod M of conductor $\mathfrak{c}_\chi | M$. For a prime $p|M$, we define a Dirichlet character χ_p modulo p^{M_p} by

$$\chi_p(d) = \begin{cases} 0 & \text{if } p|d \\ \chi(x) & \text{if } p \nmid d, \text{ where } x \equiv d \bmod p^{M_p}, x \equiv 1 \bmod q^{M_q}\ (q \neq p). \end{cases} \tag{5.35}$$

The value $\chi_p(d)$ is independent of both the choice of x and the choice of modulus $M \in \mathfrak{c}_\chi\mathbf{Z}^+ \cap p\mathbf{Z}$. With the above definition, we have $\chi = \prod_{p|M}\chi_p$. If we take $\chi_p = \mathbf{1}$ to be the constant function 1 on $\mathbf{Z}$ when $p \nmid M$, then the product can be extended over all primes p.

We review some well-known facts about Gauss sums. For χ as above, define

$$G_\chi(m) = \sum_{d \in \mathbf{Z}/M\mathbf{Z}} \chi(d)e(\tfrac{dm}{M}).$$

Assuming that either $(m, M) = 1$ or χ is primitive, we have

$$G_\chi(m) = \tau(\chi)\overline{\chi(m)}, \tag{5.36}$$

where $\tau(\chi) = G_\chi(1)$ ([IK], §3.4). In general, suppose χ^0 is the primitive character inducing χ, and write $M = \ell\mathfrak{c}_\chi$. Then ([Mi], Lemma 3.1.3)

$$G_\chi(m) = \tau(\chi^0)\sum_{a|(\ell,m)} a\,\mu(\ell/a)\chi^0(\ell/a)\overline{\chi^0(m/a)} \tag{5.37}$$

for the Möbius function

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 \cdots p_r \text{ for distinct primes } p_1, \dots, p_r, \\ 0 & \text{if } n \text{ has a square factor } > 1. \end{cases}$$

It is well-known that

$$|\tau(\chi^0)| = \mathfrak{c}_\chi^{1/2}. \tag{5.38}$$

Therefore (5.37) gives

$$|G_\chi(m)| \le \mathfrak{c}_\chi^{1/2} \sigma(|m|), \tag{5.39}$$

where, for $k > 0$, $\sigma(k) = \sum_{d|k, d>0} d$.

Proposition 5.17. *Let m_1, m_2 be nonzero integers. Then*

$$\|\phi_{(i_p)}\|^{-2} \left| \sigma_{it}(\chi_1', \chi_2', m_1) \overline{\sigma_{it}(\chi_1', \chi_2', m_2)} \right| = O(N^\varepsilon),$$

where the implied constant depends only on m_1, m_2 and ε.

Proof. Write $m = m_1$ or m_2. From (5.32),

$$\sigma_s(\chi_1', \chi_2', m) = \frac{1}{M^{1+2s}} \sum_{c|m} \frac{\overline{\chi_1'(c)}}{c^{2s}} G_{\chi_2'}(m/c),$$

where $M = \prod p^{i_p}$. Applying (5.39), this gives

$$|\sigma_{it}(\chi_1', \chi_2', m)| \le \frac{\mathfrak{c}_{\chi_2}^{1/2}}{M} \sum_{c|m} \sigma(c) \le \frac{\mathfrak{c}_{\chi_2}^{1/2}}{M} \tau(|m|)\sigma(|m|). \tag{5.40}$$

By (5.22), we have

$$\|\phi_{(i_p)}\|^{-2} = \prod_{\substack{p|N \\ i_p=0}} (1 + \frac{1}{p}) \prod_{\substack{p|N \\ 0<i_p<N_p}} p^{i_p}(1 + \frac{2}{p-1}) \prod_{\substack{p|N \\ i_p=N_p}} p^{i_p}(1 + \frac{1}{p}).$$

Therefore

$$\|\phi_{(i_p)}\|^{-2} \le M \prod_{p|N} (1 + \frac{2}{p-1}). \tag{5.41}$$

Together, these bounds give

$$\begin{aligned} \frac{|\sigma_{it}(\chi_1', \chi_2', m)|}{\|\phi_{(i_p)}\|} &\le \frac{\mathfrak{c}_{\chi_2}^{1/2} M^{1/2}}{M} \tau(|m|)\sigma(|m|) \prod_{p|N} (1 + \frac{2}{p-1})^{1/2} \\ &\ll_\varepsilon \tau(|m|)\sigma(|m|) N^{\varepsilon/2}. \end{aligned} \tag{5.42}$$

The proposition follows. $\square$

6 The kernel of $R(f)$

In this section we give the spectral formula for the kernel function of $R(f)$. We refer to [Ar1], [GJ] and [Kn] for further discussion and theoretical background. Our purpose is to show that these spectral terms converge absolutely in a strong sense (Theorems 6.10 and 6.11). This provides the justification for their use in the relative trace formula. Our treatment is based on the methods of Arthur [Ar1], [Ar2], especially Lemma 4.4 of [Ar2]. His result holds for any connected reductive algebraic group over $\mathbf{Q}$. In the setting of GL(2) it gives e.g. that for an orthonormal basis $\{\phi\} \subseteq H(0)$ (cf. (6.1)),

$$\int_{-\infty}^{\infty} \left| \sum_{\phi} E(\pi_{it}(f)\phi_{it}, x)\overline{E(\phi_{it}, y)} \right| dt < \infty.$$

We give a detailed discussion here partly to avoid referring the reader to a paper on general groups just for a result about GL(2), but also because we need to show that the absolute values can be brought inside the sum, at least for the class of functions f considered in this paper.

6.1 The spectral decomposition

The right regular representation of $G(\mathbf{A})$ on $L^2(\omega)$ decomposes in terms of cuspidal representations on GL(m) for $m \leq 2$. The continuous part of $L^2(\omega)$ is indexed by certain cuspidal representations of GL(1), i.e. Hecke characters, and the discrete part consists of irreducible cuspidal representations and, in some situations, one-dimensional representations.

Suppose χ is a Hecke character satisfying $\chi^2 = \omega$. Then defining

$$\phi_\chi(g) = \chi(\det(g)) \qquad (g \in G(\mathbf{A})),$$

we see that ϕ_χ is square integrable modulo $Z(\mathbf{A})$, with

$$\|\phi_\chi\|^2 = \text{meas}(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})) = \pi/3.$$

Note that $\phi_\chi(zg) = \omega(z)\phi_\chi(g)$ for all $z \in Z(\mathbf{A})$. Therefore ϕ_χ spans a one-dimensional subrepresentation of $L^2(\omega)$, which we denote by $\mathbf{C}_\chi$. Conversely, any one-dimensional subrepresentation of $L^2(\omega)$ arises in this way from a character satisfying $\chi^2 = \omega$.

Proposition 6.1. *The spaces $\mathbf{C}_\chi$ are mutually orthogonal, and also orthogonal to $L^2_0(\omega)$.*

Proof. Suppose V is a unitary representation of a group G. Then for any closed G-stable subspace S, the action of G preserves the decomposition $V = S \oplus S^\perp$. If W is any other closed G-stable subspace of V, then it is easy to show that $W = (W \cap S) \oplus (W \cap S^\perp)$. In particular if W is one-dimensional, then $W \subseteq S$ or $W \subseteq S^\perp$. Applying this with $S = \mathbf{C}_{\chi_1}$ and $W = \mathbf{C}_{\chi_2}$ shows the first claim, and taking $S = L^2_0(\omega)$ gives the second. Note that ϕ_χ is not cuspidal because its constant term is

$$\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi_\chi(ng)dn = \phi_\chi(g) \neq 0. \qquad \square$$

We denote by $L^2_{\mathrm{res}}(\omega)$ the Hilbert direct sum

$$L^2_{\mathrm{res}}(\omega) = \bigoplus_{\chi^2=\omega} \mathbf{C}_\chi.$$

(These characters arise from the residues of certain Eisenstein series at $s = 1/2$). If $L^2_{\mathrm{res}}(\omega)$ is nonzero, then it is infinite dimensional. To see this, note that if there exists χ with $\chi^2 = \omega$, then $L^2_{\mathrm{res}}(\omega) = \bigoplus_{\eta^2=1} \mathbf{C}_{\chi\eta}$. There are infinitely many quadratic Hecke characters η.

The direct sum

$$L^2_{\mathrm{disc}}(\omega) \stackrel{\mathrm{def}}{=} L^2_0(\omega) \oplus L^2_{\mathrm{res}}(\omega)$$

is the discrete part of the spectrum of $L^2(\omega)$. We next describe its orthogonal complement $L^2_{\mathrm{cont}}(\omega)$. For $s \in \mathbf{C}$ define

$$H(s) = \bigoplus_{\chi_1\chi_2=\omega} H(\chi_1, \chi_2, s), \tag{6.1}$$

where $H(\chi_1, \chi_2, s)$ is defined in §5.1, and this Hilbert space direct sum is taken over all ordered pairs of finite order Hecke characters whose product is ω.

Remark: Define a character (ω, s) of $B' \stackrel{\mathrm{def}}{=} Z(\mathbf{A})N(\mathbf{A})M(\mathbf{Q})M(\mathbf{R}^+)$ by

$$(\omega, s) : \left(\begin{smallmatrix} z & \\ & z \end{smallmatrix}\right)\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto \omega(z)|y|^{s+1/2}.$$

The right regular representation π_s of $G(\mathbf{A})$ on $H(s)$ is equivalent to the induced representation $\mathrm{Ind}_{B'}^{G(\mathbf{A})}(\omega, s)$. This latter viewpoint is the one taken in [GJ]. To see the equivalence, note by transitivity of induction that

$$\mathrm{Ind}_{B'}^{G(\mathbf{A})}(\omega, s) = \mathrm{Ind}_{B(\mathbf{A})}^{G(\mathbf{A})} \mathrm{Ind}_{B'}^{B(\mathbf{A})}(\omega, s). \tag{6.2}$$

By restriction, $\mathrm{Ind}_{B'}^{B(\mathbf{A})}(\omega, s)$ can be identified with $\mathrm{Ind}_{Z(\mathbf{A})M(\mathbf{Q})M(\mathbf{R}^+)}^{M(\mathbf{A})}(\omega, s)$. The quotient $(Z(\mathbf{A})M(\mathbf{Q})M(\mathbf{R}^+))\backslash M(\mathbf{A})$ is compact, so by the Peter-Weyl theorem, the functions $(\chi_1, \chi_2, s) = \chi_1(a)\chi_2(d)|\frac{a}{d}|^{s+\frac{1}{2}}$ with $\chi_1\chi_2 = \omega$ form a basis for the induced space. Thus the right-hand side of (6.2) is equal to $\mathrm{Ind}_{B(\mathbf{A})}^{G(\mathbf{A})} \bigoplus(\chi_1, \chi_2, s) = H(s)$.

Let

$$H = \int_{-\infty}^{\infty} H(it)dt$$

be the direct integral. This Hilbert space consists of all functions

$$A : i\mathbf{R} \longrightarrow \bigcup_{t \in \mathbf{R}} H(it) \qquad \text{(disjoint union)}$$

(identifying functions that are equal a.e.) satisfying:

- $A(it) \in H(it)$ for all t,
- the composition $i\mathbf{R} \xrightarrow{A} \bigcup H(it) \longrightarrow H(0)$, obtained by identifying $H(it)$ with $H(0)$ as in §5.1, is measurable,
- $\|A\|^2 \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{-\infty}^{\infty} \|A(it)\|^2 dt < \infty$.

The associated inner product is given by

$$\langle A, B \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \langle A(it), B(it) \rangle \, dt. \tag{6.3}$$

Define an action of $G(\mathbf{A})$ on H by

$$(gA)(it) = \pi_{it}(g) A(it).$$

This representation is unitary since each π_{it} is unitary:

$$\|gA\|^2 = \frac{1}{\pi} \int_{-\infty}^{\infty} \|\pi_{it}(g) A(it)\|^2 dt = \|A\|^2.$$

By §4 of [GJ], there is a $G(\mathbf{A})$-equivariant Hilbert space isomorphism[3]

$$M(it) : H(it) \longrightarrow H(-it).$$

Thus the subspace

$$\mathcal{L} = \{A \in H \,|\, A(-it) = M(it) A(it) \text{ for all } t \in \mathbf{R}\} \tag{6.4}$$

is stable under the action of $G(\mathbf{A})$. It is this representation which is isomorphic to $L^2_{\text{cont}}(\omega)$:

Theorem 6.2. *Consider the orthogonal decomposition*

$$L^2(\omega) = L^2_{\text{disc}}(\omega) \oplus L^2_{\text{cont}}(\omega).$$

There is a $G(\mathbf{A})$-equivariant isomorphism of Hilbert spaces

$$S : L^2_{\text{cont}}(\omega) \longrightarrow \mathcal{L},$$

[3]By an isomorphism of Hilbert spaces, we mean a bijective linear isometry.

which we extend to the full space $L^2(\omega)$ by taking $S = 0$ on $L^2_{\rm disc}(\omega)$, characterized by the property that for any $\psi \in L^2(\omega)$ and $\phi \in H(0)$,

$$\langle S\psi(it), \phi_{it}\rangle = \frac{1}{2}\,\langle \psi, E(\phi, it, \cdot)\rangle = \frac{1}{2}\int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \psi(g)\overline{E(\phi, it, g)}\,dg \tag{6.5}$$

for almost all t. The following Parseval identity holds for $\psi, \eta \in L^2(\omega)$:

$$\begin{aligned}\langle \psi, \eta\rangle = &\sum_{\varphi} \langle \psi, \varphi\rangle\,\overline{\langle \eta, \varphi\rangle} + \frac{3}{\pi}\sum_{\chi^2=\omega} \langle \psi, \phi_\chi\rangle\,\overline{\langle \eta, \phi_\chi\rangle} \\ &+ \frac{1}{4\pi}\sum_{\phi}\int_{-\infty}^{\infty} \langle \psi, E(\phi, it, \cdot)\rangle\,\overline{\langle \eta, E(\phi, it, \cdot)\rangle}dt.\end{aligned} \tag{6.6}$$

Here, φ (resp. ϕ) runs through an orthonormal basis for $L^2_0(\omega)$ (resp. $H(0)$).

Remarks: The fact that S is an intertwining operator can be seen from (6.5). Indeed, for any $\phi \in H(0)$,

$$\langle SR(g)\psi(it), \phi_{it}\rangle = \frac{1}{2}\,\langle R(g)\psi, E(\phi_{it}, \cdot)\rangle = \frac{1}{2}\,\left\langle \psi, R(g^{-1})E(\phi_{it}, \cdot)\right\rangle$$

$$= \frac{1}{2}\,\left\langle \psi, E(\pi_{it}(g^{-1})\phi_{it}, \cdot)\right\rangle = \left\langle S\psi, \pi_{it}(g^{-1})\phi_{it}\right\rangle = \langle \pi_{it}(g)S\psi(it), \phi_{it}\rangle$$

as claimed. Passing to the second line, we used $R(g)E(\phi_s, x) = E(\pi_s(g)\phi_s, x)$, which is clear when $\mathrm{Re}(s) > 1/2$ and holds for $\mathrm{Re}(s) = 0$ by analytic continuation. In a similar fashion, we can derive the useful identity

$$\langle \psi, E(\pi_{it}(f)\phi_{it}, \cdot)\rangle = \langle R(f)^*\psi, E(\phi_{it}, \cdot)\rangle \tag{6.7}$$

for $\psi \in L^2(\omega)$ and $f \in L^1(\overline{\omega})$.

Proof. See §4 of [GJ] for an explicit construction of S. The identity (6.5) is their (5.16). We just explain how to derive the Parseval identity from their discussion. Let $P_{\rm disc}$ (resp. $P_{\rm cont}$) be the orthogonal projection of $L^2(\omega)$ onto $L^2_{\rm disc}(\omega)$ (resp. $L^2_{\rm cont}(\omega)$). Then

$$\langle \psi, \eta\rangle = \langle P_{\rm disc}\psi, P_{\rm disc}\eta\rangle + \langle P_{\rm cont}\psi, P_{\rm cont}\eta\rangle\,.$$

We apply the usual Parseval identity in $L^2_{\rm disc}(\omega)$ to obtain the discrete part of (6.6). For the continuous part, by (6.3) we have

$$\langle P_{\rm cont}\psi, P_{\rm cont}\eta\rangle = \langle S\psi, S\eta\rangle = \frac{1}{\pi}\int_{-\infty}^{\infty} \langle S\psi(it), S\eta(it)\rangle\,dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\sum_{\phi} \langle S\psi(it), \phi_{it}\rangle\,\overline{\langle S\eta(it), \phi_{it}\rangle}\,dt.$$

Here ϕ runs through an orthonormal basis for $H(0)$, and we have applied Parseval's identity in $H(it)$. We pull the sum out (justification given below) and apply (6.5) to get

$$\langle P_{\mathrm{cont}}\psi, P_{\mathrm{cont}}\eta\rangle = \frac{1}{4\pi}\sum_{\phi}\int_{-\infty}^{\infty} \langle \psi, E(\phi, it, \cdot)\rangle\, \overline{\langle \eta, E(\phi, it, \cdot)\rangle}\, dt, \tag{6.8}$$

which gives (6.6). To justify pulling out the sum, we need to show convergence of

$$\int_{-\infty}^{\infty}\sum_{\phi} |\, \langle S\psi(it), \phi_{it}\rangle\, \overline{\langle S\eta(it), \phi_{it}\rangle}|\, dt.$$

Applying Cauchy-Schwarz to the sum, the above is

$$\le \int_{-\infty}^{\infty}\Big(\sum_{\phi} |\, \langle S\psi(it), \phi_{it}\rangle\, |^2\Big)^{1/2}\Big(\sum_{\phi} |\, \langle S\eta(it), \phi_{it}\rangle\, |^2\Big)^{1/2} dt$$

$$= \int_{-\infty}^{\infty} \|S\psi(it)\|\, \|S\eta(it)\|\, dt \quad \text{(Parseval's)}$$

$$\le \left[\int_{-\infty}^{\infty} \|S\psi(it)\|^2 dt\right]^{1/2} \left[\int_{-\infty}^{\infty} \|S\eta(it)\|^2 dt\right]^{1/2} < \infty \quad \text{(Cauchy-Schwarz)}. \quad \square$$

6.2 Kernel functions

Suppose X is a Radon measure space, and T is a bounded linear operator on $L^2(X)$. We say that a measurable function $K(x, y)$ on $X \times X$ is a **kernel function** for T if $T = T_K$, where

$$T_K\psi(x) \stackrel{\mathrm{def}}{=} \int_X K(x, y)\, \psi(y)\, dy.$$

If the equality $T\psi = T_K\psi$ is only known to hold for all ψ which are bounded and of compact support, then we say that $K(x, y)$ is a **weak kernel** for T. We shall repeatedly use the fact that K is a weak kernel for T if and only if $\langle T_K\psi_1, \psi_2\rangle = \langle T\psi_1, \psi_2\rangle$ for all ψ_1, ψ_2 which are bounded and compactly supported.

Lemma 6.3. *If $K(x, y)$ and $K'(x, y)$ are weak kernel functions for T, then $K(x, y) = K'(x, y)$ for almost all $(x, y) \in X \times X$.*

Proof. This is straightforward; see the proof of Proposition 15.1 of [KL2]. $\square$

Given an operator T on a Hilbert space, its **Hilbert-Schmidt norm** is defined by

$$\|T\|_{HS}^2 = \sum \|Te_i\|^2,$$

where $\{e_i\}$ is an orthonormal basis for the space. If the norm is finite, then it is independent of the choice of basis, and we say that T is a **Hilbert-Schmidt operator**. It is well-known that an operator T on $L^2(X)$ is Hilbert-Schmidt if and only if it has a square integrable kernel $K(x,y) \in L^2(X \times X)$ ([RS], Theorem VI.23). In this situation, if we let $\{\psi\}$ and $\{\phi\}$ be orthonormal bases for $L^2(X)$, then $\{\psi \otimes \overline{\phi}\}$ is an orthonormal basis for $L^2(X \times X)$ ([RS], p. 51) and for almost all (x,y) we have

$$K(x,y) = \sum_{\psi,\phi} \langle K, \psi \otimes \overline{\phi} \rangle \, \psi(x)\overline{\phi(y)} = \sum_{\phi}\Big(\sum_{\psi} \langle T\phi, \psi\rangle \, \psi(x)\Big)\overline{\phi(y)}$$
$$= \sum_{\varphi} T\phi(x)\overline{\phi(y)}. \tag{6.9}$$

For an integer $m \geq 0$, let $C_c^m(G(\mathbf{A}), \overline{\omega})$ denote the space of factorizable functions $f = f_\infty \prod_p f_p$ on $G(\mathbf{A})$ with the following properties:

- f has compact support mod $Z(\mathbf{A})$
- f transforms under $Z(\mathbf{A})$ by $\overline{\omega}$
- f_∞ is m-times continuously differentiable on $G(\mathbf{R})$
- Each f_p is locally constant, and for almost all p, f_p is the function supported on $Z(\mathbf{Q}_p)K_p$ defined by $f_p(zk) = \overline{\omega_p(z)}$.

Theorem 6.4. *Suppose $m \geq 3$. Then for any $f \in C_c^m(G(\mathbf{A}), \overline{\omega})$, the operator $R_0(f)$ on $L_0^2(\omega)$ is Hilbert-Schmidt. When f_∞ is bi-K_∞-invariant, $m \geq 2$ suffices.*

Proof. In the case of interest to us here, where f_∞ is bi-K_∞-invariant, we will prove that $m \geq 2$ suffices in Corollary 8.32 later on, as a consequence of a more general result where we allow f_∞ to have noncompact support. For the general case of $f \in C_c^m(G(\mathbf{A}), \overline{\omega})$, see Theorem 2.1 of [GJ] for a sketch over the adeles, and [Bu] or §3 of [Kn] for proofs over $G(\mathbf{R})^+$. As can be seen from the proof in [Kn], $m = 3$ suffices. □

Let $f \in L^1(\overline{\omega})$. Then for all $\psi \in L^2(\omega)$, we have

$$R(f)\psi(x) = \int_{\overline{G}(\mathbf{A})} f(y)\psi(xy)dy = \int_{\overline{G}(\mathbf{A})} f(x^{-1}y)\psi(y)dy$$
$$= \int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} K(x,y)\psi(y)dy$$

for the kernel function

$$K(x,y) = K_f(x,y) = \sum_{\gamma \in \overline{G}(\mathbf{Q})} f(x^{-1}\gamma y). \tag{6.10}$$

If f is continuous and compactly supported modulo the center, then because $\overline{G}(\mathbf{Q})$ is a discrete subset of $\overline{G}(\mathbf{A})$, the sum is locally finite, so $K(x,y)$ is a continuous function on $G(\mathbf{A}) \times G(\mathbf{A})$.

The expression (6.10) is the geometric form of the kernel. When $f \in C_c^m(G(\mathbf{A}), \overline{\omega})$ for m sufficiently large (we will prove in Corollary 6.12 that $m = 8$ suffices), the kernel also has a spectral expansion of the following form, valid almost everywhere in $G(\mathbf{A}) \times G(\mathbf{A})$:

$$K(x,y) = K_{\text{cont}}(x,y) + K_{\text{cusp}}(x,y) + K_{\text{res}}(x,y). \tag{6.11}$$

Here

$$K_{\text{cont}}(x,y) = \frac{1}{4\pi} \sum_{\phi} \int_{-\infty}^{\infty} E(\pi_{it}(f)\phi_{it}, x)\overline{E(\phi_{it}, y)}dt$$

for an orthonormal basis $\{\phi\}$ for $H(0)$,

$$K_{\text{cusp}}(x,y) = \sum_{\varphi} R(f)\varphi(x)\overline{\varphi(y)} \tag{6.12}$$

as in (6.9) for an orthonormal basis $\{\varphi\}$ for $L_0^2(\omega)$, and

$$\begin{aligned} K_{\text{res}}(x,y) &= \frac{3}{\pi} \sum_{\chi^2=\omega} R(f)\phi_\chi(x)\overline{\phi_\chi(y)} \\ &= \frac{3}{\pi} \sum_{\chi^2=\omega} \chi(\det x)\overline{\chi(\det y)} \int_{\overline{G}(\mathbf{A})} f(g)\chi(\det g)dg \\ &= \begin{cases} \frac{3}{\pi} \int_{\overline{G}(\mathbf{A})} f(g)dg & \text{if } \omega \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6.13}$$

To see (6.13), notice that in the integral on the previous line, if we replace g by gk for $k \in K_1(N)$, a factor of $\chi(\det k)$ comes out. So the integral vanishes unless χ is trivial on $\widehat{\mathbf{Z}}^*$. Since χ has finite order, this is possible only if χ is trivial (since $\mathbf{Q}$ has class number 1), which means that $\omega = \chi^2$ is also trivial.

If $f = f_\infty f^{\mathfrak{n}}$ is a weight 0 Hecke operator as in (4.23), then using (3.16) and (4.16), (6.13) becomes

$$\frac{3}{\pi} \int_{\overline{G}(\mathbf{R})} f_\infty(g)dg \int_{\overline{M_1(\mathfrak{n},N)}} f^{\mathfrak{n}}(m)dm = \frac{3}{\pi} h(\tfrac{i}{2}) \prod_{p|\mathfrak{n}} \sum_{j=0}^{\mathfrak{n}_p} p^j = \frac{3}{\pi} h(\tfrac{i}{2}) \sum_{d|\mathfrak{n}} d \tag{6.14}$$

for the Selberg transform h of f_∞.

For such a Hecke operator, we will derive (6.11) from the spectral decomposition in Theorem 6.2 using a nice choice of basis, and show that for this choice it is in fact valid for *all* (x,y). This is a special case of a result of Arthur ([Ar2] §4, culminating on p. 935). We need this fact because our principal objective is to derive a relative trace formula by integrating $K(x,y)$ over

$$(x,y) \in (N(\mathbf{Q})\backslash N(\mathbf{A}))^2,$$

a space which has *measure zero* in $(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}))^2$, so an almost-everywhere spectral expression for $K(x,y)$ is not adequate.

6.3 A spectral lower bound for $K_{h*h^*}(x,x)$

In this section we will take $f = h * h^*$ for a suitable function h, and give a spectral lower bound for $K_f(x,x)$ in Proposition 6.6. We begin with the following lemma.

Lemma 6.5 ([GGK], Lemma 5.2.1)**.** *Let X be a Radon measure space, and let T be an operator on $L^2(X)$. Suppose there is a continuous weak kernel function $K(x,y)$ for T. Suppose further that for all bounded compactly supported ψ,*

$$\langle T\psi, \psi\rangle \geq 0.$$

Then $K(x,x) \geq 0$ for all $x \in X$.

Proof. Suppose for some x that $\operatorname{Re} K(x,x) < 0$. By continuity there exists a compact neighborhood $U \subseteq X$ of x such that $\operatorname{Re} K(a,b) < 0$ for all $(a,b) \in U \times U$. Let ψ be the characteristic function of U. Then

$$0 \leq \langle T\psi, \psi\rangle = \int_X T\psi(a)\overline{\psi(a)}da = \int_U \int_U K(a,b)db\,da.$$

The right-hand side has negative real part, which is a contradiction. Therefore $\operatorname{Re} K(x,x) \geq 0$. By a similar argument, we find also that $\operatorname{Im} K(x,x) = 0$. □

Proposition 6.6. *Let $h \in C_c^m(G(\mathbf{A}), \overline{\omega})$ be a bi-$K_\infty \times K_1(N)$-invariant function for $m \geq 2$. Choose orthonormal bases $\{\varphi\}$ and $\{\phi\}$ for $L^2_{\mathrm{disc}}(\omega)$ and $H(0)^{K_\infty \times K_1(N)}$ respectively, consisting of continuous functions. Then for all $x \in \overline{G}(\mathbf{A})$,*

$$\sum_\varphi |R(h)\varphi(x)|^2 + \frac{1}{4\pi}\sum_\phi \int_{-\infty}^{\infty} |E(\pi_{it}(h)\phi_{it}, x)|^2 dt \leq K_{h*h^*}(x,x). \tag{6.15}$$

Here $K_{h*h^*}(x,y)$ is the geometric kernel defined in (6.10)*.*

Remark: The set $\{\phi\}$ can be extended to an orthonormal basis for all of $H(0)$ in (6.15). Indeed, because h is $K_\infty \times K_1(N)$-invariant, by Lemma 3.10 $\pi_{it}(h)\phi_{it}$ vanishes when ϕ belongs to the orthogonal complement of the finite dimensional subspace

$$H(0)^{K_\infty \times K_1(N)} = \bigoplus_{\substack{\chi_1\chi_2=\omega \\ \mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}|N}} H(\chi_1,\chi_2)^{K_\infty \times K_1(N)}.$$

We will prove the proposition in stages. It is an application of Lemma 6.5, but complicated by the fact that we do not know *a priori* that the left-hand side of (6.15) is continuous. Thus we will approximate it by a partial sum, defined as follows. Fix an orthonormal subset $Q \subseteq L^2_{\mathrm{disc}}(\omega)$, and let J be a symmetric compact subset of $\mathbf{R}$. To these we attach the following function

$$K'(x,y) = K'_{\mathrm{disc}}(x,y) + K'_{\mathrm{cont}}(x,y), \tag{6.16}$$

where

$$K'_{\mathrm{disc}}(x,y) = \sum_{\varphi\in Q} R(h)\varphi(x)\overline{R(h)\varphi(y)}$$

and

$$K'_{\mathrm{cont}}(x,y) = \frac{1}{4\pi}\sum_{\phi}\int_J E(\pi_{it}(h)\phi_{it},x)\overline{E(\pi_{it}(h)\phi_{it},y)}dt.$$

Here ϕ runs through an orthonormal basis for $H(0)^{K_\infty\times K_1(N)}$.

Lemma 6.7. *There exists a bounded linear operator T'_{cont} on $L^2(\omega)$ for which K'_{cont} is a weak kernel: for all bounded $\psi\in L^2(\omega)$ with compact support modulo $Z(\mathbf{A})G(\mathbf{Q})$,*

$$T'_{\mathrm{cont}}\psi(x) = \int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} K'_{\mathrm{cont}}(x,y)\psi(y)dy \tag{6.17}$$

for almost all x. The analogous statement for K'_{disc} also holds.

Proof. For any measurable symmetric subset $J\subseteq\mathbf{R}$, define

$$\mathcal{L}_J = \int_J H(it)dt\cap\mathcal{L},$$

where $\mathcal{L}$ was defined in (6.4). Here we regard each element of the direct integral as a function on all of $\mathbf{R}$, taking the value 0 at points outside J. It is easy to see that $\mathcal{L}_J$ is a closed $G(\mathbf{A})$-invariant subspace of $\mathcal{L}$, and we have the orthogonal decomposition

$$\mathcal{L} = \mathcal{L}_J\oplus\mathcal{L}_{\mathbf{R}-J}.$$

We denote the analogous decomposition in $L^2_{\mathrm{cont}}(\omega)\cong\mathcal{L}$ by

$$L^2_{\mathrm{cont}}(\omega) = L_J\oplus L_{\mathbf{R}-J}.$$

Define a $G(\mathbf{A})$-equivariant map $S_J : L^2(\omega)\longrightarrow\mathcal{L}_J$ by

$$S_J\psi = \begin{cases} S\psi & \text{if } \psi\in L_J \\ 0 & \text{if } \psi\in(L_J)^\perp,\end{cases}\qquad\text{i.e.}\qquad S_J\psi(it) = \begin{cases} S\psi(it) & \text{for a.e. } t\in J \\ 0 & \text{for a.e. } t\notin J.\end{cases}$$

Its restriction to L_J is an isomorphism of Hilbert spaces. The map

$$P_J \stackrel{\mathrm{def}}{=} (S_J)^*\,S_J$$

is the orthogonal projection of $L^2(\omega)$ onto L_J, so $S_J = S\circ P_J$.

Now let J be the given compact set. Define $T'_{\mathrm{cont}} = P_J\,R(h*h^*)\,P_J$. It is a bounded operator because $\|T'_{\mathrm{cont}}\|\le\|R(h*h^*)\|\le\|h*h^*\|_{L^1}$ (cf. [KL2], p. 140). For bounded compactly supported $\psi_1,\psi_2\in L^2(\omega)$,

$$\langle T'_{\mathrm{cont}}\psi_1,\psi_2\rangle = \langle P_JR(h*h^*)P_J\psi_1,\psi_2\rangle = \langle R(h^*)P_J\psi_1, R(h^*)P_J\psi_2\rangle$$

$$= \langle P_JR(h^*)\psi_1, P_JR(h^*)\psi_2\rangle = \langle SP_JR(h^*)\psi_1, SP_JR(h^*)\psi_2\rangle$$

$$= \langle S_J R(h^*)\psi_1, S_J R(h^*)\psi_2\rangle = \frac{1}{\pi}\int_J \langle SR(h^*)\psi_1(it), SR(h^*)\psi_2(it)\rangle\, dt$$

$$= \frac{1}{\pi}\int_J \langle \pi_{it}(h^*)S\psi_1(it), \pi_{it}(h^*)S\psi_2(it)\rangle\, dt$$

$$= \frac{1}{\pi}\int_J \sum_\phi \langle S\psi_1(it), \pi_{it}(h)\phi_{it}\rangle\, \overline{\langle S\psi_2(it), \pi_{it}(h)\phi_{it}\rangle}\, dt$$

$$= \frac{1}{4\pi}\int_J \sum_\phi \langle \psi_1, E(\pi_{it}(h)\phi_{it}, \cdot)\rangle\, \overline{\langle \psi_2, E(\pi_{it}(h)\phi_{it}, \cdot)\rangle}\, dt \qquad (6.18)$$

$$= \int\limits_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \left[\int\limits_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \left\{ \frac{1}{4\pi}\sum_\phi \int_J E(\pi_{it}(h)\phi_{it}, x)\overline{E(\pi_{it}(h)\phi_{it}, y)}dt \right\} \psi_1(y)dy \right] \overline{\psi_2(x)}dx.$$

The interchange of the sum and integrals is justified by Fubini's theorem, since the Eisenstein series are continuous, J is compact, the sum over ϕ is finite, and since ψ_1, ψ_2 are bounded with compact support modulo $Z(\mathbf{A})G(\mathbf{Q})$. This proves (6.17).

For K'_{disc}, the statement is much easier because $K'_{\text{disc}}(x,y)$ is square integrable over $(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}))^2$, so for almost all x the expression $\int K'_{\text{disc}}(x,y)\psi(y)dy$ is meaningful for *all* $\psi \in L^2(\omega)$, and serves to define $T'_{\text{disc}}\psi(x)$. To see the square integrability, note that by the Cauchy-Schwarz inequality,

$$|K'_{\text{disc}}(x,y)|^2 \le (\sum_{\varphi\in Q} |R(h)\varphi(x)|^2)(\sum_{\varphi\in Q} |R(h)\varphi(y)|^2).$$

Therefore

$$\iint_{(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}))^2} |K'_{\text{disc}}(x,y)|^2 dx\, dy \le (\sum_{\varphi\in Q} \|R(h)\varphi\|^2)(\sum_{\varphi\in Q} \|R(h)\varphi\|^2),$$

which is finite since $R(h)$ is a Hilbert-Schmidt operator on $L^2_{\text{disc}}(\omega)$ by Theorem 6.4. (On $L^2_{\text{res}}(\omega)$ it actually has finite rank as shown in (6.13).) □

Proposition 6.8. *Let $T' = T'_{\text{disc}} + T'_{\text{cont}}$ with notation as in the above lemma. Suppose that the orthonormal set $Q \subseteq L^2_{\text{disc}}(\omega)$ is finite, and that $J \subseteq \mathbf{R}$ is compact and symmetric. Then*

$$\langle T'\psi, \psi\rangle \le \langle R(h * h^*)\psi, \psi\rangle$$

for all bounded $\psi \in L^2(\omega)$ of compact support modulo $Z(\mathbf{A})G(\mathbf{Q})$.

Proof. Extend Q to an orthonormal basis $\widetilde{Q}$ of $L^2_{\text{disc}}(\omega)$. We have

$$\langle T'_{\text{disc}}\psi, \psi\rangle = \int\limits_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \Big(\int\limits_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \sum_{\varphi\in Q} R(h)\varphi(x)\overline{R(h)\varphi(y)}\psi(y)dy\Big)\overline{\psi(x)}dx.$$

The sum can be pulled out because of the conditions placed on ψ and since Q is finite. So the above is

$$= \sum_{\varphi \in Q} |\langle R(h)\varphi, \psi\rangle|^2 \le \sum_{\varphi \in \widetilde{Q}} |\langle R(h)\varphi, \psi\rangle|^2 = \sum_{\varphi \in \widetilde{Q}} |\langle \varphi, R(h)^*\psi\rangle|^2$$

$$= \langle P_{\mathrm{disc}} R(h)^*\psi, P_{\mathrm{disc}} R(h)^*\psi\rangle = \langle P_{\mathrm{disc}} R(h * h^*)\psi, \psi\rangle .$$

Passing to the last line, we applied Parseval's identity (6.6), while the last equality follows easily by the fact that $R(h)$ commutes with the orthogonal projection P_{disc}.

Likewise, by (6.18),

$$\langle T'_{\mathrm{cont}}\psi, \psi\rangle = \frac{1}{4\pi} \sum_{\phi} \int_J |\langle \psi, E(\pi_{it}(h)\phi_{it}, \cdot)\rangle|^2 dt$$

$$\le \frac{1}{4\pi} \sum_{\phi} \int_{-\infty}^{\infty} |\langle \psi, E(\pi_{it}(h)\phi_{it}, \cdot)\rangle|^2 dt = \frac{1}{4\pi} \sum_{\phi} \int_{-\infty}^{\infty} |\langle R(h)^*\psi, E(\phi_{it}, \cdot)\rangle|^2 dt$$

$$= \langle P_{\mathrm{cont}} R(h)^*\psi, P_{\mathrm{cont}} R(h)^*\psi\rangle = \langle P_{\mathrm{cont}} R(h * h^*)\psi, \psi\rangle .$$

Again we used Parseval's identity (6.8) in passing to the last line. We have also used (6.7). □

Proof of Proposition 6.6. Let Q be a finite subset of the given orthonormal basis $\{\varphi\}$ of $L^2_{\mathrm{disc}}(\omega)$, and let J be a symmetric compact subset of $\mathbf{R}$. Let $K'(x,y)$ be the associated partial kernel function as above, and set $f = h * h^*$. Then $K'(x,y)$ is continuous since all φ and ϕ are continuous by hypothesis. On the other hand, we saw in (6.10) that $K_f(x,y)$ is also continuous. By the above proposition, $\langle (R(f) - T')\psi, \psi\rangle \ge 0$ for all bounded $\psi \in L^2(\omega)$ of compact support modulo the center. Hence by Lemma 6.5, $K_f(x,x) - K'(x,x) \ge 0$ for all $x \in G(\mathbf{A})$. It follows that

$$\sup_{Q,J} K'(x,x) \le K_f(x,x).$$

The proposition now follows, since the supremum is precisely the left-hand side of (6.15). □

6.4 The spectral form of the kernel of $R(f)$

The following lemma, which follows from a result of Duflo and Labesse, will enable us to reduce to the special situation $f = h * h^*$ discussed above.

Lemma 6.9. *Let $r \ge 1$, and suppose $f \in C_c^{4r}(G(\mathbf{A}), \overline{\omega})$ is bi-invariant under $K_\infty \times K_1(N)$. Then there exist functions $h_1, h_2, k_1, k_2 \in C_c^{2r-2}(G(\mathbf{A}), \overline{\omega})$ which are also bi-invariant under $K_\infty \times K_1(N)$ such that*

$$f = h_1 * h_2 + k_1 * k_2.$$

Proof. Write $K' = K_\infty \times K_1(N)$. In this proof only, we normalize so that $\text{meas}(K') = 1$. A function $a(g)$ on $G(\mathbf{A})$ is said to be K'-central if $a(kg) = a(gk)$ for all $k \in K'$. For any function $a(g)$ we define

$$\tilde{a}(g) = \int_{K'} a(kg)dk.$$

Obviously $\tilde{a}$ is left K'-invariant. If a is K'-central, then $\tilde{a}(g)$ is also right K'-invariant.

Define an action of $\mathfrak{g}_{\mathbf{R}} = \text{Lie}(G(\mathbf{R}))$ on the smooth functions by

$$X * f(g) = \left.\frac{d}{dt}\right|_{t=0} f(\exp(-tX)g). \tag{6.19}$$

This extends naturally to an action of the universal enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$. By [DL] (I.1.11), there exist K'-central functions $a \in C_c^{2r-2}(G(\mathbf{A}),\overline{\omega})$, $b \in C_c^\infty(G(\mathbf{A}),\overline{\omega})$, and a differential operator $D \in U(\mathfrak{g}_{\mathbf{C}})$ of order 2, such that

$$f = a * (D^{r+1} * f) + b * f.$$

Let $c = D^{r+1} * f$. Because f is C^{4r}, c is $C^{4r-(2r+2)} = C^{2r-2}$. It follows from (6.19) that c is right K'-invariant. By the left K'-invariance of f,

$$f(x) = \int_{K'} f(kx)dk = \int_{K'}\int_{\overline{G}} a(g)c(g^{-1}kx)dg\,dk + \int_{K'}\int_{\overline{G}} b(g)f(g^{-1}kx)dg\,dk$$

$$= \int_{\overline{G}}\int_{K'} a(kg)c(g^{-1}x)dk\,dg + \int_{\overline{G}}\int_{K'} b(kg)f(g^{-1}x)dk\,dg = (\tilde{a}*c)(x) + (\tilde{b}*f)(x).$$

Because $\tilde{a}$ is bi-K'-invariant, it is easy to verify that $\tilde{a} * c = \tilde{a} * \tilde{c}$. Therefore we can take $h_1 = \tilde{a}$, $h_2 = \tilde{c}$, $k_1 = \tilde{b}$ and $k_2 = f$. □

Theorem 6.10. *Let $f = f_\infty f^{\mathfrak{n}}$, where $f_\infty \in C_c^m(G(\mathbf{R})^+//K_\infty)$ for $m \geq 8$. Let $\mathcal{F}_{\mathbf{A}}$ be an orthonormal basis for $L_0^2(\omega)^{K_\infty \times K_1(N)}$, chosen as in Proposition 4.8. Then both*

$$\sum_{\varphi \in \mathcal{F}_{\mathbf{A}}} R(f)\varphi(x)\overline{\varphi(y)} \quad \text{and} \quad \sum_{\varphi \in \mathcal{F}_{\mathbf{A}}} |R(f)\varphi(x)\overline{\varphi(y)}|$$

are bounded on any compact subset of $G(\mathbf{A}) \times G(\mathbf{A})$ and continuous in x and y separately.

Proof. It suffices to prove the assertion for the expression with the absolute values. Because $m \geq 8$, by Lemma 6.9 there exist $h_1, h_2, k_1, k_2 \in C_c^2(G(\mathbf{A}),\overline{\omega})$ such that $f = h_1 * h_2 + k_1 * k_2$. By linearity and the triangle inequality, it suffices to prove the theorem for $f = h_1 * h_2$.

By Proposition 4.8, for $\varphi \in \mathcal{F}_{\mathbf{A}}$ we can write

$$R(f)\varphi = \lambda\varphi, \quad R(h_1)\varphi = \lambda_1\varphi, \quad R(h_2)\varphi = \lambda_2\varphi.$$

Note that φ is also an eigenvector of $R(h_1^*)$. The eigenvalue is $\overline{\lambda_1}$ since

$$\langle R(h_1^*)\varphi, \varphi \rangle = \langle \varphi, R(h_1)\varphi \rangle = \overline{\lambda_1} \langle \varphi, \varphi \rangle .$$

Furthermore, $\lambda = \lambda_1 \lambda_2$ since

$$\lambda \langle \varphi, \varphi \rangle = \langle R(f)\varphi, \varphi \rangle = \langle R(h_2)\varphi, R(h_1^*)\varphi \rangle = \lambda_1 \lambda_2 \langle \varphi, \varphi \rangle .$$

This implies that

$$R(f)\varphi(x)\overline{\varphi(y)} = \lambda_1\lambda_2\varphi(x)\overline{\varphi(y)} = R(h_1)\varphi(x)\overline{R(h_2^*)\varphi(y)}.$$

By Cauchy-Schwarz, for any subset S of $\mathcal{F}_{\mathbf{A}}$,

$$\begin{aligned} \sum_{\varphi \in S} |R(f)\varphi(x)\overline{\varphi(y)}| &= \sum_{\varphi\in S} |R(h_1)\varphi(x)\overline{R(h_2^*)\varphi(y)}| \\ &\le \Big(\sum_{\varphi\in S} |R(h_1)\varphi(x)|^2\Big)^{1/2} \Big(\sum_{\varphi \in S} |R(h_2^*)\varphi(y)|^2\Big)^{1/2} \qquad (6.20) \\ &\le K_{h_1 * h_1^*}(x,x)^{1/2} K_{h_2^* * h_2}(y,y)^{1/2}. \end{aligned}$$

The last inequality holds by Proposition 6.6. Because the two kernels are continuous, the above is bounded on any compact set.

Now we show that $\sum_\varphi |R(f)\varphi(x)\overline{\varphi(y)}|$ is continuous in y for fixed x. Let U be any compact subset of $G(\mathbf{A})$. Fix $x \in G(\mathbf{A})$. It suffices to show that the series converges uniformly as a function of $y \in U$. Let C be an upper bound for $K_{h_2^* * h_2}(y,y)^{1/2}$ on U. Fix $\varepsilon > 0$. We know that $\sum_\varphi |R(h_1)\varphi(x)|^2 < \infty$. Hence for any ordering $\varphi_1, \varphi_2, \dots$ of $\{\varphi\}$, there exists $N > 0$ such that

$$\sum_{n \ge N} |R(h_1)\varphi_n(x)|^2 < \frac{\varepsilon^2}{C^2}.$$

Therefore by (6.20),

$$\sum_{n\ge N} |R(f)\varphi(x)\overline{\varphi(y)}| \le C\Big(\sum_{n\ge N} |R(h_1)\varphi_n(x)|^2\Big)^{1/2} < \varepsilon.$$

Hence the series converges uniformly for $y \in U$, as needed. Similarly for fixed y, the sum is continuous in x. □

Theorem 6.11. *Let $f = f_\infty f^{\mathrm{n}}$ be as in the previous theorem. Then both*

$$K_{\mathrm{cont}}(x,y) = \frac{1}{4\pi} \sum_\phi \int_{-\infty}^{\infty} E(\pi_{it}(f)\phi_{it}, x)\overline{E(\phi_{it},y)}dt \qquad (6.21)$$

and

$$\frac{1}{4\pi} \sum_\phi \int_{-\infty}^{\infty} |E(\pi_{it}(f)\phi_{it}, x)\overline{E(\phi_{it},y)}|dt \qquad (6.22)$$

are bounded on any compact subset of $G(\mathbf{A}) \times G(\mathbf{A})$ and continuous in x and y separately. Here ϕ runs through an orthonormal basis for $H(0)^{K_\infty \times K_1(N)}$.

Proof. The proof is similar to that of the previous theorem. We can assume $f = h_1 * h_2$. For $j \geq 1$, let $R_j = [-j, -j+1] \cup [j-1, j]$, and define

$$G_j(x,y) = \sum_\phi \frac{1}{4\pi} \int_{R_j} \left| E(\pi_{it}(f)\phi_{it}, x)\overline{E(\phi_{it}, y)} \right| dt.$$

It is a continuous function of x and y. Note that (6.22) is equal to $\sum_j G_j(x,y)$. It suffices to show that for fixed x this series converges uniformly for y in a compact set. Write

$$\pi_{it}(h_1)\phi_{it} = \lambda_1^\phi(t)\phi_{it}, \qquad \pi_{it}(h_2)\phi_{it} = \lambda_2^\phi(t)\phi_{it}.$$

For any set S of natural numbers, $\sum_{j\in S} G_j(x,y)$ is

$$\leq \frac{1}{4\pi} \sum_{j\in S} \sum_\phi \Big(\int_{R_j} |\lambda_1^\phi(t) E(\phi, it, x)|^2 dt \Big)^{1/2} \Big(\int_{R_j} |\lambda_2^\phi(t) E(\phi, it, y)|^2 dt \Big)^{1/2}$$

$$\leq \Big(\frac{1}{4\pi} \sum_{j\in S} \sum_\phi \int_{R_j} |\lambda_1^\phi(t) E(\phi, it, x)|^2 dt \Big)^{1/2} \Big(\frac{1}{4\pi} \sum_{j\in S} \sum_\phi \int_{R_j} |\lambda_2^\phi(t) E(\phi, it, y)|^2 dt \Big)^{1/2}$$

$$\leq K_{h_1 * h_1^*}(x,x)^{1/2} K_{h_2^* * h_2}(y,y)^{1/2}$$

by Proposition 6.6. The proof now proceeds as before. □

Now we derive the spectral formula for the kernel $K(x,y)$ of $R(f)$. Because

$$R(f) = R(f)P_{\text{disc}} + R(f)P_{\text{cont}},$$

it suffices to give kernel functions for each of the operators on the right-hand side. The operator $R(f)P_{\text{disc}}$ is Hilbert-Schmidt, so its kernel is given by

$$K_{\text{disc}}(x,y) = \sum_{\substack{\{\varphi\} \subseteq L^2(\omega) \\ \text{o.n.b.}}} R(f)P_{\text{disc}}\varphi(x)\overline{\varphi(y)}.$$

Because $R(f)P_{\text{disc}}$ annihilates all $\varphi \in L^2_{\text{cont}}(\omega)$, the above is equal to the sum $K_{\text{cusp}}(x,y) + K_{\text{res}}(x,y)$ as in (6.12) and (6.13).

For $R(f)P_{\text{cont}}$, suppose $\psi_1, \psi_2 \in L^2(\omega)$ are bounded and compactly supported modulo $Z(\mathbf{A})G(\mathbf{Q})$. Then

$$\langle R(f)P_{\text{cont}}\psi_1, \psi_2 \rangle = \langle P_{\text{cont}}\psi_1, P_{\text{cont}}R(f^*)\psi_2 \rangle \tag{6.23}$$

$$= \frac{1}{4\pi} \sum_\phi \int_{-\infty}^{\infty} \langle \psi_1, E(\phi_{it}, \cdot) \rangle \langle E(\phi_{it}, \cdot), R(f^*)\psi_2 \rangle \, dt$$

$$= \int_{\overline{G}} \left[\int_{\overline{G}} \left\{ \frac{1}{4\pi} \sum_\phi \int_{-\infty}^{\infty} E(\pi_{it}(f)\phi_{it}, x)\overline{E(\phi_{it}, y)} dt \right\} \psi_1(y) dy \right] \overline{\psi_2(x)} dx.$$

We used Parseval's identity (6.8) when passing to the second line, and (6.7) when passing to the third line. The convergence is absolute by Theorem 6.11 and the conditions on ψ_1, ψ_2, so the rearrangement of the sum and integrals is justified. It follows that the expression in the braces, which coincides with (6.21), is a weak kernel function for $R(f)P_{\mathrm{cont}}$.

Corollary 6.12. *Suppose $f_\infty \in C_c^m(G(\mathbf{R})^+//K_\infty)$ for $m \geq 8$, and let $f = f_\infty f^{\mathrm{n}}$. Then for all $x, y \in G(\mathbf{A})$,*

$$K(x,y) = K_{\mathrm{cusp}}(x,y) + K_{\mathrm{res}}(x,y) + K_{\mathrm{cont}}(x,y),$$

where we choose bases as in Theorems 6.10 and 6.11. Each function on the right is separately continuous in each variable.

Proof. Denote the right-hand side by $\Psi(x,y)$. As we have just shown, Ψ is a weak kernel function for $R(f)$. By Lemma 6.3 we conclude that $K(x,y) = \Psi(x,y)$ almost everywhere in $G(\mathbf{A})\times G(\mathbf{A})$. We know that $K(x,y)$ is continuous. By the above theorems, $\Psi(x,y)$ is continuous in x and y separately. By Lemma 6.13 below, it follows that $\Psi(x,y) = K(x,y)$ for all x and y. □

Lemma 6.13. *Let X and Y be two positive Borel measure spaces. Let D be a measurable function on $X \times Y$ such that $D(x,y) = 0$ almost everywhere and $D(x,y)$ is a continuous function of x and y separately. Then $D(x,y) = 0$ for all x and y.*

Proof. Because $\int_X \int_Y |D(x,y)|dydx = 0$, the set $\{x \in X| \int_Y |D(x,y)|dy > 0\}$ has measure zero. Let $S \subseteq X$ denote its complement. For fixed $x' \in S$, $D(x',y) = 0$ for almost all $y \in Y$. By the continuity of $y \mapsto D(x',y)$, $D(x',y) = 0$ for all $y \in Y$. Therefore $S \times Y \subseteq \{(x,y)|D(x,y) = 0\}$. This means that for any $y \in Y$, $D(x,y) = 0$ for all $x \in S$, i.e. for almost all $x \in X$. Now by the continuity of $x \mapsto D(x,y)$, it follows that $D(x,y) = 0$ for all $x \in X$ and all $y \in Y$. □

7 A Fourier trace formula for GL(2)

For integers $m_1, m_2, \mathtt{n} > 0$, we will compute a variant of the Kuznetsov/Bruggeman trace formula, involving Fourier coefficients at m_1, m_2, the eigenvalues of $T_\mathtt{n}$, and Kloosterman sums.

Let $f = f_\infty \times f^\mathtt{n}$, with $f_\infty \in C_c^m(G^+//K_\infty)$ for $m \geq 8$ to allow for use of Corollary 6.12 (though for the convergence of the cuspidal term in Proposition 7.5 we will take $m \geq 12$). For real numbers $y_1, y_2 > 0$ and $K(x,y)$ as in (6.10), consider the expression

$$I = \frac{1}{\sqrt{y_1 y_2}} \iint\limits_{(N(\mathbf{Q})\backslash N(\mathbf{A}))^2} K(n_1 \left(\begin{smallmatrix} y_1 & \\ & 1 \end{smallmatrix}\right), n_2 \left(\begin{smallmatrix} y_2 & \\ & 1 \end{smallmatrix}\right)) \, \overline{\theta_{m_1}(n_1)} \, \theta_{m_2}(n_2) \, dn_1 dn_2, \tag{7.1}$$

where

$$\theta_m(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)) = \theta_m(x) = \theta(-mx)$$

for the standard character θ defined by (2.5), and dn_j is the Haar measure of total volume 1. We will compute the relative trace formula obtained by evaluating the above in two ways, using the geometric and spectral expressions for the kernel. The result is a primitive Kuznetsov formula given as Theorem 7.13. The variables y_1, y_2 give us some extra flexibility. To obtain a more refined formula, we will set

$$y_1 m_1 = y_2 m_2 = w \tag{7.2}$$

in the primitive formula, and then integrate w from 0 to ∞. The result is Theorem 7.14, which is a generalized Kuznetsov formula.

7.1 Convergence of the spectral side

According to Corollary 6.12,

$$K(x,y) = K_{\text{cusp}}(x,y) + K_{\text{cont}}(x,y) + K_{\text{res}}(x,y),$$

where each term on the right is separately continuous in each variable. Each term is also bounded on the compact set $(N(\mathbf{Q})\backslash N(\mathbf{A}))^2$ by Theorems 6.10 and 6.11, and hence integrable there. Furthermore, the sums defining K_{cusp} and K_{cont} can be pulled out of the double integral for the same reason.

The justification for integrating over w will be handled later.

7.2 Cuspidal contribution

Here we will compute the cuspidal term

$$I_{\text{cusp}} = \frac{1}{\sqrt{y_1 y_2}} \iint\limits_{(N(\mathbf{Q})\backslash N(\mathbf{A}))^2} K_{\text{cusp}}(n_1 \begin{pmatrix} y_1 & \\ & 1 \end{pmatrix}, n_2 \begin{pmatrix} y_2 & \\ & 1 \end{pmatrix}) \overline{\theta_{m_1}(n_1)} \theta_{m_2}(n_2) dn_1 dn_2.$$

By Lemma 3.10, $R_0(f)$ annihilates the orthogonal complement of $L^2_0(\omega)^{K_\infty\times K_1(N)}$. Let $\mathcal{F}_{\mathbf{A}}$ be the eigenbasis of $L^2_0(\omega)^{K_\infty\times K_1(N)}$ defined in Proposition 4.8, so that for $\varphi\in\mathcal{F}_{\mathbf{A}}$ we have $R(f)\varphi(x)=\sqrt{\mathtt{n}}\,h(t)\lambda_{\mathtt{n}}(\varphi)\phi(x)$. Then $K_{\text{cusp}}(x,y)$ equals

$$\sqrt{\mathtt{n}}\sum_{\varphi_j\in\mathcal{F}_{\mathbf{A}}}\frac{h(t_j)\,\lambda_{\mathtt{n}}(\varphi_j)\,\varphi_j(x)\overline{\varphi_j(y)}}{\|\varphi_j\|^2}=\sqrt{\mathtt{n}}\sum_{u_j\in\mathcal{F}}\frac{h(t_j)\lambda_{\mathtt{n}}(u_j)\,\varphi_{u_j}(x)\overline{\varphi_{u_j}(y)}}{\|u_j\|^2}$$

for $\mathcal{F}$ as in (4.25). As explained in Section 7.1 above, I_{cusp} is absolutely convergent, and by Fubini's Theorem

$$I_{\text{cusp}}=\sqrt{\frac{\mathtt{n}}{y_1y_2}}\sum_j\frac{h(t_j)\lambda_{\mathtt{n}}(u_j)}{\|u_j\|^2}\int_{\mathbf{Q}\backslash\mathbf{A}}\varphi_{u_j}\left(\left(\begin{smallmatrix}y_1&x\\0&1\end{smallmatrix}\right)\right)\overline{\theta_{m_1}(x)}\,dx\int_{\mathbf{Q}\backslash\mathbf{A}}\overline{\varphi_{u_j}\left(\left(\begin{smallmatrix}y_2&x\\0&1\end{smallmatrix}\right)\right)}\theta_{m_2}(x)dx.$$

Lemma 7.1. *Let u be a Maass cusp form with Fourier expansion as in* (4.8). *Then for $r\in\mathbf{Q}$ and $y\in\mathbf{R}^*$,*

$$\int_{\mathbf{Q}\backslash\mathbf{A}}\varphi_u(\left(\begin{smallmatrix}y&x\\0&1\end{smallmatrix}\right))\overline{\theta_r(x)}dx=\begin{cases}a_r(u)y^{1/2}K_{it}(2\pi|r|y) & \textit{if } r\in\mathbf{Z}-\{0\}\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Using the fundamental domain $[0,1]\times\widehat{\mathbf{Z}}$ for $\mathbf{Q}\backslash\mathbf{A}$ and (4.18), we have

$$\int_{\mathbf{Q}\backslash\mathbf{A}}\varphi_u(\left(\begin{smallmatrix}y&x\\0&1\end{smallmatrix}\right))\overline{\theta_r(x)}dx=\int_0^1u(x+iy)\theta_\infty(rx)dx\int_{\widehat{\mathbf{Z}}}\theta_{\text{fin}}(rx)dx.$$

The second integral on the right vanishes unless $r\in\mathbf{Z}$, in which case it is equal to 1. Assuming $r\in\mathbf{Z}$, this becomes $\int_0^1u(x+iy)e^{-2\pi irx}dx$, and the assertion then follows by substituting the Fourier expansion (4.8) of u. □

Lemma 7.2. *Let $\{u_j\}$ be an orthogonal basis for $L^2_0(N,\omega')$ consisting of cusp forms. Let t_j be the spectral parameter of u_j. Then for any $M>0$,*

$$\Big|\{j:\,|t_j|\le M\}\Big|<\infty. \tag{7.3}$$

Remark: Much more is known. According to Weyl's Law (which in this context follows from the Selberg trace formula),

$$\Big|\{j:\,|t_j|\le M\}\Big|=\frac{\text{vol}(\Gamma_0(N)\backslash\mathbf{H})}{4\pi}M^2+O(N^{1/2}M\log(NM)) \tag{7.4}$$

([IK] p. 391, [Sel4] p. 668).

Proof. Let $h(iz)\in PW^4(\mathbf{C})^{\text{even}}$. By Proposition 3.6, there exists a function $f_\infty\in C_c^2(G^+//K_\infty)$ whose Selberg transform is $h(t)$. Let $f'=f_\infty\times f^1$, where f^1 is the Hecke operator on $G(\mathbf{A}_{\text{fin}})$ with $\mathtt{n}=1$. By Proposition 4.8, the operator $R_0(f')$ is diagonalizable with eigenvalues $h(t_j)$. By Theorem 6.4, this operator is Hilbert-Schmidt. Therefore

$$\sum_j|h(t_j)|^2<\infty.$$

On the other hand, if (7.3) fails to hold, the set $\{t_j : |t_j| \le M\}$ has a limit point $P \in \mathbf{C}$. Choosing h so that $h(P) \neq 0$ would then contradict the above summability. It remains to show that such h exists. If $P = 0$, we can let h be the Mellin transform of a nonzero element $\Phi \in C_c^\infty(\mathbf{R}^+)^w$ that assumes only nonnegative real values. Then $h(0) = \int_0^\infty \Phi(y)\frac{dy}{y} > 0$, as needed. Now suppose $P \neq 0$. Let $h_1 \in PW(\mathbf{C})^{\text{even}}$ be nonzero, with $h_1(Q) \neq 0$, say. By continuity, we may assume that $Q \neq 0$. Then we can take $h(z) = h_1(\frac{Q}{P}z)$. □

Corollary 7.3. *The set of exceptional spectral parameters $t_j \notin \mathbf{R}$ is finite.*

Proof. If t_j is exceptional, then by Proposition 4.7, $t_j = -is$ for some real $s \in (-\frac{1}{2}, \frac{1}{2})$. In particular, $|t_j| < \frac{1}{2}$, and the above lemma shows that the set of such t_j is finite. □

Lemma 7.4. *With t_j as in Lemma 7.2, we have*

$$\sum_j |h(t_j)| < \infty$$

for any function $h(iz) \in PW^m(\mathbf{C})^{even}$ with $m \ge 10$.

Remark: More is known. Using Weyl's Law (7.4), it is straightforward to show that $1 + |t_j| \gg j^{1/2}$. Therefore $\sum |h(t_j)| \ll \sum (1 + |t_j|)^{-m} \ll \sum j^{-m/2} < \infty$ if $m > 2$.

Proof. Let $f' = f_\infty \times f^1$ be the global function attached to h as in the proof of Lemma 7.2. Let $\varphi_j = \frac{\varphi_{u_j}}{\|\varphi_{u_j}\|} \in L_0^2(\omega)$ be the unit vector attached to u_j. Noting that $f_\infty \in C_c^8(G^+//K_\infty)$ by Proposition 3.6, we can write $f' = a * b + c * d$ for bi-K_∞-invariant functions $a, b, c, d \in C_c^2(G(\mathbf{A}), \overline{\omega})$ by Lemma 6.9. Then by (4.24),

$$\begin{aligned}
\sum_j |h(t_j)| &= \sum_j |\langle R_0(f')\varphi_j, \varphi_j\rangle| = \sum_j |\langle R_0(a)R_0(b)\varphi_j, \varphi_j\rangle + \langle R_0(c)R_0(d)\varphi_j, \varphi_j\rangle| \\
&\le \sum_j |\langle R_0(b)\varphi_j, R_0(a^*)\varphi_j\rangle| + \sum_j |\langle R_0(d)\varphi_j, R_0(c^*)\varphi_j\rangle| \\
&\le \sum_j \|R_0(b)\varphi_j\|\|R_0(a^*)\varphi_j\| + \sum_j \|R_0(d)\varphi_j\|\|R_0(c^*)\varphi_j\| \\
&\le \Big(\sum_j \|R_0(b)\varphi_j\|^2\Big)^{1/2} \Big(\sum_j \|R_0(a^*)\varphi_j\|^2\Big)^{1/2} \\
&\qquad + \Big(\sum_j \|R_0(d)\varphi_j\|^2\Big)^{1/2} \Big(\sum_j \|R_0(c^*)\varphi_j\|^2\Big)^{1/2}.
\end{aligned}$$

The above is finite since all four operators are Hilbert-Schmidt by Theorem 6.4. □

Proposition 7.5. *Given $h(iz) \in PW^m(\mathbf{C})^{even}$ with $m \geq 10$, let f_∞ be its inverse spherical transform in $C^{m-2}(G^+//K_\infty)$ as in Proposition 3.6, and let $f = f_\infty \times f^{\mathtt{n}}$. Then the integral I_{cusp} is absolutely convergent. It vanishes unless $m_1, m_2 \in \mathbf{Z} - \{0\}$. Let $\mathcal{F}$ be an orthogonal eigenbasis for $L^2_0(N, \omega')$ as in (4.25). Then for $m_1, m_2 \in \mathbf{Z}^+$, we have*

$$I_{\text{cusp}} = \sqrt{\mathtt{n}} \sum_{u_j \in \mathcal{F}} \frac{\lambda_{\mathtt{n}}(u_j)\, a_{m_1}(u_j)\overline{a_{m_2}(u_j)}}{\|u_j\|^2} h(t_j) K_{it_j}(2\pi m_1 y_1)\, K_{it_j}(2\pi m_2 y_2), \quad (7.5)$$

the sum converging absolutely. Now suppose $m \geq 12$. Then letting $I_{\text{cusp}}(w)$ denote the above when $w = y_1 m_1 = y_2 m_2$, we have

$$\int_0^\infty I_{\text{cusp}}(w)dw = \frac{\pi\sqrt{\mathtt{n}}}{8} \sum_{u_j \in \mathcal{F}} \frac{\lambda_{\mathtt{n}}(u_j)\, a_{m_1}(u_j)\overline{a_{m_2}(u_j)}}{\|u_j\|^2} \frac{h(t_j)}{\cosh(\pi t_j)}, \quad (7.6)$$

the sum and integral converging absolutely.

Remark: In fact, (7.6) converges absolutely for any function h (holomorphic or not) satisfying a bound of the form $h(t) \ll \frac{1}{(1+|t|)^m}$ for $m > 3$. Indeed, granting Weyl's Law, one finds as in the proof below that (7.6) is $\ll \sum |h(t_j)|(1+|t_j|) \ll \sum (1+|t_j|)^{-m+1} \ll \sum j^{(-m+1)/2} < \infty$ if $m > 3$. By the fact that Weyl's Law is an exact asymptotic, any improvement allowing smaller m (say $m > 2$) must come from a strengthening of the estimate (7.8).

Proof. The absolute convergence of I_{cusp} (with absolute values inside the sum defining K_{cusp}) is a consequence of the continuity result of Theorem 6.10 and the compactness of $N(\mathbf{Q})\backslash N(\mathbf{A})$. The equality (7.5) and fact that m_1, m_2 must be integers follow immediately from Lemma 7.1 and the discussion preceding it. The second Bessel factor does not need the complex conjugate because t_j is purely imaginary or purely real (cf. Proposition 3.8), so that $K_{it_j}(x)$ is real for real x by (4.9) (if t is real, consider $w \mapsto w^{-1}$ in that equation).

Equation (7.6) follows formally from (7.5) by the identity

$$\int_0^\infty K_{it}(2\pi w)^2 dw = \frac{\Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}-it)}{8} = \frac{\pi}{8\cosh(\pi t)} \quad (7.7)$$

([GR], 6.576), which is valid whenever $\text{Im}(t) = \text{Re}(it) < \frac{1}{2}$, which holds here by Proposition 3.8. As justification, we have to prove that the integral

$$\int_0^\infty \sqrt{\mathtt{n}} \sum_{u_j \in \mathcal{F}} \frac{|\lambda_{\mathtt{n}}(u_j)\, a_{m_1}(u_j)\overline{a_{m_2}(u_j)}|}{\|u_j\|^2} |h(t_j) K_{it_j}(2\pi w)^2| dw$$

converges. By (7.7) and the fact that $|K_{it}(2\pi w)|^2 = K_{it}(2\pi w)^2$, this amounts to showing that the right-hand side of (7.6) is absolutely convergent.

Recall that $|\lambda_{\mathtt{n}}(u_j)|$ is bounded by a constant depending only on $\mathtt{n}$. (For an elementary proof, see [Ro], Proposition 2.9. Currently the best known bound is

$\tau(\mathtt{n})\mathtt{n}^{7/64}$ due to Kim and Sarnak [KS]. According to the Ramanujan conjecture, $|\lambda_{\mathtt{n}}(u_j)| \le \tau(\mathtt{n})$.)

For any Maass cusp form u with spectral parameter t, and $m \neq 0$, we have the well-known elementary bound

$$\frac{|a_n(u)|^2}{\|u\|^2} \ll \psi(N)(|t| + \frac{|n|}{N})\, e^{\pi|t|},$$

where the implied constant is absolute (see Theorem 3.2 of [Iw2]). This gives

$$\frac{|a_{m_1}(u)\overline{a_{m_2}(u)}|}{\|u\|^2} \ll_{N,m_1,m_2} (1+|t|)e^{\pi|t|}. \tag{7.8}$$

When t is real, the exponential factor is negated by $\cosh(\pi t) \gg e^{\pi|t|}$ in the denominator of (7.6). For the finitely many non-real t_j, we have $|t_j| \le \frac{1}{2}$. Hence

$$\sum_{u_j \in \mathcal{F}} \frac{|\lambda_{\mathtt{n}}(u_j)\, a_{m_1}(u_j)\overline{a_{m_2}(u_j)}h(t_j)|}{\|u_j\|^2 \cosh(\pi t_j)} \ll \sum_j |h(t_j)|(1+|t_j|). \tag{7.9}$$

Let $\tilde{h}(t) = t^2 h(t)$. Assuming $h(iz)$ is Paley-Wiener of order $m \ge 12$, it is easy to show that $\tilde{h}(iz) \in PW^{10}(\mathbf{C})^{\text{even}}$. Note that

$$\sum_{|t_j|\ge 2} |h(t_j)|(1+|t_j|) < \sum_{|t_j|\ge 2} |t_j|^2 |h(t_j)| = \sum_{|t_j|\ge 2} |\tilde{h}(t_j)|.$$

By Lemma 7.4 above, the latter expression is finite. In view of (7.3) with $M = 2$, this implies that the right-hand side of (7.9) is finite. □

7.3 Residual contribution

By (6.13), K_{res} vanishes when ω is nontrivial. Otherwise by (6.14) we have

$$I_{\text{res}}(f, m_1, m_2, w) = \frac{1}{\sqrt{y_1 y_2}} \iint_{(\mathbf{Q}\backslash\mathbf{A})^2} K_{\text{res}}(\left(\begin{smallmatrix} y_1 & x_1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} y_2 & x_2 \\ 0 & 1 \end{smallmatrix}\right))\theta_{m_1}(x_1)\,\overline{\theta_{m_2}(x_2)}\, dx_1\, dx_2$$

$$= \frac{3h(\frac{i}{2})}{\pi\sqrt{y_1 y_2}} \Big(\sum_{d|\mathtt{n}} d\Big) \int_{\mathbf{Q}\backslash\mathbf{A}} \overline{\theta(m_1 x)} dx \int_{\mathbf{Q}\backslash\mathbf{A}} \theta(m_2 x) dx.$$

Both integrals vanish since $m_1, m_2 \neq 0$. Thus the residual spectrum makes no contribution to this trace formula.

7.4 Continuous contribution

We continue to assume that $f = f_\infty \times f^{\mathtt{n}}$ satisfies the hypotheses of (7.1). Using the decomposition $H(0) = \bigoplus H(\chi_1, \chi_2)$, the continuous kernel is given by

$$K_{\text{cont}}(g_1, g_2) = \frac{1}{4\pi} \sum_{\chi_1,\chi_2} \sum_{\phi} \int_{-\infty}^{\infty} E(\pi_{it}(f)\phi_{it}, g_1)\overline{E(\phi_{it}, g_2)} dt. \tag{7.10}$$

Here χ_1, χ_2 ranges through all (ordered) pairs of finite order Hecke characters satisfying $\chi_1\chi_2 = \omega$, and ϕ ranges through an orthonormal basis for $H(\chi_1,\chi_2)^{K_\infty \times K_1(N)}$. By Corollary 5.11, the latter space is nonzero if and only if $\mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}|N$. In particular, both sums above are finite.

For $\phi \in H(\chi_1,\chi_2)^{K_\infty\times K_1(N)}$ we know that

$$\pi_{it}(f)\phi_{it} = \sqrt{\mathtt{n}}\,\lambda_{\mathtt{n}}(\chi_1,\chi_2,it)h(t)\phi_{it}$$

by Proposition 5.2. Hence for the orthogonal basis $\mathcal{B}(\chi_1,\chi_2)$ given in Corollary 5.11,

$$K_{\mathrm{cont}}(g_1,g_2) =$$

$$\frac{\sqrt{\mathtt{n}}}{4\pi}\sum_{\chi_1,\chi_2}\sum_{\phi\in\mathcal{B}(\chi_1,\chi_2)}\frac{1}{\|\phi\|^2}\int_{-\infty}^{\infty} h(t)\lambda_{\mathtt{n}}(\chi_1,\chi_2,it)E(\phi,it,g_1)\overline{E(\phi,it,g_2)}dt. \quad (7.11)$$

We need to integrate the above over $(N(\mathbf{Q})\backslash N(\mathbf{A}))^2$. For $m\in\mathbf{Q}$ and real $y>0$, let

$$a_m^\phi(s,y) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} E(\phi,s,n\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right))\,\overline{\theta_m(n)}dn$$

be the adelic Fourier coefficient of $E(\phi,s,n\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right))$. The above coincides with the m^{th} Fourier coefficient of $E_\phi(s,z)$, which was denoted $a_m(s,y)$ earlier. Indeed, using the fundamental domain $[0,1]\times\widehat{\mathbf{Z}}$ for $\mathbf{Q}\backslash\mathbf{A}$,

$$\begin{aligned} a_m^\phi(s,y) &= \int_0^1\int_{\widehat{\mathbf{Z}}} E(\phi,s,\left(\begin{smallmatrix} y & x\\ 0 & 1\end{smallmatrix}\right)\times\left(\begin{smallmatrix} 1 & u\\ 0 & 1\end{smallmatrix}\right))e(-mx)\theta_{\mathrm{fin}}(mu)\,du\,dx\\ &= \int_0^1 E(\phi,s,\left(\begin{smallmatrix} y & x\\ 0 & 1\end{smallmatrix}\right)\times 1_{\mathrm{fin}})e(-mx)dx\int_{\widehat{\mathbf{Z}}}\theta_{\mathrm{fin}}(mu)\,du. \end{aligned}$$

Because $\widehat{\mathbf{Z}} = \ker\theta_{\mathrm{fin}}$, this is

$$= \begin{cases} \displaystyle\int_0^1 E_\phi(s,x+iy)e(-mx)\,dx & m\in\mathbf{Z}\\ 0 & \text{otherwise}\end{cases}$$

$= a_m(s,y)$, as claimed.

Therefore by the formula for Fourier coefficients given in §5.6, the continuous contribution to the trace formula is

$$\begin{aligned} I_{\mathrm{cont}} &= \frac{1}{\sqrt{y_1y_2}}\iint_{(\mathbf{Q}\backslash\mathbf{A})^2} K_{\mathrm{cont}}(\left(\begin{smallmatrix} y_1 & x_1\\ & 1\end{smallmatrix}\right),\left(\begin{smallmatrix} y_2 & x_2\\ & 1\end{smallmatrix}\right))\overline{\theta_{m_1}(x_1)}\theta_{m_2}(x_2)dx_1dx_2\\ &= \frac{\sqrt{\mathtt{n}}}{4\pi\sqrt{y_1y_2}}\sum_{\chi_1,\chi_2}\sum_{\phi}\int_{-\infty}^{\infty}\frac{\lambda_{\mathtt{n}}(\chi_1,\chi_2,it)}{\|\phi\|^2}h(t)a_{m_1}^\phi(it,y_1)\,\overline{a_{m_2}^\phi(it,y_2)}\,dt \end{aligned}$$

$$= \sqrt{\mathtt{n}}\sum_{\chi_1,\chi_2}\sum_{(i_p)}\int_{\mathbf{R}}\frac{\lambda_{\mathtt{n}}(\chi_1,\chi_2,it)\sigma_{it}(\chi_1',\chi_2',m_1)\overline{\sigma_{it}(\chi_1',\chi_2',m_2)}(\frac{m_1}{m_2})^{it}K_{it}(2\pi m_1y_1)K_{it}(2\pi m_2y_2)h(t)}{\|\phi_{(i_p)}\|^2|\Gamma(\frac12+it)|^2|L_N(1+2it,\chi_1\overline{\chi_2})|^2}dt. \quad (7.12)$$

The notation is as in §5.6, and is also recalled in Theorem 7.14 below. We have used the fact that $|C_{(i_p)}| = 1$ and that $K_{it}(x)$ is real.

We explained in §7.1 why the above expression is absolutely convergent. This can also be seen directly, using the following lemma.

Lemma 7.6. *For $y > 0$ and $s = \sigma + it$ with $\sigma \geq 0$,*

$$\frac{|K_s(y)|}{|\Gamma(\frac{1}{2}+s)|} \leq \left(\frac{2}{y}\right)^{\sigma} \frac{(1+2|s|)}{y\sqrt{\pi}}.$$

Proof. By Basset's formula for $K_\nu(z)$ (eq. (1) on p. 172 of [Wa]),

$$\frac{K_s(y)}{\Gamma(\frac{1}{2}+s)} = \left(\frac{2}{y}\right)^{s} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\cos(yx)}{(x^2+1)^{\frac{1}{2}+s}} dx,$$

valid for $\sigma > 0$. Integrating by parts, we have

$$\frac{K_s(y)}{\Gamma(\frac{1}{2}+s)} = \left(\frac{2}{y}\right)^{s} \frac{(1+2s)}{y\sqrt{\pi}} \int_0^\infty \frac{x\sin(yx)}{(x^2+1)^{\frac{3}{2}+s}} dx.$$

This equality is valid on $\sigma > -\frac{1}{2}$ since the right-hand side is convergent for such s. Therefore if $\sigma \geq 0$,

$$\frac{|K_s(y)|}{|\Gamma(\frac{1}{2}+s)|} \leq \left(\frac{2}{y}\right)^{\sigma} \frac{(1+2|s|)}{y\sqrt{\pi}} \int_0^\infty \frac{x}{(x^2+1)^{\frac{3}{2}}} dx = \left(\frac{2}{y}\right)^{\sigma} \frac{(1+2|s|)}{y\sqrt{\pi}}. \qquad \square$$

By the lemma, the combined contribution of the Bessel and Gamma functions in the integrand of (7.12) is $\ll (1+2|t|)^2$. By Corollary 2.3,

$$L_N(1+2it, \chi_1\overline{\chi_2})^{-1} = L(1+2it, \overline{\chi_1'}\chi_2')^{-1} \ll_\varepsilon N^\varepsilon(\log(3+2|t|))^7. \tag{7.13}$$

Thus, the absolute convergence of the integral (7.12) follows from

$$h(t) \ll (1+|t|)^{-4} \tag{7.14}$$

(which holds since $h(iz)$ is Paley-Wiener of order $m \geq 10 > 4$).

In fact we can prove the following asymptotic bound for I_{cont}.

Proposition 7.7. *For any $\varepsilon > 0$, the quantity (7.12) is $\ll N^{\frac{1}{2}+\varepsilon}$. The implied constant is ineffective, but depends only on ε, $\mathtt{n}$, m_1, m_2, f_∞, and y_1, y_2.*

Proof. By Proposition 5.17, we have

$$\frac{\left|\sigma_{it}(\chi_1', \chi_2', m_1)\overline{\sigma_{it}(\chi_1', \chi_2', m_2)}\right|}{\|\phi_{(i_p)}\|^2} \ll_{m_1,m_2,\varepsilon} N^\varepsilon. \tag{7.15}$$

It is clear that

$$|\lambda_{\mathtt{n}}(\chi_1, \chi_2, it)| = \left|\sum_{d|\mathtt{n}} (\tfrac{\mathtt{n}}{d^2})^{it}\overline{\chi_1(d_N)\chi_2((\tfrac{\mathtt{n}}{d})_N)}\right| \leq \tau(\mathtt{n}) \ll_{\mathtt{n}} 1.$$

Therefore by Lemma 7.6 and (7.13)-(7.15), the integral occurring in (7.12) is

$$\ll N^{3\varepsilon}\int_{-\infty}^{\infty} h(t)(1+2|t|)^2(\log(3+2|t|))^{14}dt = O(N^{3\varepsilon}), \tag{7.16}$$

with the implied constant depending on $f_\infty, \varepsilon, y_1, y_2, m_1, m_2$, and $\mathtt{n}$. Thus, recalling Corollaries 5.10 and 5.11, (7.12) is

$$\ll N^{3\varepsilon} \sum_{\substack{(i_p)\\ 0\le i_p\le N_p}} \sum_{\substack{(\nu_{1p},\nu_{2p})\\ \nu_{2p}\le i_p\le N_p-\nu_{1p}}} \sum_{\substack{\chi_1\chi_2=\omega\\ \mathfrak{c}_{\chi_j}=\prod_p p^{\nu_{jp}}\\ (j=1,2)}} 1.$$

The set of tuples (i_p) is in 1-1 correspondence with the set of positive divisors $M=\prod p^{i_p}$ of N, and likewise $\{(\nu_{1p})\}\leftrightarrow\{\nu_1|\frac{N}{M}\}$ and $\{(\nu_{2p})\}\leftrightarrow\{\nu_2|M\}$. Hence the above triple sum can be rewritten

$$\sum_{M|N}\sum_{\nu_2|M}\sum_{\nu_1|\frac{N}{M}}\sum_{\substack{\chi_1\chi_2=\omega\\ \mathfrak{c}_{\chi_1}=\nu_1\\ \mathfrak{c}_{\chi_2}=\nu_2}} 1.$$

The number of terms M is $\ll N^\varepsilon$, and the same is true for ν_1 and ν_2. Thus for fixed M,ν_1,ν_2, it remains to count the number of pairs (χ_1,χ_2). Because $\nu_1\nu_2|N$, there exists $j\in\{1,2\}$ such that $\nu_j\le N^{1/2}$. The number of possibilities for the character χ_j of conductor ν_j is $|(\mathbf{Z}/\nu_j\mathbf{Z})^*|\le N^{1/2}$. Once χ_j is chosen, its counterpart is determined by $\chi_1\chi_2=\omega$. Thus the number of pairs (χ_1,χ_2) is $\le N^{1/2}$. This proves that (7.12) is $\ll N^{\frac12+6\varepsilon}$. $\square$

Lastly, let $I_{\mathrm{cont}}(w)$ denote the quantity in (7.12) with $w=m_1y_1=m_2y_2$. Using (7.7), we see that $\int_0^\infty I_{\mathrm{cont}}(w)dw$ is equal to

$$\frac{\sqrt{\mathtt{n}}}{8}\sum_{\chi_1\chi_2=\omega}\sum_{(i_p)}\int_{-\infty}^{\infty}\frac{\lambda_{\mathtt{n}}(\chi_1,\chi_2,it)\sigma_{it}(\chi_1',\chi_2',m_1)\overline{\sigma_{it}(\chi_1',\chi_2',m_2)}(\frac{m_1}{m_2})^{it}h(t)}{\|\phi_{(i_p)}\|^2|L_N(1+2it,\chi_1\overline{\chi_2})|^2}dt. \tag{7.17}$$

The exchange of the dt and dw integrals is justified by the absolute convergence of (7.17), which is proven in the same way as Proposition 7.7, giving the following.

Proposition 7.8. *For any $\varepsilon>0$, the quantity* (7.17) *is* $\ll_\varepsilon N^{\frac12+\varepsilon}$.

7.5 Geometric side

In this section, we take f as in (7.1), although we can relax the requirement on m. In Proposition 7.9, we will require $m\ge5$, and in Propositions 7.11 and 7.12 we need $m>2$.

For positive integers m_1,m_2, we need to evaluate the integral (7.1)

$$I=I_{f,m_1,m_2,y_1,y_2}=\frac{1}{\sqrt{y_1y_2}}\int_{\mathbf{Q}\backslash\mathbf{A}}\int_{\mathbf{Q}\backslash\mathbf{A}}K\left(\left(\begin{smallmatrix}y_1&t_1\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}y_2&t_2\\0&1\end{smallmatrix}\right)\right)\theta(m_1t_1-m_2t_2)dt_1\,dt_2,$$

using $K(g_1,g_2)=\sum_{\gamma\in\overline{G}(\mathbf{Q})} f(g_1^{-1}\gamma g_2)$. This was carried out in detail with a different choice of f_∞ in [KL1], and we follow the same procedure here.

Let $H=N\times N$, and endow it with an action on $\overline{G}$ by

$$(n_1,n_2)\cdot\gamma = n_1^{-1}\gamma n_2.$$

We break the sum over $\gamma\in\overline{G}(\mathbf{Q})$ into $H(\mathbf{Q})$-orbits to get

$$I=\sum_{\delta\in N(\mathbf{Q})\backslash\overline{G}(\mathbf{Q})/N(\mathbf{Q})}\int_{H_\delta(\mathbf{Q})\backslash H(\mathbf{A})} f\big(\left(\begin{smallmatrix} y_1 & t_1\\ & 1\end{smallmatrix}\right)^{-1}\delta\left(\begin{smallmatrix} y_2 & t_2\\ & 1\end{smallmatrix}\right)\big)\frac{\overline{\theta_{m_1}(t_1)}\theta_{m_2}(t_2)}{\sqrt{y_1y_2}}\,d(t_1,t_2).$$

Here H_δ denotes the stabilizer of δ, and $d(t_1,t_2)$ denotes the quotient measure coming from the Haar measure dt_1dt_2 on $H(\mathbf{A})\cong\mathbf{A}\times\mathbf{A}$, the latter being normalized as in Section 2.1. The interchange of the sum and the integral is justified because the function $\sum_\gamma |f(x^{-1}\gamma y)|$ is continuous and hence integrable over the compact set $H(\mathbf{Q})\backslash H(\mathbf{A})$.

We let $I_\delta(f)$ denote the integral attached to δ as above. The following set of representatives δ is obtained from the Bruhat decomposition:

$$\{\left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right)|\ \gamma\in\mathbf{Q}^*\}\bigcup\{\left(\begin{smallmatrix}0 & -\mu\\ 1 & 0\end{smallmatrix}\right)|\ \mu\in\mathbf{Q}^*\}.$$

An orbit δ is **relevant** if the character $\overline{\theta_{m_1}}\theta_{m_2}$ is trivial on $H_\delta(\mathbf{A})$. The orbital integral $I_\delta(f)$ vanishes if δ is not relevant. It is straightforward to show that the relevant orbits are

$$\{\left(\begin{smallmatrix}m_2/m_1 & 0\\ 0 & 1\end{smallmatrix}\right)\}\bigcup\{\left(\begin{smallmatrix}0 & -\mu\\ 1 & 0\end{smallmatrix}\right)|\ \mu\in\mathbf{Q}^*\}.$$

See [KL1] for details.

7.5.1 First cell term

Proposition 7.9. *Let $\delta=\left(\begin{smallmatrix}m_2/m_1 & \\ & 1\end{smallmatrix}\right)$ for $m_1,m_2>0$. Then $I_\delta(f)$ is nonzero only if $m_1m_2=b^2\mathtt{n}$ for some positive integer $b|\gcd(m_1,m_2)$. If this condition is met, then*

$$I_\delta(f)=\frac{\psi(N)\overline{\omega'(m_1/b)}}{b\sqrt{y_1y_2}}\int_{-\infty}^{\infty}V\left(\frac{t^2+m_1^2y_1^2+m_2^2y_2^2}{m_1y_1m_2y_2}-2\right)e^{2\pi i t}dt \qquad (7.18)$$

for V as in (3.5). Letting $I_\delta(w)$ denote the above quantity for $w=y_1m_1=y_2m_2$, we have

$$\int_0^\infty I_\delta(w)dw=\frac{\psi(N)\sqrt{\mathtt{n}}}{2\,\omega'(\sqrt{m_1\mathtt{n}/m_2})}V(0). \qquad (7.19)$$

Proof. For this choice of δ, we find as in [KL1] that

$$H_\delta(\mathbf{Q})=\left\{\left(\begin{pmatrix}1 & \frac{m_2}{m_1}t\\ & 1\end{pmatrix},\begin{pmatrix}1 & t\\ & 1\end{pmatrix}\right)|\ t\in\mathbf{Q}\right\}.$$

Now note that

$$\begin{pmatrix} y_1 & t_1 \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{m_2}{m_1} & \\ & 1 \end{pmatrix} \begin{pmatrix} y_2 & t_2 \\ & 1 \end{pmatrix} = \begin{pmatrix} \frac{m_2 y_2}{m_1 y_1} & \frac{1}{y_1}(\frac{m_2}{m_1} t_2 - t_1) \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} m_1 y_1 & \\ & m_1 y_1 \end{pmatrix}^{-1} \begin{pmatrix} m_2 y_2 & m_2 t_2 - m_1 t_1 \\ & m_1 y_1 \end{pmatrix}.$$

Here we view $m_j \in \mathbf{Q}^* \subseteq \mathbf{A}^*$ and $y_j \in \mathbf{R}^+$. Thus because f is invariant under $Z(\mathbf{R})^+ Z(\mathbf{Q})$,

$$I_\delta(f) = \iint\limits_{\{(\frac{m_2}{m_1} t_2, t_2) \in \mathbf{Q}^2\} \backslash \mathbf{A}^2} f(\begin{pmatrix} m_2 y_2 & m_2 t_2 - m_1 t_1 \\ 0 & m_1 y_1 \end{pmatrix}) \frac{\theta(m_1 t_1 - m_2 t_2)}{\sqrt{y_1 y_2}} d(t_1, t_2).$$

Here $d(t_1, t_2)$ is the quotient measure coming from $dt_1 dt_2$. In $\mathbf{A}^2$, let $t = m_2 t_2 - m_1 t_1$. Then the map $\mathbf{A}^2 \to \mathbf{A}^2$ defined by $(t_1, t_2) \mapsto (t, t_2)$ induces a homeomorphism between the quotient spaces

$$\left\{ \left(\tfrac{m_2}{m_1} t_2, t_2 \right) | t_2 \in \mathbf{Q} \right\} \backslash \mathbf{A}^2 \longrightarrow \{(0, t_2) | t_2 \in \mathbf{Q}\} \backslash \mathbf{A}^2 = \mathbf{A} \times (\mathbf{Q} \backslash \mathbf{A}).$$

Noting that $dt\, dt_2 = |m_1|_{\mathbf{A}} dt_1 dt_2 = dt_1 dt_2$, we see that the quotient measure is $d(t_1, t_2) = dt\, dt_2$, where we use dt_2 now to represent the quotient measure on $\mathbf{Q} \backslash \mathbf{A}$. Thus

$$I_\delta(f) = \int_{\mathbf{A}} \int_{\mathbf{Q} \backslash \mathbf{A}} f(\left(\begin{smallmatrix} m_2 y_2 & t \\ & m_1 y_1 \end{smallmatrix} \right)) \frac{\theta(-t) dt_2\, dt}{\sqrt{y_1 y_2}} = \int_{\mathbf{A}} f(\left(\begin{smallmatrix} m_2 y_2 & t \\ & m_1 y_1 \end{smallmatrix} \right)) \frac{\theta(-t)}{\sqrt{y_1 y_2}} dt.$$

The integral factors as $I_\delta(f)_{\text{fin}} I_\delta(f)_\infty$. As shown in Proposition 3.3 of [KL1], the finite part vanishes unless $m_1 m_2 = b^2 \mathtt{n}$ for some positive integer $b | \gcd(m_1, m_2)$, in which case

$$I_\delta(f)_{\text{fin}} = \frac{\psi(N)}{b \omega'(m_1/b)}.$$

The archimedean part is

$$I_\delta(f)_\infty = \frac{1}{\sqrt{y_1 y_2}} \int_{-\infty}^{\infty} f_\infty(\begin{pmatrix} m_2 y_2 & t \\ & m_1 y_1 \end{pmatrix}) e^{2\pi i t} dt,$$

and (7.18) follows upon using (3.7).

Set $w = y_1 m_1 = y_2 m_2$ in (7.18), so $\frac{1}{\sqrt{y_1 y_2}} = \frac{\sqrt{m_1 m_2}}{w}$. Then

$$I_\delta(w) = \frac{\psi(N) \sqrt{m_1 m_2}}{b \omega'(m_1/b)} \frac{1}{w} \int_{-\infty}^{\infty} V(\frac{t^2}{w^2}) e^{2\pi i t} dt.$$

Replacing t by wt and dt by $\frac{dt}{w}$, and using $\frac{\sqrt{m_1 m_2}}{b} = \sqrt{\mathtt{n}}$, we have

$$\int_0^\infty I_\delta(w) dw = \frac{\psi(N) \sqrt{\mathtt{n}}}{\omega'(\sqrt{m_1 \mathtt{n}/m_2})} \int_0^\infty \int_{-\infty}^\infty V(t^2) e^{2\pi i w t} dt\, dw.$$

Let $r(t) = V(t^2)$, a compactly supported continuous even function. Note that

$$\hat{r}(w) = \int_{-\infty}^{\infty} V(t^2)e^{-2\pi iwt}dt = \int_{-\infty}^{\infty} V(t^2)e^{2\pi iwt}dt$$

is also an even function. By Fourier inversion,

$$\frac{1}{2}V(0) = \frac{1}{2}r(0) = \frac{1}{2}\int_{-\infty}^{\infty} \hat{r}(w)dw = \int_0^{\infty} \hat{r}(w)dw,$$

which proves (7.19). We recall that Fourier inversion is valid so long as r and $\hat{r}$ are both integrable. Because $f_\infty \in C_c^m(G^+//K_\infty)$ for $m \geq 5$, V and r are compactly supported and twice continuously differentiable by Proposition 3.2. It follows by a standard argument (see Proposition 8.8 below) that $\hat{r}(w) \ll (1+w^2)^{-1}$ and hence is integrable. □

7.5.2 Second cell terms

We will need a few facts and definitions. For the function $k(z_1, z_2)$ attached to f_∞ in (3.8), define its **Zagier transform** by

$$\mathcal{Z}k(s,t) = \iint_{\mathbf{H}} k(z+t, \frac{-1}{z})y^s dz,$$

where $dz = \frac{dx\,dy}{y^2}$. Using Proposition 3.3, we see that

$$\mathcal{Z}k(s,t) = \iint_{\mathbf{H}} V\Big(\frac{|z^2+tz+1|^2}{y^2}\Big)y^s dz = \iint_{\mathbf{H}} V\Big(\frac{|z^2+1-\frac{t^2}{4}|^2}{y^2}\Big)y^s dz,$$

where the second expression comes from completing the square and replacing z by $z - \frac{t}{2}$. We refer to Proposition 4 of [Za] (where the above is denoted $V(s,t)$) for the absolute convergence and other information. In Section 5 of [Za], it is computed in terms of the Selberg transform $h(t)$.

We will only be interested in the case $s = 1$, so we set

$$Z(t) = \mathcal{Z}k(1,t) = \iint_{\mathbf{H}} V\left(\frac{|z^2+1-\frac{t^2}{4}|^2}{y^2}\right)\frac{dy}{y}dx. \qquad (7.20)$$

This is expressed in terms of the Selberg transform $h(t)$ in (4.12) of [Za]. Since V is compactly supported, $Z(t)$ is also compactly supported as a function of $t \in \mathbf{R}$. Indeed, writing $u = x^2$, $v = y^2$ and $w = -(1-\frac{t^2}{4})$, we have

$$\frac{|z^2+1-\frac{t^2}{4}|^2}{y^2} = \frac{(u-v-w)^2+4uv}{v} = \frac{(-u-v+w)^2+4vw}{v} \geq 4w = t^2-4.$$

Thus, if $|t|$ is sufficiently large, $t^2 - 4$ exceeds the supremum of $\mathrm{Supp}(V)$, and the integrand of (7.20) is 0.

The orbital integral attached to $\delta = \left(\begin{smallmatrix} 0 & -\mu \\ 1 & 0 \end{smallmatrix}\right)$ involves the Fourier transform

$$\widehat{Z}(a) = \int_{-\infty}^{\infty} Z(t)e^{-2\pi iat}dt = \int_{-\infty}^{\infty} Z(t)e^{2\pi iat}dt.$$

Proposition 7.10. *For $a \neq 0$, we have*

$$\widehat{Z}(a) = \frac{i}{4a}\int_{-\infty}^{\infty} J_{2it}(4\pi a)\frac{h(t)\,t}{\cosh(\pi t)}\,dt$$

for the J-Bessel function, and the Selberg transform $h(t)$ of f_∞.

Proof. This is due to Zagier. The proof is explained in §2.1 of [Joy]. Another account is given in [LiX], Lemma 3.4. The absolute convergence of the integral holds by the fact that $h \in PW^m(\mathbf{C})^{\text{even}}$ for $m > 2$ (see the proof of Proposition 7.12 below). □

Also, for any $c \in \mathfrak{c}_{\omega'}\mathbf{Z}^+$, we will need the following generalized Kloosterman sum:

$$S_{\omega'}(a,b;\mathtt{n};c) = \sum_{\substack{d,d'\in\mathbf{Z}/c\mathbf{Z},\\ dd'=\mathtt{n}}} \overline{\omega'(d)}e(\frac{ad+bd'}{c}). \tag{7.21}$$

We will describe some basic properties of these sums in Section 9.

Proposition 7.11. *Let $\delta = \left(\begin{smallmatrix} 0 & -\mu \\ 1 & 0\end{smallmatrix}\right)$ for $\mu \in \mathbf{Q}^*$. Then $I_\delta(f)$ is nonzero only if $\mu = \frac{\mathtt{n}}{c^2}$ for some positive integer $c \in N\mathbf{Z}$. Under this condition,*

$$I_\delta(f) = \psi(N)\frac{S_{\omega'}(m_2,m_1;\mathtt{n};c)}{\sqrt{y_1y_2}}\iint\limits_{\mathbf{R}\times\mathbf{R}} k(z_1,\frac{-\mathtt{n}}{c^2z_2})e^{2\pi i(m_2x_2-m_1x_1)}dx_1dx_2, \tag{7.22}$$

where $k(z_1,z_2) = f_\infty(g_1^{-1}g_2)$ as in (3.8). Taking $I_\delta(w)$ to be the above quantity when $w = m_1y_1 = m_2y_2$, the integral $\int_0^\infty I_\delta(w)dw$ equals

$$\frac{i\sqrt{\mathtt{n}}\psi(N)}{4}\frac{S_{\omega'}(m_2,m_1;\mathtt{n};c)}{c}\int_{-\infty}^{\infty} J_{2it}\left(\frac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right)\frac{h(t)\,t}{\cosh(\pi t)}dt. \tag{7.23}$$

Proof. For $\delta = \left(\begin{smallmatrix} & -\mu \\ 1 & \end{smallmatrix}\right)$, we find that $H_\delta(\mathbf{A}) = \{(1,1)\}$. Given $y_1,y_2 > 0$, we need to integrate

$$f(\left(\begin{smallmatrix} y_1 & x_1 \\ & 1\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} & -\mu \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} y_2 & x_2 \\ & 1\end{smallmatrix}\right))\theta(m_1x_1-m_2x_2). \tag{7.24}$$

Again the integral factors as $I_\delta(f)_{\text{fin}}I_\delta(f)_\infty$. Because f_∞ is supported on $G(\mathbf{R})^+$, the archimedean integral vanishes unless $\mu > 0$. Under this assumption, the finite part was shown in Proposition 3.7 of [KL1] to vanish unless $\mu = \frac{\mathtt{n}}{c^2}$ for some $c \in N\mathbf{Z}^+$, in which case

$$I_\delta(f)_{\text{fin}} = \frac{\psi(N)}{\omega'(-1)}S_{\omega'}(m_2,m_1;\mathtt{n};c) = \psi(N)S_{\omega'}(m_2,m_1;\mathtt{n},c). \tag{7.25}$$

From (3.8), $f_\infty(\left(\begin{smallmatrix} y_1 & x_1 \\ & 1\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} & -\mu \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} y_2 & x_2 \\ & 1\end{smallmatrix}\right)) = k(z_1,\frac{-\mu}{z_2})$, so the archimedean part can be written as

$$I_\delta(f)_\infty = \frac{1}{\sqrt{y_1y_2}}\iint_{\mathbf{R}\times\mathbf{R}} k(z_1,\frac{-\mu}{z_2})e(m_2x_2-m_1x_1)dx_1dx_2.$$

Setting $\mu = \frac{\mathtt{n}}{c^2}$, (7.22) follows immediately.

Now write $x_1 = \sqrt{\mu m_2/m_1}\, t_1$ and $x_2 = \sqrt{\mu m_1/m_2}\, t_2$, so that $dx_1 dx_2 = \mu dt_1 dt_2 = \frac{\mathtt{n}}{c^2} dt_1 dt_2$. Then

$$\left(x_1 + iy_1, \frac{-\mu}{x_2 + iy_2}\right) = \sqrt{\frac{\mu m_2}{m_1}} \left(t_1 + \frac{iy_1 m_1}{\sqrt{\mu m_1 m_2}}, \frac{-1}{t_2 + \frac{i m_2 y_2}{\sqrt{\mu m_1 m_2}}}\right).$$

The scalar in front does not affect the value of k by (3.9). Set $w = m_1 y_1 = m_2 y_2$, so that $\frac{1}{\sqrt{y_1 y_2}} = \frac{\sqrt{m_1 m_2}}{w}$. The archimedean part of the integral of $I_\delta(w)$ is the product of $\frac{\mathtt{n}\sqrt{m_1 m_2}}{c^2}$ with

$$\int_0^\infty \iint_{\mathbf{R}\times\mathbf{R}} k\Big(t_1 + \frac{iwc}{\sqrt{\mathtt{n} m_1 m_2}}, \frac{-1}{t_2 + \frac{iwc}{\sqrt{\mathtt{n} m_1 m_2}}}\Big) e^{2\pi i \frac{\sqrt{\mathtt{n} m_1 m_2}}{c}(t_2 - t_1)} dt_1 dt_2 \frac{dw}{w}.$$

Arguing formally for the moment, we exchange the order of integration. Substitute $t = t_1 - t_2$ for t_1, x for t_2, and y for $\frac{wc}{\sqrt{\mathtt{n} m_1 m_2}}$. Then because $\frac{dw}{w}$ is a multiplicative Haar measure, the above integral is

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\infty k\left(x + iy + t, \frac{-1}{x+iy}\right) \frac{1}{y} dy\, dx\, e^{-2\pi i t \frac{\sqrt{\mathtt{n} m_1 m_2}}{c}} dt \tag{7.26}$$

$$= \widehat{Z}\Big(\frac{\sqrt{\mathtt{n} m_1 m_2}}{c}\Big) = \frac{ic}{4\sqrt{\mathtt{n} m_1 m_2}} \int_{-\infty}^\infty J_{2it}\Big(\frac{4\pi\sqrt{\mathtt{n} m_1 m_2}}{c}\Big) \frac{h(t)\, t}{\cosh(\pi t)} dt$$

by Proposition 7.10. Formula (7.23) now follows upon multiplying by $\frac{\mathtt{n}\sqrt{m_1 m_2}}{c^2}$ and the finite part (7.25). The exchange of the order of integration is justified by Fubini's Theorem. Indeed, as explained after (7.20), $\mathcal{Z}|k|(1,t)$ is compactly supported as a function of t. Therefore the triple integral (7.26) is absolutely convergent. $\square$

The geometric side is equal to the main term from Proposition 7.9 plus the sum over $c \in N\mathbf{Z}^+$ of the term in Proposition 7.11.

Proposition 7.12. *We have the following bound for the sum of the Kloosterman terms on the refined geometric side:*

$$\sum_{c \in N\mathbf{Z}^+} \frac{i\sqrt{\mathtt{n}}\psi(N)}{4} \frac{S_{\omega'}(m_2, m_1; \mathtt{n}; c)}{c} \int_{-\infty}^\infty J_{2it}\left(\frac{4\pi\sqrt{\mathtt{n} m_1 m_2}}{c}\right) \frac{h(t)\, t}{\cosh(\pi t)} dt \tag{7.27}$$

$$= O(N^\varepsilon),$$

where the implied constant depends on $\mathtt{n}$, h, m_1, m_2, *and* $0 < \varepsilon < 1$.

Proof. (See also Theorem 16.8 on page 414 of [IK] for the case $\mathtt{n} = 1$, $\omega = 1$.) We will show below that

$$\int_{-\infty}^\infty J_{2it}\left(\frac{4\pi\sqrt{\mathtt{n} m_1 m_2}}{c}\right) \frac{h(t)\, t}{\cosh(\pi t)} dt \ll \left(\frac{2\pi\sqrt{\mathtt{n} m_1 m_2}}{c}\right)^{1-\varepsilon}. \tag{7.28}$$

In Theorem 9.1 we will prove the Weil-type bound

$$|S_{\omega'}(m_2, m_1; \mathtt{n}; c)| \leq \tau(\mathtt{n})\tau(c)(m_2\mathtt{n}, m_1\mathtt{n}, c)^{1/2}c^{1/2}\mathfrak{c}_{\omega'}^{1/2}.$$

Together these statements imply that (7.27) is

$$\ll \psi(N)\mathfrak{c}_{\omega'}^{1/2} \sum_{c\in N\mathbf{Z}^+} \frac{\tau(c)}{c^{3/2-\varepsilon}} \leq \frac{\psi(N)N^{1/2}\tau(N)}{N^{3/2-\varepsilon}} \sum_{c\in\mathbf{Z}^+} \frac{\tau(c)}{c^{3/2-\varepsilon}}. \tag{7.29}$$

Using $\tau(N) \ll N^\varepsilon$ and $\psi(N) = N\prod_{p|N}(1+\frac{1}{p}) \ll N^{1+\varepsilon}$, this gives

$$(7.27) \ll \frac{N^{1+\varepsilon}N^\varepsilon}{N^{1-\varepsilon}} = N^{3\varepsilon},$$

as needed.

It remains to establish (7.28). We let $s = \sigma + it$ be a complex variable. Let $\sigma_0 < \frac{1}{2}$ be a fixed positive number. The restriction on σ_0 is to ensure that $\cosh(-i\pi s)$ is nonzero on the strip $0 \leq \sigma \leq \sigma_0$. From the integral representation

$$J_s(x) = \frac{(x/2)^s}{\Gamma(s+1/2)\sqrt{\pi}} \int_0^\pi \cos(x\cos\theta)\sin^{2s}\theta\, d\theta \qquad (\operatorname{Re} s > -\tfrac{1}{2})$$

([AAR], Corollary 4.11.2), we see that for $\sigma \geq 0$,

$$J_s(x) \ll \left(\frac{x}{2}\right)^\sigma \frac{1}{|\Gamma(s+\frac{1}{2})|}$$

for an absolute implied constant. By the hypotheses on $h(t)$, there exists a positive constant C such that

$$h(-is) \ll \frac{C^\sigma}{(1+|t|)^M},$$

for any $m \geq M \geq 2$. By these asymptotics, the integrand of (7.28) is

$$J_{2s}\left(\frac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right)\frac{h(-is)(-is)}{\cosh(-is\pi)} \tag{7.30}$$

$$\ll \left(\frac{2\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right)^{2\sigma} \frac{|s|C^\sigma}{(1+|t|)^M} \frac{1}{|\Gamma(2s+\frac{1}{2})\cosh(-is\pi)|}.$$

By [AAR], Corollary 1.4.4, for $0 \leq \sigma \leq \frac{1}{2}$ and $|t| \geq 1$,

$$|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}(1+O(1/|t|))$$

for an absolute implied constant. Thus for $0 \leq \sigma \leq \sigma_0 < \frac{1}{2}$ and $|t| \geq 1$,

$$\frac{1}{|\Gamma(2s+\frac{1}{2})\cosh(-is\pi)|} \ll |t|^{-2\sigma}\frac{e^{\pi|t|}}{|e^{(t-i\sigma)\pi}+e^{(-t+i\sigma)\pi}|} = O_{\sigma_0}(1).$$

The left hand side is continuous and hence bounded on the compact set $0 \le \sigma \le \sigma_0$, $|t| \le 1$. Thus the expression is bounded on the whole strip $0 \le \sigma \le \sigma_0$. (We have imposed $\sigma_0 < \frac{1}{2}$ in order to avoid the zero of $\cosh(-is\pi) = \cos(s\pi)$ at $s = \frac{1}{2}$.) Hence for such σ,

$$J_{2s}\left(\frac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right)\frac{h(-is)(-is)}{\cosh(-is\pi)} \ll_{\sigma_0} \left(\frac{2\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right)^{2\sigma}\frac{|s|C^{\sigma}}{(1+|t|)^M}. \tag{7.31}$$

Let T be an arbitrary large real number, and let $\mathcal{R}_T$ be the contour defined by the rectangle with vertices $A = -iT$, $B = \sigma_0 - iT$, $C = \sigma_0 + iT$ and $D = iT$, with counter-clockwise orientation. By the Cauchy residue theorem,

$$\int_{\mathcal{R}_T} J_{2s}\left(\frac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right)\frac{h(-is)(-is)}{\cosh(-is\pi)}dt = 0.$$

By the estimate (7.31) with $M \ge 2$, we see that

$$\lim_{T\to\infty}\int_{\overline{AB}}(7.30)ds = \lim_{T\to\infty}\int_{\overline{CD}}(7.30)ds = 0,$$

and that (7.30) is absolutely integrable over $i\mathbf{R}$ and $\sigma_0 + i\mathbf{R}$. Taking $T \to \infty$,

$$\int_{-\infty}^{\infty} J_{2it}\left(\frac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right)\frac{h(t)\,t}{\cosh(\pi t)}dt = \int_{\operatorname{Re}s=\sigma_0} J_{2s}\left(\frac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right)\frac{h(-is)(-is)}{\cosh(-is\pi)}dt$$
$$\ll_{\sigma_0}\left(\frac{2\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right)^{2\sigma_0}$$

by (7.31) with $M > 2$. Taking $\sigma_0 = \frac{1}{2} - \frac{\varepsilon}{2}$, we obtain (7.28). □

7.6 Final formulas

The formulas given below follow upon equating the geometric side with the spectral side in the two cases (primitive and refined) computed above.

Theorem 7.13 (Pre-KTF)**.** *Let $\mathcal{F}$ be an orthogonal eigenbasis of $T_{\mathtt{n}}$ for the space $L_0^2(N,\omega')$ of cusp forms of weight 0, chosen as in* (4.25)*. Let $h(iz) \in PW^{12}(\mathbf{C})^{even}$, and let $f_\infty \in C_c^{10}(G^+//K_\infty)$, V, and k be the associated functions as in* (3.15)*,* (3.5) *and* (3.8)*. Then for any positive integers m_1, m_2, and real $y_1, y_2 > 0$, we have*

$$\sqrt{\mathtt{n}}\sum_{u_j\in\mathcal{F}}\frac{\lambda_{\mathtt{n}}(u_j)\,a_{m_1}(u_j)\overline{a_{m_2}(u_j)}}{\|u_j\|^2}h(t_j)K_{it_j}(2\pi m_1y_1)\,K_{it_j}(2\pi m_2y_2)$$

$$+\sqrt{\mathtt{n}}\sum_{\chi_1,\chi_2}\sum_{(i_p)}\int_{-\infty}^{\infty}\frac{\lambda_{\mathtt{n}}(\chi_1,\chi_2,it)h(t)(\frac{m_1}{m_2})^{it}K_{it}(2\pi m_1y_1)K_{it}(2\pi m_2y_2)}{\|\phi_{(i_p)}\|^2\,|\Gamma(\frac{1}{2}+it)L_N(1+2it,\chi_1\overline{\chi_2})|^2}$$
$$\times\,\sigma_{it}(\chi_1',\chi_2',m_1)\overline{\sigma_{it}(\chi_1',\chi_2',m_2)}\,dt$$

$$= T(m_1,m_2,\mathtt{n})\frac{\sqrt{\mathtt{n}}\,\psi(N)\overline{\omega'(\sqrt{\frac{\mathtt{n}m_1}{m_2}})}}{\sqrt{m_1m_2y_1y_2}}\int_{-\infty}^{\infty}V\left(\frac{t^2+m_1^2y_1^2+m_2^2y_2^2}{m_1y_1m_2y_2}-2\right)e^{2\pi it}dt$$

$$+\psi(N)\sum_{c\in N\mathbf{Z}^+}\frac{S_{\omega'}(m_2,m_1;\mathtt{n};c)}{\sqrt{y_1y_2}}\iint\limits_{\mathbf{R}\times\mathbf{R}}k(z_1,\frac{-\mathtt{n}}{c^2z_2})e^{2\pi i(m_2x_2-m_1x_1)}dx_1dx_2,$$

where:

- χ_1,χ_2 *range through all ordered pairs of finite order Hecke characters with* $\chi_1\chi_2=\omega$ *and whose conductors satisfy* $\mathfrak{c}_{\chi_1}\mathfrak{c}_{\chi_2}|N$.
- $L_N(s,\chi_1\overline{\chi_2})$ *is the partial L-function defined in* (2.11).
- $\lambda_{\mathtt{n}}(\chi_1,\chi_2,it)=\sum_{d|\mathtt{n}}\left(\frac{\mathtt{n}}{d^2}\right)^{it}\overline{\chi_1(d_N)\chi_2((\frac{\mathtt{n}}{d})_N)}$, *where* d_N *is the idele which agrees with* d *at all places* $p|N$ *and is* 1 *at all other places.*
- (i_p) *runs through all sequences* $(i_p)_{p|N}$ *with*

$$\mathrm{ord}_p(\mathfrak{c}_{\chi_2})\le i_p\le N_p-\mathrm{ord}_p(\mathfrak{c}_{\chi_1}).$$

- χ_1' *is the Dirichlet character of modulus* $N_1=\prod_{\substack{p|N\\ i_p<N_p}}p^{N_p}$ *attached to* χ_1 *as in* (2.8).
- χ_2' *is the Dirichlet character of modulus* $N_2=\prod_{\substack{p|N\\ i_p>0}}p^{N_p}$ *attached to* χ_2 *as in* (2.8).
- $z_1=x_1+iy_1,\ z_2=x_2+iy_2$.
- *All other notation is given in Theorem 7.14 below.*

Remark: The hypothesis that h be Paley-Wiener of order 12 arises from the following places. We need the inverse Selberg transform f_∞ to be in C_c^8 in order to apply Corollary 6.12, whose hypothesis stems from the restrictions in Lemma 6.9. By Proposition 3.6, we are only able to guarantee this if $h\in PW^{10}$. Furthermore, we needed $z^2h(iz)\in PW^{10}$ to prove the convergence of the cuspidal term. As remarked there, assuming Weyl's Law would render this step unnecessary. In computing the main geometric term, we required V to be twice differentiable to justify using Fourier inversion. For this it would be enough for f_∞ to be in C_c^5 (see Proposition 3.2), or for h to be Paley-Wiener of order $m>4$ (see Proposition 8.16 below). We will discuss weakening the hypotheses in Section 8.

For the refined version of the KTF given below, we have multiplied each term by $\frac{8}{\pi\sqrt{\mathtt{n}}}$, and we have used formula (3.17) for $V(0)$. We have also expressed everything in purely classical (non-adelic) terms, replacing the sum over pairs χ_1, χ_2 of Hecke characters of conductor dividing N by a sum over pairs $\widetilde{\chi}_1, \widetilde{\chi}_2$ of Dirichlet characters of modulus N. Indeed, the two correspond bijectively by (2.8). Furthermore, $L_N(s, \chi_1\overline{\chi_2}) = L(s, \widetilde{\chi}_1^{-1}\widetilde{\chi}_2)$ by (2.10). Lastly, we point out that by that fact that χ_{1p} is unramified when $i_p = N_p$, $\widetilde{\chi}_1$ is induced (in the sense of (2.7)) from the Dirichlet character χ_1' of modulus N_1 attached to χ_1 in the above theorem, and likewise $\widetilde{\chi}_2$ is induced from χ_2'.

Theorem 7.14 (KTF). *Let $h(iz) \in PW^{12}(\mathbf{C})^{\text{even}}$ (see the remark above). Let $\mathcal{F}$ be an orthogonal eigenbasis of $T_{\mathtt{n}}$ for the space $L_0^2(N, \omega')$ of cusp forms of weight 0, chosen as in (4.25). Then for any positive integers m_1, m_2, we have*

$$\sum_{u_j \in \mathcal{F}} \frac{\lambda_{\mathtt{n}}(u_j)\, a_{m_1}(u_j)\overline{a_{m_2}(u_j)}}{\|u_j\|^2} \frac{h(t_j)}{\cosh(\pi t_j)}$$

$$+\frac{1}{\pi} \sum_{\widetilde{\chi}_1, \widetilde{\chi}_2} \sum_{(i_p)} \int_{-\infty}^{\infty} \frac{\lambda_{\mathtt{n}}(\chi_1', \chi_2', it)\sigma_{it}(\chi_1', \chi_2', m_1)\overline{\sigma_{it}(\chi_1', \chi_2', m_2)}(\frac{m_1}{m_2})^{it} h(t)}{\|\phi_{(i_p)}\|^2\, |L(1+2it, \widetilde{\chi}_1^{-1}\widetilde{\chi}_2)|^2} dt$$

$$= T(m_1, m_2, \mathtt{n})\psi(N)\overline{\omega'\big(\sqrt{\tfrac{m_1 \mathtt{n}}{m_2}}\big)} \frac{1}{\pi^2} \int_{-\infty}^{\infty} h(t) \tanh(\pi t)\, t\, dt$$

$$+\frac{2i\psi(N)}{\pi} \sum_{c \in N\mathbf{Z}^+} \frac{S_{\omega'}(m_2, m_1; \mathtt{n}; c)}{c} \int_{-\infty}^{\infty} J_{2it}\left(\frac{4\pi\sqrt{\mathtt{n} m_1 m_2}}{c}\right) \frac{h(t)\, t}{\cosh(\pi t)} dt,$$

where:

- $\psi(N) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)] = N \prod_{p|N}(1 + \frac{1}{p})$.
- *The Petersson norm is given by* $\|u_j\|^2 = \dfrac{1}{\psi(N)} \displaystyle\int_{\Gamma_0(N)\backslash \mathbf{H}} |u_j(x+iy)|^2 \frac{dx\, dy}{y^2}$.
- *For $u_j \in \mathcal{F}$, $\Delta u_j = (\frac{1}{4} + t_j^2)u_j$ and $T_{\mathtt{n}} u_j = \lambda_{\mathtt{n}}(u_j)u_j$.*
- $T(m_1, m_2, \mathtt{n}) = \begin{cases} 1 & \text{if } m_1 m_2 = b^2 \mathtt{n} \text{ for some integer } b | \gcd(m_1, m_2) \\ 0 & \text{otherwise.} \end{cases}$
Equivalently, $T(a_1, a_2, a_3) \in \{0, 1\}$ is nonzero if and only if $a_i a_j / a_k$ is a perfect square for all distinct $i, j, k \in \{1, 2, 3\}$.
- *$\widetilde{\chi}_1, \widetilde{\chi}_2$ range through all ordered pairs of Dirichlet characters modulo N for which $\widetilde{\chi}_1\widetilde{\chi}_2 = \omega'$ and whose conductors satisfy $\mathfrak{c}_{\widetilde{\chi}_1}\mathfrak{c}_{\widetilde{\chi}_2} | N$.*
- *(i_p) runs through all sequences $(i_p)_{p|N}$ with*
$$\mathrm{ord}_p(\mathfrak{c}_{\widetilde{\chi}_2}) \le i_p \le \mathrm{ord}_p(N) - \mathrm{ord}_p(\mathfrak{c}_{\widetilde{\chi}_1}).$$
- $\|\phi_{(i_p)}\|^2 = \displaystyle\prod_{\substack{p|N \\ i_p = 0}} \frac{p}{(p+1)} \prod_{\substack{p|N \\ 0 < i_p < N_p}} \frac{p-1}{p^{i_p}(p+1)} \prod_{\substack{p|N \\ i_p = N_p}} \frac{1}{p^{N_p - 1}(p+1)}.$

- χ_1' *is the Dirichlet character mod* $N_1 = \prod_{\substack{p|N \\ i_p<N_p}} p^{N_p}$ *inducing* $\widetilde{\chi}_1$ *as in* (2.7).

- χ_2' *is the Dirichlet character mod* $N_2 = \prod_{\substack{p|N \\ i_p>0}} p^{N_p}$ *inducing* $\widetilde{\chi}_2$ *as in* (2.7).

- $\lambda_{\mathtt{n}}(\chi_1', \chi_2', it) = \sum_{d|\mathtt{n}} \left(\frac{\mathtt{n}}{d^2}\right)^{it} \overline{\chi_1'(d)\chi_2'(\frac{\mathtt{n}}{d})}$. *(This is the same as* $\lambda_{\mathtt{n}}(\widetilde{\chi}_1, \widetilde{\chi}_2, it)$ *since* $\widetilde{\chi}_1, \widetilde{\chi}_2$ *are induced from* χ_1', χ_2' *and* $(\mathtt{n}, N) = 1$. *It is also the same as* $\lambda_{\mathtt{n}}(\chi_1, \chi_2, it)$ *from the previous theorem, by* (2.9).*)*

- $M = \prod_{p|N} p^{i_p}$ *is also a modulus for* χ_2'.

- $\sigma_{it}(\chi_1', \chi_2', m) = \frac{1}{M^{1+2it}} \sum_{c|m} \frac{\overline{\chi_1'(c)}}{c^{2it}} \sum_{d \in \mathbf{Z}/M\mathbf{Z}} \chi_2'(d) e(\frac{dm}{Mc})$. *The sum over* d *is also expressed in terms of the primitive character inducing* χ_2' *in* (5.37).

- $S_{\omega'}(m_2, m_1; \mathtt{n}; c) = \sum_{\substack{d,d' \in \mathbf{Z}/c\mathbf{Z} \\ dd' = \mathtt{n}}} \overline{\omega'(d)} e(\frac{m_2 d + m_1 d'}{c})$.

7.7 Classical derivation

When we take $\mathtt{n} = 1$ in the above theorem, we obtain the "classical" Kuznetsov formula

$$\sum_{u_j \in \mathcal{F}} \frac{a_{m_1}(u_j)\overline{a_{m_2}(u_j)}}{\|u_j\|^2} \frac{h(t_j)}{\cosh(\pi t_j)}$$

$$+\frac{1}{\pi} \sum_{\widetilde{\chi}_1, \widetilde{\chi}_2} \sum_{(i_p)} \int_{-\infty}^{\infty} \frac{\sigma_{it}(\chi_1', \chi_2', m_1)\overline{\sigma_{it}(\chi_1', \chi_2', m_2)}(\frac{m_1}{m_2})^{it} h(t)}{\|\phi_{(i_p)}\|^2 \, |L(1+2it, \widetilde{\chi}_1^{-1}\widetilde{\chi}_2)|^2} dt$$

$$= \delta(m_1, m_2)\psi(N)\overline{\omega'\left(\sqrt{m_1/m_2}\right)} \frac{1}{\pi^2} \int_{-\infty}^{\infty} h(t) \tanh(\pi t) \, t \, dt \tag{7.32}$$

$$+\frac{2i\psi(N)}{\pi} \sum_{c \in N\mathbf{Z}^+} \frac{S_{\omega'}(m_2, m_1; c)}{c} \int_{-\infty}^{\infty} J_{2it}\left(\frac{4\pi\sqrt{m_1 m_2}}{c}\right) \frac{h(t)\, t}{\cosh(\pi t)} dt.$$

Conversely, Theorem 7.14 can also be derived from (7.32). To see this, start by choosing the orthogonal basis $\mathcal{F}$ to consist of Hecke eigenvectors with $a_1(u_j) = 1$. (Such a basis is easily constructed by a Gram-Schmidt procedure; cf. [KL1], Lemma 3.10.) With this normalization, by (4.10), for all $u_j \in \mathcal{F}$ we have

$$\lambda_{\mathtt{n}}(u_j) a_{m_1}(u_j) = \sum_{\ell | \gcd(\mathtt{n}, m_1)} \overline{\omega'(\ell)} \, a_{\frac{\mathtt{n} m_1}{\ell^2}}(u_j). \tag{7.33}$$

If we denote the classical formula (7.32) by $\mathrm{CK}(m_1, m_2)$, then the sum

$$\sum_{\ell | \gcd(\mathtt{n}, m_1)} \overline{\omega'(\ell)} \, \mathrm{CK}(\tfrac{\mathtt{n} m_1}{\ell^2}, m_2) \tag{7.34}$$

is precisely Theorem 7.14. The proof of this assertion involves proving four identities, one for each of the four terms (cuspidal, Eisenstein, main, Kloosterman) of (7.32). Indeed each term can be summed individually over ℓ as in (7.34) to recover the corresponding term in Theorem 7.14. For the cuspidal term, this is immediate from (7.33). For the Kloosterman term, after summing over ℓ, one applies a generalization of Selberg's identity for Kloosterman sums, given in (9.4) below, to obtain the corresponding term in Theorem 7.14. For the main term, the desired identity follows from

$$\sum_{\ell|(\mathtt{n},m_1)} \overline{\omega'(\ell)}\, \delta(\tfrac{\mathtt{n}m_1}{\ell^2}, m_2)\overline{\omega'(\sqrt{\tfrac{\mathtt{n}m_1}{\ell^2 m_2}})} = \overline{\omega'(\sqrt{\tfrac{\mathtt{n}m_1}{m_2}})} \sum_{\ell|(\mathtt{n},m_1)} \delta(\tfrac{\mathtt{n}m_1}{\ell^2}, m_2)$$

$$= T(m_1, m_2, \mathtt{n})\overline{\omega'(\sqrt{\tfrac{\mathtt{n}m_1}{m_2}})}.$$

The manipulations required for the Eisenstein term are a bit more involved. The goal is to prove that for any integers n, m prime to N,

$$\lambda_n(\chi_1', \chi_2', it)\sigma_{it}(\chi_1', \chi_2', m)m^{it} = \sum_{\ell|(n,m)} \overline{\omega'(\ell)}\sigma_{it}(\chi_1', \chi_2', \tfrac{mn}{\ell^2})(\tfrac{nm}{\ell^2})^{it}. \tag{7.35}$$

Dividing both sides by $\frac{(nm)^{it}}{M^{1+2it}}$, using $\overline{\omega'(\ell)} = \overline{\chi_1'(\ell)\chi_2'(\ell)}$, and simplifying what remains, one reduces the problem to showing that

$$\sum_{d|n}\sum_{c|m} \frac{\overline{\chi_1'(dc)\chi_2'(\frac{n}{d})}}{(dc)^{2it}} \sum_{b \bmod M} \chi_2'(b)e(\frac{bm}{Mc})$$

$$= \sum_{\ell|(n,m)} \sum_{r|\frac{nm}{\ell^2}} \frac{\overline{\chi_1'(\ell r)\chi_2'(\ell)}}{(\ell r)^{2it}} \sum_{b \bmod M} \chi_2'(b)e(\frac{bmn}{M\ell^2 r}).$$

Setting $dc = \ell r = a$, it then suffices to show that for each divisor $a|nm$,

$$\sum_{\substack{d|n, d|a,\\ \frac{a}{d}|m}} \overline{\chi_2'(\tfrac{n}{d})} \sum_{b \bmod M} \chi_2'(b)e(\frac{bmd}{Ma}) = \sum_{\substack{\ell|(m,n), \ell|a,\\ a|\frac{nm}{\ell}}} \overline{\chi_2'(\ell)} \sum_{b \bmod M} \chi_2'(b)e(\frac{bmn}{M\ell a}). \tag{7.36}$$

Proposition 7.15. *Given $a|mn$ as above, define*

$$D(a) = \{(d,c)|\, dc = a, d|n, c|m\}$$

and

$$D'(a) = \{(\ell, r)|\, \ell r = a, \ell|(n,m), r|\tfrac{nm}{\ell^2}\}.$$

Then the map

$$(d,c) \mapsto \left(\tfrac{(n,a)c}{a}, \tfrac{a^2}{(n,a)c}\right) = \left(\tfrac{(n,a)}{d}, \tfrac{ad}{(n,a)}\right)$$

defines a bijection from $D(a)$ to $D'(a)$, with inverse

$$(\ell, r) \mapsto \left(\tfrac{(n,a)r}{a}, \tfrac{a^2}{(n,a)r}\right) = \left(\tfrac{(n,a)}{\ell}, \tfrac{a\ell}{(n,a)}\right).$$

The proof of the proposition is left to the reader. Using it, we see that the left-hand side of (7.36) is equal to

$$\sum_{\substack{\ell|(m,n),\ell|a,\\ a|\frac{nm}{\ell}}} \overline{\chi_2'(\frac{\ell n}{(n,a)})} \sum_{b \bmod M} \chi_2'(b)e(\frac{bm(n,a)}{M\ell a}).$$

Replacing b by $b\frac{n}{(n,a)}$ (which is valid since $\frac{n}{(n,a)}$ is prime to M), we obtain the right-hand side of (7.36), as needed.

8 Validity of the KTF for a broader class of h

We have shown that the Kuznetsov trace formula is valid for the restricted class of functions h with $h(iz) \in PW^{12}(\mathbf{C})^{even}$. For certain applications it is useful to allow for a wider class of functions. For example, the Gaussian $h(t) = e^{-t^2/T^2}$ has rapid decay for real t, but $h(iz)$ is not Paley-Wiener of any order, due to faster than exponential growth when z is real. Here we consider functions h satisfying:

$$\begin{cases} h \text{ is even,} \\ h(t) \text{ is holomorphic in the region } |\operatorname{Im} t| < A, \\ h(t) \ll (1+|t|)^{-B} \text{ in the region } |\operatorname{Im} t| < A, \end{cases} \tag{8.1}$$

for positive real constants A and B.

Theorem 8.1. *If A, B are sufficiently large, then the KTF (Theorem 7.14) is valid for all functions h satisfying* (8.1)*.*

Remarks: (1) We will not obtain the optimum values for A, B. Kuznetsov's original paper established the formula in the case $N = 1$ for any $A > \frac{1}{2}$ and $B > 2$, [Ku]. According to [IK] Theorem 16.3, these parameters work for any level N. This is the range used by Selberg in his original work on the trace formula [Sel2]. Since, as proven by Selberg, we have $|\operatorname{Im}(t)| < \frac{1}{4}$ for the cuspidal spectral parameters t, it is plausible that $A > \frac{1}{4}$ would suffice. This has been proven to be the case when $N = 1$ by Yoshida, [Y]. However, allowing $A < \frac{1}{2}$ results in poorer control over the size of the Kloosterman term. See Proposition 8.24 below and the remarks following it.

(2) Given the above theorem, one can use the following idea of Kuznetsov to show that in fact $B > 2$ suffices. Briefly, suppose h satisfies (8.1) for some A sufficiently large as in the theorem, and some $B > 2$. Choose $\alpha > 0$ very small, but still large enough that $\frac{1}{2\alpha} < A$. Define, for $r \in \mathbf{R}$,

$$h_r(t) = -\frac{1}{2}\left(h\Big(\frac{r+\frac{i}{2}}{\alpha}\Big) + h\Big(\frac{r-\frac{i}{2}}{\alpha}\Big)\right) \frac{\cosh(\pi r)\cosh(\pi\alpha t)}{\cosh(\pi(r-\alpha t))\cosh(\pi(r+\alpha t))}.$$

Then $h_r(t)$ satisfies (8.1) for any B (it has exponential decay as $|\mathrm{Re}(t)| \to \infty$), and for $A = \frac{1}{2\alpha}$. Therefore if α is chosen suitably, the KTF is valid for $h_r(t)$ by the theorem. In each term of this KTF, we integrate r over $\mathbf{R}$ and use the identity

$$\int_{-\infty}^{\infty} h_r(t)dr = h(t),$$

which is valid for h analytic on $|\operatorname{Im} t| \le \frac{1}{2\alpha}$ ([IK] Lemma 16.4, [Ku] (6.1)), to conclude that the KTF is valid for h. It is not too hard to justify this process by showing that everything is absolutely convergent, using the fact that $B > 2$.

We prove Theorem 8.1 at the end of §8.3. We will make use of the KTF already established for Paley-Wiener functions of sufficiently high order, and a limiting procedure. Given h as in Theorem 8.1, let f be the corresponding function on $G(\mathbf{R})^+$, i.e. the inverse Selberg transform of h. It might not be smooth or compactly supported modulo $Z(\mathbf{R})$. In §8.2, we will define a family of compactly supported C^m functions f_T on $G(\mathbf{R})^+$ for $T > 1$ and some $m > 0$, such that $f_T \to f$ pointwise as $T \to \infty$. We let $h_T \in PW^m(\mathbf{C})^{\text{even}}$ be the Selberg transform of f_T. The KTF holds for h_T if m is sufficiently large, and we show that

$$\lim_{T\to\infty}(\text{Spec. side of KTF for } h_T) = \text{Spec. side of KTF for } h$$

and

$$\lim_{T\to\infty}(\text{Geo. side of KTF for } h_T) = \text{Geo. side of KTF for } h,$$

thus establishing the KTF for h.

We note that Finis, Lapid and Müller have used a different limiting method for GL(2), and (in large part) beyond, to extend Arthur's trace formula to a space of smooth functions allowing non-compact support even at nonarchimedean places ([FL], [FLM]).

In §8.1, we extend the basic integral transforms of §3 to allow for non-compactly supported functions. We then discuss the relationships between the various functions f, V, Q, h, and establish bounds for certain of their derivatives. In §8.2 - §8.3, we define h_T as in the above discussion, and apply a limiting process to the KTF for h_T. In the final two sections, we prove a needed auxiliary result, namely that for a test function $f = f_\infty \times f_{\text{fin}}$ with f_{fin} Schwartz-Bruhat and f_∞ bi-K_∞-invariant, twice differentiable, and of mild polynomial decay, the operator $R_0(f)$ is Hilbert-Schmidt.

Notation. Throughout this section, all the constants implicit in $\ll$ may depend on A, B and h (and hence V, f, Q etc.) unless otherwise stated. The notation C_ℓ will denote a constant depending on ℓ, A, B and h, and may have different values in different places.

8.1 Preliminaries

We start by setting out some necessary trivialities.

Proposition 8.2. *Let I be an interval on the real line. Suppose f is a measurable function on $\mathbf{R} \times I$ with $f(t, y)$ continuous in y for a.e. $t \in \mathbf{R}$. Suppose $|f(t, y)| \le F(t)$ for some function $F \in L^1(\mathbf{R})$. Then $\int_{\mathbf{R}} f(t, y)dt$ is a continuous function of $y \in I$.*

Proof. For any $\varepsilon \neq 0$ with $y + \varepsilon \in I$,

$$\int_{\mathbf{R}} f(t, y + \varepsilon)dt - \int_{\mathbf{R}} f(t, y)dt = \int_{\mathbf{R}} (f(t, y + \varepsilon) - f(t, y))dt.$$

The integrand is bounded by $2F(t)$. By the dominated convergence theorem, the integral goes to 0 as $\varepsilon \to 0$. Thus $\int_{\mathbf{R}} f(t, y)dt$ is a continuous function of $y \in I$. $\square$

Proposition 8.3. *Let I be an interval. Suppose $f(t,y)$ is a measurable function on $\mathbf{R}\times I$ such that for a.e. $t\in\mathbf{R}$ the partial derivative $f_y(t,y)$ exists and is continuous in y. Suppose further that $|f(t,y)|\leq F_0(t)$ a.e. and $|f_y(t,y)|\leq F_1(t)$ a.e. for some $F_0,F_1\in L^1(\mathbf{R})$. Then $\int_{\mathbf{R}} f(t,y)dy$ and $\int_{\mathbf{R}} f_y(t,y)dt$ are continuous functions of $y\in I$ and*

$$\frac{d}{dy}\int_{\mathbf{R}} f(t,y)dt=\int_{\mathbf{R}} f_y(t,y)dt.$$

(Here, we may view $f_y(t,y)$ as a function on $\mathbf{R}$ by prescribing arbitrary values on the measure 0 set of t for which the derivative is undefined.)

Proof. The continuity of the integrals follows from the previous proposition. Let $y_0\in I$ be fixed. Because the following double integral is absolutely convergent, we can apply Fubini's theorem:

$$\int_{y_0}^{y}\int_{\mathbf{R}} f_y(t,x)dt\,dx=\int_{\mathbf{R}}\int_{y_0}^{y} f_y(t,x)dx\,dt$$

$$=\int_{\mathbf{R}}(f(t,y)-f(t,y_0))dt=\int_{\mathbf{R}} f(t,y)dt-\int_{\mathbf{R}} f(t,y_0)dt.$$

Differentiating with respect to y, the assertion follows by the fundamental theorem of calculus. $\square$

By induction, we have the following.

Corollary 8.4. *Let I be an interval. Suppose $f(t,y)$ is a measurable function on $\mathbf{R}\times I$ such that for $k=0,1,\ldots,\ell$:*
(i) $\frac{\partial^k f(t,y)}{\partial^k y}$ exists and is continuous in y for a.e. $t\in\mathbf{R}$,
(ii) there exists $F_k\in L^1(\mathbf{R})$ such that $|\frac{\partial^k f(t,y)}{\partial y^k}|\leq F_k(t)$ a.e.
Then $\frac{d^\ell}{dy^\ell}\int_{\mathbf{R}} f(t,y)dt$ is a continuous function of $y\in I$, and

$$\frac{d^\ell}{dy^\ell}\int_{\mathbf{R}} f(t,y)dt=\int_{\mathbf{R}}\frac{\partial^\ell f(t,y)}{\partial y^\ell}dt.$$

(We may view the integrand $\frac{\partial^\ell f(t,y)}{\partial y^\ell}$ as a function on $\mathbf{R}$ by assigning artibarary values on the measure 0 set for which the derivative is undefined.)

Proposition 8.5. *Let a,b and c be positive real numbers. Suppose f is a continuous function on $\mathbf{R}$ satisfying $f(x)\ll_{a,b}|x|^{-a}$ for $|x|>b$. Then $f(x)\ll_{a,b,c}(c+|x|)^{-a}$ for all x.*

Proof. It is easy to show that $|x|^{-a}\ll(c+|x|)^{-a}$ for $|x|>b$. By the continuity of f, $f(x)\ll 1\leq(b+c)^a(c+|x|)^{-a}$ for $|x|\leq b$. The proposition follows. $\square$

Proposition 8.6. *Suppose f is a continuous function on an interval $[a,b)$ with a continuous derivative on (a,b). Suppose $\lim_{x\to a^+}f'(x)=A$. Then f has a continuous derivative on $[a,b)$ with $f'(a)=A$.*

Proof. By definition, $f'(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$. Since f is continuous at a, we can apply L'Hospital's rule, giving $f'(a) = \lim_{x\to a^+} f'(x)$, as needed. □

Proposition 8.7. *Let $\ell \geq 1$ be an integer, let I be an interval, and let $g \in C^\ell(I)$ be real-valued. Let J be an interval containing the image of g. Let $f \in C^\ell(J)$. Then*

$$\frac{d^\ell}{dt^\ell} f(g(t)) = \sum_{r=1}^{\ell} \sum_{\substack{a_1+a_2+\cdots+a_r=\ell \\ \ell\geq a_1\geq\cdots\geq a_r\geq 1}} A_{a_1,\ldots,a_r} f^{(r)}(g(t)) g^{(a_1)}(t)\cdots g^{(a_r)}(t),$$

where $A_{a_1,\ldots,a_r}$ are nonnegative integers independent of f, g.

Proof. Induction. □

Proposition 8.8. *Suppose ϕ is a function on $\mathbf{R}$ which is ℓ-times continuously differentiable, with $\phi^{(k)}(\pm\infty) = 0$ and $\phi^{(k)} \in L^1(\mathbf{R})$ for $k = 1, \ldots, \ell$. Then for such k and real $t \neq 0$,*

$$|\hat{\phi}(t)| \leq \frac{1}{|2\pi t|^k} \int_{\mathbf{R}} |\phi^{(k)}(x)| dx,$$

where $\hat{\phi}(t) = \int_{\mathbf{R}} \phi(y) e^{-2\pi i t y} dy$ is the Fourier transform of ϕ.

Proof. See Lemma 19.11 and Proposition 8.15 of [KL2]. □

V revisited

We now re-examine the integral transforms of Section 3, without the hypothesis of compact support. Let $C^m(G^+//K_\infty)$ denote the set of bi-K_∞-invariant complex-valued functions with continuous m-th derivative. Let $C^m(\mathbf{R}^+)^w$ be the set of $a : \mathbf{R}^+ \longrightarrow \mathbf{C}$ with continuous m-th derivative, satisfying $a(y) = a(y^{-1})$.

For $f \in C^m(G^+//K_\infty)$ and $u \geq 0$, we define

$$V(u) = V(y + y^{-1} - 2) = f(\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}). \tag{8.2}$$

In the other direction,

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = V(\frac{a^2+b^2+c^2+d^2}{ad-bc} - 2). \tag{8.3}$$

Proposition 8.9. *For $y \in \mathbf{R}^+$, the substitution*

$$u = y + y^{-1} - 2$$

defines a linear injection: $C^m(\mathbf{R}^+)^w \longrightarrow C^{m'}([0,\infty))$ when $3m' \leq m+1$. Any function in the image of the map is C^m on $(0,\infty)$.

Proof. See Proposition 3.1. The proof given there does not actually use the hypothesis of compact support. □

The Harish-Chandra transform revisited

Given $f \in C^m(G^+//K_\infty)$, its Harish-Chandra transform is the function on $\mathbf{R}^+$ defined by

$$(\mathcal{H}f)(y) = y^{-1/2} \int_{\mathbf{R}} f(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix})dx,$$

provided the integral is absolutely convergent. If $V \in C^{m'}([0, \infty))$ is the function associated to f as above, then

$$\mathcal{H}f(y) = \int_{\mathbf{R}} V(y + y^{-1} - 2 + x^2)dx.$$

The Mellin transform revisited

Let Φ be a measurable complex-valued function on $\mathbf{R}^+$. Its Mellin transform is the function of $\mathbf{C}$ defined by

$$(\mathcal{M}\Phi)(s) = \mathcal{M}_s\Phi = \int_0^\infty \Phi(y)y^s \tfrac{dy}{y},$$

provided the integral is absolutely convergent. For example, starting with $f \in C^m(G^+//K_\infty)$ with compact support or just sufficient decay, one can define $\Phi = \mathcal{H}f$ and $h(t) = \mathcal{M}_{it}\Phi$. However, our interest here is to go in the other direction, starting from h. Thus we shall need to consider conditions under which the inverse Mellin transform exists. Throughout this section, η denotes a complex-valued function satisfying:

$$\begin{cases} \eta(s) \text{ is a holomorphic function in } A_1 < \operatorname{Re} s < A_2, \\ \eta(s) \ll (1 + |s|)^{-B} \text{ in the same strip,} \end{cases} \tag{8.4}$$

for some real numbers $A_1 < A_2$ and $B > 0$.

Proposition 8.10. *Suppose $B > 1$ and σ is a real number satisfying $A_1 < \sigma < A_2$. For $y > 0$, define*

$$\Phi_\sigma(y) = \frac{1}{2\pi i} \int_{\operatorname{Re} s=\sigma} \eta(s)y^{-s}ds. \tag{8.5}$$

The integral is absolutely convergent and independent of σ. Therefore we can define

$$\Phi(y) = \frac{1}{2\pi i} \int_{\operatorname{Re} s=\sigma} \eta(s)y^{-s}ds. \tag{8.6}$$

Furthermore if $A_1 = -A_2$ and η is an even function, then $\Phi(y) = \Phi(y^{-1})$.

Proof. The absolute convergence of (8.5) follows from $B > 1$ and (8.4). Let $\sigma_0 < \sigma_1$ be two real numbers in the open interval (A_1, A_2). For $\alpha > 0$, let Γ_α be the rectangle with vertices $\sigma_0 \pm \alpha i$, $\sigma_1 \pm \alpha i$, and counterclockwise orientation. By

Cauchy's theorem, $\int_{\Gamma_\alpha} \eta(s)y^{-s}ds = 0$. By (8.4), $\int_{\sigma_0}^{\sigma_1} \eta(\sigma \pm i\alpha)y^{-(\sigma \pm i\alpha)}d\sigma \to 0$ as $\alpha \to \infty$. It follows that Φ_σ is independent of σ.

Finally, suppose η is an even function. Letting $\sigma = 0$ in (8.6),

$$\Phi(y^{-1}) = \frac{1}{2\pi i}\int_{i\mathbf{R}} \eta(s)y^s ds = \frac{1}{2\pi i}\int_{i\mathbf{R}} \eta(-s)y^{-s}ds = \frac{1}{2\pi i}\int_{i\mathbf{R}} \eta(s)y^{-s}ds = \Phi(y).$$

□

Proposition 8.11. *Suppose $B > 1$ and fix $s = \sigma + i\tau$ with $A_1 < \sigma < A_2$. Let Φ be the function defined by* (8.6)*. Then $\mathcal{M}\Phi(s)$ is absolutely convergent and equal to $\eta(s)$.*

Proof. Write $y = e^{2\pi v}$. Then by (8.6),

$$\Phi(y) = \frac{1}{2\pi}\int_{\mathbf{R}} \eta(\sigma + it)e^{-2\pi v\sigma}e^{-2\pi ivt}dt = \frac{1}{2\pi e^{2\pi v\sigma}}\widehat{\eta_\sigma}(v), \tag{8.7}$$

where $\eta_\sigma(t) = \eta(\sigma + it)$. Because $B > 1$, (8.4) shows that $\eta_\sigma \in L^1(\mathbf{R})$. Let $0 < r < \min(A_2 - \sigma, \sigma - A_1)$. Then by Cauchy's integral formula,

$$\eta^{(k)}(s) = \frac{k!}{2\pi i}\int_{|z-s|=r} \frac{\eta(z)dz}{(z-s)^{k+1}} \ll \int_{|z-s|=r} \frac{|dz|}{r^{k+1}(1+|z|)^B} \ll_r \frac{1}{(1+|s|)^B}.$$

In order to remove the dependence on r in the estimates that follow, we take $r = \frac{1}{2}\min(A_2 - \sigma, \sigma - A_1)$.

From the above, we see that $\eta_\sigma^{(k)}(t) = \eta^{(k)}(\sigma + it) \in L^1(\mathbf{R})$ and $\eta_\sigma^{(k)}(\pm\infty) = 0$. By Proposition 8.8,

$$\widehat{\eta_\sigma}(v) \ll |v|^{-2}\int_{\mathbf{R}} |\eta_\sigma''(t)|dt \ll_\sigma v^{-2} \quad (v \neq 0). \tag{8.8}$$

Thus $\widehat{\eta_\sigma} \in L^1(\mathbf{R})$, so given $s = \sigma + i\tau$ with $\sigma \in (A_1, A_2)$, (8.7) gives

$$\int_0^\infty |\Phi(y)y^s|\frac{dy}{y} = \int_{\mathbf{R}} |\widehat{\eta_\sigma}(v)|dv < \infty.$$

This shows that $\mathcal{M}\Phi(s)$ is absolutely convergent.

Because η_σ is continuous and integrable, and $\widehat{\eta_\sigma} \in L^1(\mathbf{R})$, we may apply Fourier inversion, giving:

$$\begin{aligned}\eta(s) = \eta_\sigma(\tau) &= \int_{\mathbf{R}} \widehat{\eta_\sigma}(v)e^{2\pi iv\tau}dv = 2\pi\int_{\mathbf{R}} \Phi(e^{2\pi v})e^{2\pi v\sigma}e^{2\pi iv\tau}dv \\ &= 2\pi\int_{\mathbf{R}} \Phi(e^{2\pi v})e^{2\pi v(\sigma+i\tau)}dv = \int_0^\infty \Phi(y)y^s\,\tfrac{dy}{y}.\end{aligned}$$

□

Relationship between h and V

Throughout this section we assume that h satisfies (8.1). We take

$$\eta(s) = h(-is),$$

which satisfies (8.4) with $A_1 = -A$ and $A_2 = A$.

Proposition 8.12. *Suppose $B > 1$, and σ is a real number with $|\sigma| < A$. For $y > 0$, define*

$$\Phi(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=\sigma} h(-is) y^{-s} ds. \tag{8.9}$$

Then Φ belongs to $C(\mathbf{R}^+)^w$, is independent of σ, and

$$\mathcal{M}_{it}\Phi = h(t)$$

for all complex numbers t with $|\operatorname{Im}(t)| < A$. If we also define $\Phi(0) = 0$, then Φ is continuous on $[0, \infty)$.

Proof. In view of Proposition 8.11, it only remains to verify the continuity of Φ at $y = 0$. This will be done in greater generality in the next proposition. □

Proposition 8.13. *Suppose $0 \le \ell < \min(B-1, A)$ is an integer, and σ is a real number with $|\sigma| < A$. Then the function Φ defined in (8.9) has a continuous ℓ-th derivative on $[0, \infty)$. In fact, for $y > 0$,*

$$\Phi^{(\ell)}(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=\sigma} \left(\prod_{k=0}^{\ell-1} (-s-k) \right) h(-is) y^{-s-\ell} ds. \tag{8.10}$$

The above integral is absolutely convergent and independent of σ. For $y = 0$, $\Phi^{(\ell)}(0) = 0$. Lastly,

$$\Phi^{(\ell)}(y) \ll_\ell (1+y)^{-A-\ell}. \tag{8.11}$$

Proof. Suppose $0 \le j \le \ell < B - 1$, and write $s = \sigma + it$. We have

$$\left| \left(\prod_{k=0}^{j-1} (-s-k) \right) h(-is) \right| \le C_j \frac{\prod_{k=0}^{j-1}(|s|+k)}{(1+|s|)^B} \le C_j \frac{\prod_{k=0}^{j-1}(A+|t|+k)}{(1+|t|)^B}. \tag{8.12}$$

Letting $\xi_j(s) = \left(\prod_{k=0}^{j-1}(-s-k) \right) h(-is)$, we see from the first inequality in (8.12) that $\xi_j(s) \ll_j \frac{1}{(1+|s|)^{B-j}}$, so it satisfies (8.1) with B replaced by $B-j > 1$. By Proposition 8.10, the right-hand side of (8.10) is absolutely convergent and independent of σ.

Given $y > 0$, let $y_0 = y/2$, $y_1 = 2y$. Then $y \in I = [y_0, y_1]$. Define $Y = y_0$ if $\sigma + j > 0$, $Y = y_1$ if $\sigma + j \le 0$. Then by (8.12), $|\xi_j(s) y^{s-j}| \le F_j(t)$, where $F_j(t) = C_j \frac{\prod_{k=0}^{j-1}(A+t+k)}{(1+|t|)^B} Y^{-\sigma-j} \ll (1+|t|)^{-(B-j)}$ is integrable since $B - j > 1$. Thus by Corollary 8.4,

$$\frac{d^\ell}{dy^\ell}\Phi(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=\sigma} h(-is) \frac{d^\ell y^{-s}}{dy^\ell} ds,$$

where the integral is continuous in $y > 0$. This proves (8.10). Furthermore, taking $y = e^{2\pi v}$,

$$\frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma} \xi_\ell(s) y^{-s-\ell} ds = \frac{e^{-2\pi(\sigma+\ell)v}}{2\pi}\int_{\mathbf{R}} \xi_\ell(\sigma+it)e^{-2\pi i t v}dt.$$

Choose σ such that $-A < \sigma < -\ell$. Then

$$\lim_{y\to 0^+}\frac{d^\ell}{dy^\ell}\Phi(y) = \lim_{v\to-\infty}\frac{e^{-2\pi(\sigma+\ell)v}}{2\pi}\int_{\mathbf{R}} \xi_\ell(\sigma+it)e^{-2\pi i t v}dt = 0.$$

The ℓ-differentiability of $\Phi(y)$ and the continuity of the ℓ-th derivative at $y = 0$ now follow by Proposition 8.6.

To obtain the bound (8.11), first suppose $y > 0$. Then

$$\int_{\operatorname{Re} s=\sigma}\left|\left(\prod_{k=0}^{\ell-1}(-s-k)\right)h(-is)y^{-s-\ell}\right||ds| \ll_\ell y^{-\sigma-\ell}\int_{\mathbf{R}}\frac{dt}{(1+|t|)^{B-\ell}} \ll_\ell y^{-\sigma-\ell},$$

since $B-\ell > 1$. The implied constant is independent of σ, so we can let $\sigma \to A^-$ to obtain $\Phi^{(\ell)}(y) \ll y^{-A-\ell}$. The desired bound then follows by Proposition 8.5. ☐

For $u \geq 0$, define

$$Q(u) = \Phi(y), \tag{8.13}$$

where $y = y(u) = \frac{2+u+\sqrt{4u+u^2}}{2} > 0$. Note that in the other direction, $u = y + y^{-1} - 2$.

Proposition 8.14. *For $y(u)$ as above, and any nonnegative integer ℓ,*

$$y^{(\ell)}(u) \ll u^{-\ell+1} \quad \textit{for } u > 1.$$

Proof. We have

$$y'(u) = \frac{1}{2}\left(1 + \frac{2+u}{\sqrt{u(4+u)}}\right) \ll 1,$$

$$y''(u) = -\frac{2}{(u(4+u))^{3/2}} \ll u^{-3} \ll u^{-1}.$$

For $\ell \geq 3$,

$$y^{(\ell)}(u) = -2\sum_{i=0}^{\ell-2}\binom{\ell-2}{i}\frac{d^i u^{-3/2}}{du^i}\frac{d^{\ell-2-i}(u+4)^{-3/2}}{du^{\ell-2-i}} \ll u^{-\ell-1} \ll u^{-\ell+1},$$

where $\binom{n}{i}$ is the binomial coefficient. ☐

Proposition 8.15. *The function $Q(u)$ is continuous on $[0,\infty)$. Suppose $0 \leq \ell < \min(B-1, A)$. Then $Q(u)$ is ℓ-times continuously differentiable on the open interval $(0,\infty)$, where it satisfies*

$$Q^{(\ell)}(u) \ll_\ell (1+u)^{-A-\ell}. \tag{8.14}$$

Remark: In Corollary 8.18 below, we will show that if $\ell < \min(B-2, A-1)$, then the above assertions also hold at the endpoint $u = 0$.

Proof. The continuity of $Q(u) = \Phi(y)$ is immediate from that of Φ and $y(u)$. Because $\ell < \min(B-1, A)$, Φ has a continuous ℓ-th derivative by Proposition 8.13. By Proposition 8.9, Q has a continuous ℓ-th derivative on $(0, \infty)$.

When $\ell = 0$, the bound (8.14) is immediate from (8.11) and the fact that $y(u) \sim u$. Suppose $\ell > 0$. By Proposition 8.7, (8.11), and Proposition 8.14, for $u > 1$ we have

$$\frac{d^\ell}{du^\ell} Q(u) = \frac{d^\ell}{du^\ell} \Phi(y(u)) \ll_\ell \sum_{r=1}^{\ell} \sum_{\substack{a_1+a_2+\cdots+a_r=\ell \\ \ell \geq a_1 \geq \cdots \geq a_r \geq 1}} \Phi^{(r)}(y(u)) y^{(a_1)}(u) \cdots y^{(a_r)}(u)$$

$$\ll_\ell \sum_{r=1}^{\ell} \sum_{\substack{a_1+a_2+\cdots+a_r=\ell \\ \ell \geq a_1 \geq \cdots \geq a_r \geq 1}} (1+y(u))^{-A-r} u^{-a_1+1} \cdots u^{-a_r+1} \ll_\ell u^{-A-\ell}$$

since $y(u) \sim u$. The bound (8.14) follows for all $u > 0$ by Proposition 8.5. □

Proposition 8.16. *Suppose $B > 2$ and $A > 1$. Then the function*

$$V(u) = -\frac{1}{\pi} \int_{\mathbf{R}} Q'(u + w^2) dw \tag{8.15}$$

is absolutely convergent and continuous for $u \geq 0$. In fact, for any nonnegative integer $\ell < \min(B-2, A-1)$, $V(u)$ has a continuous ℓ-th derivative given by

$$V^{(\ell)}(u) = -\frac{1}{\pi} \int_{\mathbf{R}} Q^{(\ell+1)}(u + w^2) dw, \tag{8.16}$$

the integral converging absolutely. Furthermore, for all $u \geq 0$,

$$V^{(\ell)}(u) \ll_\ell (1+u)^{-A-\ell-\frac{1}{2}}. \tag{8.17}$$

Remark: When $u = 0$, the integrands of (8.15) and (8.16) may be undefined at $w = 0$, but the integrals still make sense.

Proof. Suppose $0 \leq k < \min(B-2, A-1)$. Then $k+1 < \min(B-1, A)$, so by Proposition 8.15, Q is $(k+1)$-times continuously differentiable on $(0, \infty)$, and

$$|Q^{(k+1)}(u + w^2)| \leq C_k (1 + u + w^2)^{-A-k-1} \leq C_k (1 + w^2)^{-A-k-1} \tag{8.18}$$

for $u > 0$, where C_k is a positive constant. Now apply Corollary 8.4 with $y = u$, $t = w$, $f(t, y) = Q'(u + w^2)$, and $F_k(w)$ equal to the right-hand side of (8.18). The equality (8.16) and its continuity and absolute convergence follow.

To obtain the bound (8.17), we observe that

$$|V^{(\ell)}(u)| \leq \frac{1}{\pi} \int_{\mathbf{R}} |Q^{(\ell+1)}(u + w^2)| dw \ll_\ell \int_{\mathbf{R}} (1 + u + w^2)^{-A-\ell-1} dw$$

$$= (1+u)^{-A-\ell-1}\int_{\mathbf{R}}(1+((1+u)^{-\frac{1}{2}}w)^2)^{-A-\ell-1}dw$$
$$= (1+u)^{-A-\ell-\frac{1}{2}}\int_{\mathbf{R}}(1+w^2)^{-A-\ell-1}dw \ll (1+u)^{-A-\ell-\frac{1}{2}}. \qquad \square$$

Proposition 8.17. *Suppose $B > 2$ and $A > 1$. Then for all $u \geq 0$,*

$$\int_{\mathbf{R}} V(u+x^2)dx = Q(u), \tag{8.19}$$

the integral converging absolutely.

Proof. Under the given hypothesis, we can take $\ell = 1$ in (8.14) to give

$$\int_{\mathbf{R}}\int_{\mathbf{R}} |Q'(u+x^2+w^2)|dwdx \ll_{\ell} \int_{\mathbf{R}}\int_{\mathbf{R}}(1+u+x^2+w^2)^{-A-1}dwdx$$
$$= \int_0^{\infty}\int_0^{2\pi}(1+u+r^2)^{-A-1}r\,d\theta dr = 2\pi\int_0^{\infty}(1+u+r^2)^{-A-1}rdr < \infty.$$

(The bound for the integrand we applied is valid whenever $u+x^2+w^2 > 0$, i.e. for almost all x, w.) Therefore the integral in (8.19) is absolutely convergent. It defines a continuous function of $u \geq 0$ by Proposition 8.2, since $V(u+x^2) \leq (1+x^2)^{-A-\frac{1}{2}}$ by (8.17), the latter function being integrable. Furthermore, assuming $u > 0$,

$$\int_{\mathbf{R}} V(u+x^2)dx = -\frac{1}{\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} Q'(u+x^2+w^2)dwdx$$
$$= -2\int_0^{\infty} Q'(u+r^2)rdr = -Q(u+r^2)\big|_{r=0}^{r=\infty} = Q(u). \tag{8.20}$$

In the last step we used (8.14) with $\ell = 0$. This proves (8.19) for $u > 0$. Our use of the fundamental theorem of calculus in (8.20) may not be valid when $u = 0$, due to a possible discontinuity of the integrand at $r = 0$ in that case. However, because both sides of the proposed equality (8.19) are continuous functions of $u \geq 0$ which agree for all $u > 0$, they are equal when $u = 0$ as well. $\square$

Corollary 8.18. *If $\ell < \min(B-2, A-1)$, then $Q \in C^{\ell}([0,\infty))$, and (8.14) holds for all $u \geq 0$.*

Proof. By (8.17), $V^{(\ell)}(1+u+x^2) \ll (1+x^2)^{-A-\ell-\frac{1}{2}}$. Since the latter is integrable over $\mathbf{R}$, we can differentiate (8.19) under the integral sign (cf. Corollary 8.4) to obtain the result. $\square$

Proposition 8.19. *Suppose $B > 2$ and $A > 1$. Let f be the function on $G(\mathbf{R}^+)$ corresponding to V as in (8.3). Then for $|\operatorname{Im} t| < A$, the Selberg transform of f is absolutely convergent and equal to h:*

$$(\mathcal{S}f)(it) = \mathcal{M}_{it}\mathcal{H}f = h(t).$$

Proof. By Proposition 8.17, (3.10), and (8.13), $(\mathcal{H}f)(y) = Q(u) = \Phi(y)$. By Proposition 8.12, $\mathcal{M}_{it}\Phi = h(t)$. $\square$

8.2 Smooth truncation

In this section, we suppose $h(t)$ satisfies (8.1) for some $B > 2$ and $A > 1$, and continue with the same notation from the previous section. We will need to truncate V in a way that preserves its differentiability. This requires a smooth bump function.

Let $\rho : \mathbf{R} \longrightarrow [0, 1]$ be a smooth function such that:

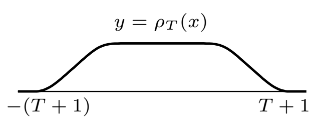
(i) $\rho(x) = 0$ for $x \leq 0$,

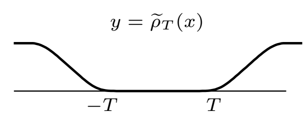
(ii) $\rho(x) = 1$ for $x \geq 1$.

For $T > 0$, define

$$\rho_T(x) = \begin{cases} 1 & \text{if } |x| < T, \\ \rho(T + 1 - |x|) & \text{if } T \leq |x| \leq T + 1, \\ 0 & \text{if } |x| > T + 1. \end{cases}$$

Then ρ_T is a smooth bump function with support in $[-(T + 1), T + 1]$. Letting $\widetilde{\rho}_T = 1 - \rho_T$, the graphs of ρ_T and $\widetilde{\rho}_T$ are given below:





For $j \geq 1$, $\rho_T^{(j)}(x) = 0$ unless $T \leq |x| \leq T + 1$. Thus $\rho_T^{(j)} \ll_j \chi_{[T,T+1]}$ on $\mathbf{R}_{\geq 0}$, where χ_I denotes the characteristic function of the set I, and by construction the implied constant is independent of T.

For $u \geq 0$, define

$$V_T(u) = V(u)\rho_T(\log(1 + u)).$$

Define

$$\widetilde{V}_T(u) = V(u) - V_T(u) = V(u)\widetilde{\rho}_T(\log(1 + u)).$$

Let f_T (resp. $\widetilde{f}_T$) be the bi-K_∞-invariant function on $G(\mathbf{R})^+$ corresponding to V_T (resp. $\widetilde{V}_T$) as in (8.3). Because V_T is compactly supported, the support of f_T is compact modulo the center.

Given the functions $\Phi(y) = Q(u)$ attached to h as in the previous section, we let $\Phi_T(y)$, (resp. $\widetilde{\Phi}_T(y)$) be the Harish-Chandra transform of f_T (resp. $\widetilde{f}_T$), and set $Q_T(u) = \Phi_T(y)$ and $\widetilde{Q}_T(u) = \widetilde{\Phi}_T(y)$, where $u = y + y^{-1} - 2$. Lastly, we define $h_T(t)$ (resp. $\widetilde{h}_T(t)$) to be the Selberg transform of f_T (resp. $\widetilde{f}_T$) as in Proposition 8.19. By the linearity of the various integral transforms, in each case we have the relation $\widetilde{\Box}_T = \Box - \Box_T$.

Suppose $\ell < \min(B - 2, A - 1)$, so that by Proposition 8.16, $V \in C^\ell([0, \infty))$. Then $V_T \in C_c^\ell([0, \infty))$, so by Proposition 3.2, $f_T \in C_c^\ell(G^+//K_\infty)$, and by Proposition 3.6, $h_T \in PW^\ell(\mathbf{C})^{\text{even}}$.

Proposition 8.20. *Suppose $B > 2$, $A > 1$, and $0 \le \ell < \min(B-2, A-1)$. Then for $u \ge 0$,*

$$\widetilde{V}_T^{(\ell)}(u) \ll_\ell (1+u)^{-A-\ell-\frac{1}{2}}\chi_{[T,\infty)}(\log(1+u)). \tag{8.21}$$

Proof. For $j \ge 1$, Proposition 8.7 gives

$$\frac{d^j}{du^j}\widetilde{\rho}_T(\log(1+u)) \ll_j \sum_{r=1}^{j} \sum_{\substack{a_1+a_2+\cdots+a_r=j \\ r\ge a_1\ge a_2\cdots\ge a_r\ge 1}} \rho_T^{(r)}(\log(1+u))(1+u)^{-a_1-a_2-\cdots-a_r}$$

$$\ll (1+u)^{-j}\chi_{[T,T+1]}(\log(1+u)).$$

By the bound (8.17), for $u \ge 0$ we have

$$\frac{d^\ell}{du^\ell}\widetilde{V}_T(u) = \frac{d^\ell}{du^\ell}V(u)\widetilde{\rho}_T(\log(1+u)) = \sum_{j=0}^{\ell}\binom{\ell}{j}V^{(\ell-j)}(u)\widetilde{\rho}_T^{(j)}(\log(1+u))$$

$$\ll_\ell (1+u)^{-A-\ell-\frac{1}{2}}\chi_{[T,\infty)}(\log(1+u)) + \sum_{j=1}^{\ell}(1+u)^{-A-\ell+j-\frac{1}{2}}(1+u)^{-j}\chi_{[T,T+1]}(\log(1+u))$$

$$\ll_\ell (1+u)^{-A-\ell-\frac{1}{2}}\chi_{[T,\infty)}(\log(1+u)).$$ □

Proposition 8.21. *Suppose $B > 2$, $A > 1$. Then for $u \ge 0$,*

$$\widetilde{Q}_T(u) = \int_{\mathbf{R}} \widetilde{V}_T(u+w^2)dw.$$

In fact, if $0 \le \ell < \min(B-2, A-1)$, then $\widetilde{Q}_T$ has a continuous ℓ-th derivative on $[0,\infty)$ given by

$$\widetilde{Q}_T^{(\ell)}(u) = \int_{\mathbf{R}} \widetilde{V}_T^{(\ell)}(u+w^2)dw, \tag{8.22}$$

the integral being absolutely convergent and continuous. Further,

$$|\widetilde{Q}_T^{(\ell)}(u)| \le \frac{E_{\ell,T}(u)}{(1+u)^{A+\ell}}, \tag{8.23}$$

where $E_{\ell,T}(u) \ll_\ell 1$ is a nonzero measurable function with $\lim_{T\to 0} E_{\ell,T}(u) = 0$.

Proof. Let $C_\ell > 0$ be the implied constant in (8.21). Then for $0 \le k \le \ell$,

$$\widetilde{V}_T^{(k)}(u+w^2) \le C_k(1+u+w^2)^{-A-k-\frac{1}{2}} \le C_k(1+w^2)^{-A-k-\frac{1}{2}}.$$

Letting $F_k(w)$ denote the latter expression, we apply Corollary 8.4 to conclude that (8.22) holds and is absolutely convergent and continuous.

It remains to establish the bound (8.23). By the previous proposition,

$$\left|\int_{\mathbf{R}} \widetilde{V}_T^{(\ell)}(u+w^2)dw\right| \le \int_{\mathbf{R}} C_\ell\chi_{[T,\infty)}(\log(1+u+w^2))(1+u+w^2)^{-A-\ell-\frac{1}{2}}dw$$

$$= (1+u)^{-A-\ell-\frac{1}{2}} C_\ell \int_{\mathbf{R}} \chi_{[T,\infty)}(\log(1+u+w^2))(1+((1+u)^{-\frac{1}{2}}w)^2)^{-A-\ell-\frac{1}{2}}\,dw$$

$$= (1+u)^{-A-\ell} C_\ell \int_{\mathbf{R}} \chi_{[T,\infty)}(\log(1+u+(1+u)w^2))(1+w^2)^{-A-\ell-\frac{1}{2}}\,dw.$$

Let

$$E_{\ell,T}(u) = C_\ell \int_{\mathbf{R}} \chi_{[T,\infty)}(\log(1+u+(1+u)w^2))(1+w^2)^{-A-\ell-\frac{1}{2}}\,dw.$$

Note that

$$|E_{\ell,T}(u)| \le C_\ell \int_{\mathbf{R}} (1+w^2)^{-A-\ell-\frac{1}{2}}\,dw < \infty.$$

By the dominated convergence theorem, $\lim_{T\to\infty} E_{\ell,T}(u) = 0$. This completes the proof. □

Corollary 8.22. *Suppose $B > 2$, $A > 1$, and $0 \le \ell < \min(B-2, A-1)$. Then for $v \in \mathbf{R}$ we have*

$$\left| \frac{d^\ell}{dv^\ell} \widetilde{Q}_T(e^{2\pi v} + e^{-2\pi v} - 2) \right| \le \frac{\hat{E}_{\ell,T}(|v|)}{(e^{2\pi v} + e^{-2\pi v})^A},$$

where $\hat{E}_{\ell,T}(|v|) \ll_\ell 1$ is a nonzero measurable function with $\lim_{T\to\infty} \hat{E}_{\ell,T}(|v|) = 0$.

Proof. When $\ell = 0$, the assertion is immediate from (8.23), taking $\hat{E}_{0,T}(|v|) = C_0 E_{0,T}(e^{2\pi v} + e^{-2\pi v} - 2)$ for a sufficiently large constant C_0. Suppose now that $\ell > 0$. Using Proposition 8.7 and the fact that $\frac{d^i}{dv^i}(e^{2\pi v} + e^{-2\pi v} - 2) \ll_i e^{2\pi v} + e^{-2\pi v}$, we have

$$\frac{d^\ell}{dv^\ell} \widetilde{Q}_T(e^{2\pi v} + e^{-2\pi v} - 2) \ll_\ell \sum_{r=1}^{\ell} \widetilde{Q}_T^{(r)}(e^{2\pi v} + e^{-2\pi v} - 2) \sum_{\substack{a_1+a_2+\cdots+a_r=\ell \\ \ell \ge a_1 \ge \cdots \ge a_r \ge 1}} (e^{2\pi v} + e^{-2\pi v})^r.$$

By the bound (8.23), this is

$$\ll_\ell \sum_{r=1}^{\ell} \frac{E_{r,T}(e^{2\pi v} + e^{-2\pi v} - 2)(e^{2\pi v} + e^{-2\pi v})^r}{(e^{2\pi v} + e^{-2\pi v} - 1)^{A+r}} \ll \frac{\sum_{r=1}^{\ell} E_{r,T}(e^{2\pi v} + e^{-2\pi v} - 2)}{(e^{2\pi v} + e^{-2\pi v})^A}.$$

Thus we can take $\hat{E}_{\ell,T}(|v|) = C_\ell \sum_{r=1}^{\ell} E_{r,T}(e^{2\pi v} + e^{-2\pi v} - 2)$ for a sufficiently large constant C_ℓ. □

Proposition 8.23. *Suppose $B > 2$, $A > 1$, and $0 \le \ell < \min(B-2, A-1)$. Let $0 < A' < A$. Then there exists a positive real number $\mathcal{E}_{\ell,T}$ such that for $|\operatorname{Im} t| \le A'$,*

$$|\widetilde{h}_T(t)| \le \frac{\mathcal{E}_{\ell,T}}{(1+|t|)^\ell} \quad \textit{and} \quad \lim_{T\to\infty} \mathcal{E}_{\ell,T} = 0.$$

Proof. Write $t = x + i\beta$ with $|\beta| \le A'$. Then

$$\widetilde{h}_T(t) = \mathcal{M}_{it}\widetilde{\Phi}_T = 2\pi \int_{\mathbf{R}} \widetilde{Q}_T(e^{2\pi v} + e^{-2\pi v} - 2)e^{2\pi itv}dv$$

$$= 2\pi \int_{\mathbf{R}} \widetilde{Q}_T(e^{2\pi v} + e^{-2\pi v} - 2)e^{-2\pi v\beta}e^{2\pi ivx}dv.$$

Since this is a Fourier transform, we can bound it using Proposition 8.8. First, by the above Corollary,

$$\frac{d^\ell}{dv^\ell}(\widetilde{Q}_T(e^{2\pi v} + e^{-2\pi v} - 2)e^{-2\pi v\beta})$$

$$= \sum_{i=0}^{\ell} \binom{\ell}{i}(-2\pi\beta)^{\ell-i}e^{-2\pi v\beta}\frac{d^i}{dv^i}\widetilde{Q}_T(e^{2\pi v} + e^{-2\pi v} - 2)$$

$$\ll_\ell e^{2\pi A'v}\sum_{i=0}^{\ell}\frac{\hat{E}_{i,T}(|v|)dv}{(e^{2\pi v} + e^{-2\pi v})^A}.$$

Let C_ℓ be the implied constant in the above inequality.

By Proposition 8.8, for $|x| = |\operatorname{Re}(t)| \ge 1$,

$$|\widetilde{h}_T(t)| \le \frac{1}{|2\pi x|^\ell}\int_{\mathbf{R}}\left|\frac{d^\ell}{dv^\ell}(\widetilde{Q}_T(e^{2\pi v} + e^{-2\pi v} - 2)e^{-2\pi v\beta})\right|dv.$$

Since $\frac{1+|t|}{|x|} \le \frac{1+|x|+|\beta|}{|x|} = |x|^{-1} + 1 + |x|^{-1}|\beta| \le 2 + A$, the above is

$$\le \frac{(2+A)^\ell}{(1+|t|)^\ell}\frac{C_\ell}{(2\pi)^\ell}\sum_{i=0}^{\ell}\int_{\mathbf{R}}\frac{e^{2\pi A'v}\hat{E}_{i,T}(|v|)dv}{(e^{2\pi v} + e^{-2\pi v})^A},$$

which converges since $A' < A$. On the other hand, if $|x| = |\operatorname{Re}(t)| \le 1$, then $1 + |t| \le 1 + |x| + |\beta| \le 2 + A$, so $1 \le \frac{(2+A)^\ell}{(1+|t|)^\ell}$, and

$$|\widetilde{h}_T(t)| \le 2\pi\int_{\mathbf{R}}|\widetilde{Q}_T(e^{2\pi v} + e^{-2\pi v} - 2)|e^{-2\pi v\beta}dv$$

$$\le \frac{2\pi(2+A)^\ell}{(1+|t|)^\ell}\int_{\mathbf{R}}\frac{e^{2\pi A'v}\hat{E}_{0,T}(|v|)dv}{(e^{2\pi v} + e^{-2\pi v})^A}.$$

Hence if we define

$$\mathcal{E}_{\ell,T} = 2\pi(2+A)^\ell\int_{\mathbf{R}}\frac{e^{2\pi A'v}\hat{E}_{0,T}(|v|)dv}{(e^{2\pi v} + e^{-2\pi v})^A} + \frac{C_\ell(2+A)^\ell}{(2\pi)^\ell}\sum_{i=0}^{\ell}\int_{\mathbf{R}}\frac{e^{2\pi A'v}\hat{E}_{i,T}(|v|)dv}{(e^{2\pi v} + e^{-2\pi v})^A},$$

then $|\widetilde{h}_T(t)| \le \frac{\mathcal{E}_{\ell,T}}{(1+|t|)^\ell}$ for all t in the strip $|\operatorname{Im}(t)| \le A'$, as needed.

Using $\hat{E}_{i,T}(|v|) \ll_\ell 1$, by the dominated convergence theorem we have

$$\lim_{T\to\infty} \mathcal{E}_{\ell,T} = 2\pi(2+A)^\ell \int_{\mathbf{R}} \frac{e^{2\pi A' v} \lim_{T\to\infty} \hat{E}_{0,T}(|v|)dv}{(e^{2\pi v}+e^{-2\pi v})^A}$$

$$+\frac{C_\ell(2+A)^\ell}{(2\pi)^\ell} \sum_{i=0}^{\ell} \int_{\mathbf{R}} \frac{e^{2\pi A' v} \lim_{T\to\infty} \hat{E}_{i,T}(|v|)}{(e^{2\pi v}+e^{-2\pi v})^A} dv = 0. \qquad \square$$

8.3 Comparing the KTF for h and h_T

We set the following notation for the various terms in the KTF, together with their absolute value counterparts (see Theorem 7.14 for notation):

$$\mathrm{Spec}_1(h) = \sum_{u_j \in \mathcal{F}} \frac{\lambda_{\mathtt{n}}(u_j)\, a_{m_1}(u_j) \overline{a_{m_2}(u_j)}}{\|u_j\|^2} \frac{h(t_j)}{\cosh(\pi t_j)}$$

$$\mathrm{Spec}_1^a(h) = \sum_{u_j \in \mathcal{F}} \left| \frac{\lambda_{\mathtt{n}}(u_j)\, a_{m_1}(u_j) \overline{a_{m_2}(u_j)}}{\|u_j\|^2} \frac{h(t_j)}{\cosh(\pi t_j)} \right|$$

$$\mathrm{Spec}_2(h) = \frac{1}{\pi} \sum_{\widetilde{\chi}_1, \widetilde{\chi}_2} \sum_{(i_p)} \int_{-\infty}^{\infty} \frac{\lambda_{\mathtt{n}}(\chi_1',\chi_2',it)\sigma_{it}(\chi_1',\chi_2',m_1)\overline{\sigma_{it}(\chi_1',\chi_2',m_2)}(\frac{m_1}{m_2})^{it} h(t)}{\|\phi_{(i_p)}\|^2\, |L(1+2it,\widetilde{\chi}_1\widetilde{\chi}_2^{-1})|^2} dt,$$

$$\mathrm{Spec}_2^a(h) = \frac{1}{\pi} \sum_{\widetilde{\chi}_1, \widetilde{\chi}_2} \sum_{(i_p)} \int_{-\infty}^{\infty} \frac{|\lambda_{\mathtt{n}}(\chi_1',\chi_2',it)\sigma_{it}(\chi_1',\chi_2',m_1)\overline{\sigma_{it}(\chi_1',\chi_2',m_2)}(\frac{m_1}{m_2})^{it} h(t)|}{\|\phi_{(i_p)}\|^2\, |L(1+2it,\widetilde{\chi}_1\widetilde{\chi}_2^{-1})|^2} dt,$$

$$\mathrm{Geo}_1(h) = T(m_1,m_2,\mathtt{n})\psi(N)\overline{\omega'\big(\sqrt{\tfrac{m_1\mathtt{n}}{m_2}}\big)} \frac{1}{\pi^2} \int_{-\infty}^{\infty} h(t)\tanh(\pi t)\, t\, dt,$$

$$\mathrm{Geo}_1^a(h) = T(m_1,m_2,\mathtt{n})\psi(N) \left|\omega'\big(\sqrt{\tfrac{m_1\mathtt{n}}{m_2}}\big)\right| \frac{1}{\pi^2} \int_{-\infty}^{\infty} |h(t)\tanh(\pi t)\, t|\, dt,$$

$$\mathrm{Geo}_2(h) = \frac{2i\psi(N)}{\pi} \sum_{c \in N\mathbf{Z}^+} \frac{S_{\omega'}(m_2,m_1;\mathtt{n};c)}{c} \int_{-\infty}^{\infty} J_{2it}\left(\tfrac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right) \frac{h(t)\,t}{\cosh(\pi t)} dt,$$

$$\mathrm{Geo}_2^a(h) = \frac{2\psi(N)}{\pi} \sum_{c \in N\mathbf{Z}^+} \frac{|S_{\omega'}(m_2,m_1;\mathtt{n};c)|}{c} \int_{-\infty}^{\infty} \left| J_{2it}\left(\tfrac{4\pi\sqrt{\mathtt{n}m_1m_2}}{c}\right) \frac{h(t)\,t}{\cosh(\pi t)} \right| dt.$$

Proposition 8.24. *Suppose h satisfies* (8.1) *for some $A > \frac{1}{4}$ and $B > 2$. Then: (i)* $\mathrm{Spec}_2^a(h) < \infty$*, (ii)* $\mathrm{Geo}_1^a(h) < \infty$*, and (iii)* $\mathrm{Geo}_2^a(h) < \infty$.

Remarks: (1) Allowing $\frac{1}{4} < A < \frac{1}{2}$ rather than $A \ge \frac{1}{2}$ comes at a price, namely, the Kloosterman term may no longer be $O(N^\varepsilon)$ (which holds when $A \ge \frac{1}{2}$), but instead only $O(N^{\frac{1}{2}-\delta})$ for any $0 < \delta < 2(A - \frac{1}{4})$.

(2) It is known that $\mathrm{Spec}_1^a(h) < \infty$ under the above conditions as well. See the remark after Proposition 7.5, where we explain why $B > 3$ suffices.

Proof. The proof of assertion (i) follows the same outline as that of Prop 7.6, although Lemma 7.5 is not needed since $\mathrm{Spec}_2^a(h)$ does not involve Bessel functions. One obtains an estimate like (7.16), but without the factor of $(1+2|t|)^2$, so that $B > 1$ suffices. The holomorphy of h is not needed here, so any value of A is allowable. Assertion (ii) is trivial since $h(t)$ is integrable and $|\tanh(\pi t)| \leq 1$. Assertion (iii) follows from the proof of Prop 7.11. It requires $A > \frac{1}{4}$ and $B > 2$. At the end of the proof, one can take $\sigma_0 = \frac{1}{4} + \varepsilon < \min(A, \frac{1}{2})$ to obtain an exponent of $\frac{1}{2} + 2\varepsilon$ in place of $1 - \varepsilon$ in (7.28). Then in place of (7.29), for any $\varepsilon' > 0$ we can obtain the bound

$$\ll \psi(N)\mathfrak{c}_{\omega'}^{\frac{1}{2}} \sum_{c \in N\mathbf{Z}^+} \frac{\tau(c)}{c^{1+2\varepsilon}} \ll \frac{N^{1+\frac{\varepsilon'}{2}} N^{\frac{1}{2}} N^{\frac{\varepsilon'}{2}}}{N^{1+2\varepsilon}} \sum_{c \in \mathbf{Z}^+} \frac{\tau(c)}{c^{1+2\varepsilon}} = O(N^{\frac{1}{2}+\varepsilon'-2\varepsilon}).$$

This proves the proposition. The bound asserted in the remark follows upon observing that $\delta = 2\varepsilon - \varepsilon'$ can assume any positive number less than $2(A - \frac{1}{4})$ when $A < \frac{1}{2}$. □

Proposition 8.25. *Suppose $\ell \geq 12$ is an integer for which*

$$\mathrm{Spec}_1^a(r_\ell) < \infty, \tag{8.24}$$

where $r_\ell(t) = \frac{1}{(1+|t|)^\ell}$. Let $B > \ell + 2$ and $A > \ell + 1$ be real constants. Then for any function h satisfying (8.1) *with these values, the KTF is valid:*

$$\mathrm{Spec}_1(h) + \mathrm{Spec}_2(h) = \mathrm{Geo}_1(h) + \mathrm{Geo}_2(h).$$

Proof. Using the fact (Proposition 4.7) that all of the spectral parameters of $L_0^2(N, \omega')$ satisfy $|\mathrm{Im}(t_j)| < \frac{1}{2}$, we apply Proposition 8.23 with $\frac{1}{2} \leq A' < A$, giving

$$\mathrm{Spec}_1^a(h - h_T) = \mathrm{Spec}_1^a(\widetilde{h}_T) \leq \mathcal{E}_{\ell,T}\, \mathrm{Spec}_1^a(r_\ell) \tag{8.25}$$

for r_ℓ as in (8.24). Noting that $h_T(iz) \in PW^\ell(\mathbf{C})^{\mathrm{even}}$ with $\ell \geq 12$ (see the discussion just before Proposition 8.20), Proposition 7.5 gives $\mathrm{Spec}_1^a(h_T) < \infty$. Thus by (8.25),

$$\mathrm{Spec}_1^a(h) = \mathrm{Spec}_1^a(h_T + \widetilde{h}_T) \leq \mathrm{Spec}_1^a(h_T) + \mathrm{Spec}_1^a(\widetilde{h}_T) < \infty.$$

Hence $\mathrm{Spec}_1(h)$ exists. Because $\lim\limits_{T\to\infty} \mathcal{E}_{\ell,T} \to 0$, using (8.25) we have

$$\lim_{T\to\infty} |\mathrm{Spec}_1(h) - \mathrm{Spec}_1(h_T)| = \lim_{T\to\infty} |\mathrm{Spec}_1(\widetilde{h}_T)| \leq \lim_{T\to\infty} \mathrm{Spec}_1^a(\widetilde{h}_T) = 0.$$

Hence

$$\lim_{T\to\infty} \mathrm{Spec}_1(h_T) = \mathrm{Spec}_1(h).$$

Now let X denote Spec_2^a, Geo_1^a, or Geo_2^a. Then by Proposition 8.23,

$$X(\widetilde{h}_T) \leq \mathcal{E}_{\ell,T} X(r_\ell).$$

Noting that $r_\ell \ll h_\ell$, where the function $h_\ell = \frac{1}{(1+t^2)^{\ell/2}}$ satisfies (8.1) with $A = 1$ and $B = \ell > 2$, we conclude from Proposition 8.24 that the above expression is finite. By the same reasoning as for Spec_1, we see that $X(h)$ exists, and is equal to $\lim_{T\to\infty} X(h_T)$. Because the KTF is valid for h_T, it follows that

$$\mathrm{Geo}_1(h) + \mathrm{Geo}_2(h) = \lim_{T\to\infty}(\mathrm{Geo}_1(h_T) + \mathrm{Geo}_2(h_T))$$

$$= \lim_{T\to\infty}(\mathrm{Spec}_1(h_T) + \mathrm{Spec}_2(h_T)) = \mathrm{Spec}_1(h) + \mathrm{Spec}_2(h).$$

This completes the proof. □

Proof of Theorem 8.1. The theorem follows immediately by the fact that one can take $\ell = 12$ in Proposition 8.25. See the remark after Proposition 7.5, or for a self-contained proof of this fact, see Proposition 8.34 below. □

8.4 $R_0(f)$ for f not smooth or compactly supported

In this section and the next, f will denote a function on $G(\mathbf{A})$, rather than on $G(\mathbf{R})^+$. The purpose of these sections is to prove that $R_0(f)$ is a Hilbert-Schmidt operator under certain mild assumptions on f. See Proposition 8.31. The discussion that follows is independent of the material in the previous sections. In particular, we do not assume (8.1) or equivalent bounds unless explicitly stated.

Throughout this section let $f = f_\infty \times f_{\mathrm{fin}}$ be a complex-valued function on $G(\mathbf{A})$, with f_∞ bi-K_∞-invariant, supported on $G(\mathbf{R})^+$, and satisfying

$$f_\infty(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \ll \frac{(ad-bc)^{\alpha/2}}{(a^2+b^2+c^2+d^2+2(ad-bc))^{\alpha/2}} \tag{8.26}$$

for some $\alpha > 2$. Here α plays the role of the weight `k` in §18-19 of [KL2]. The above is equivalent to

$$V(u) \ll \frac{1}{(u+4)^{\alpha/2}}, \tag{8.27}$$

where V is the function attached to f_∞ in (8.2). We do not assume that f_∞ is smooth, although eventually we will require it to be twice differentiable. We assume that f_{fin} is locally constant and compactly supported modulo $Z(\mathbf{A}_{\mathrm{fin}})$, and that $f(zg) = \overline{\omega(z)}f(g)$ for all $z \in Z(\mathbf{A})$ and $g \in G(\mathbf{A})$. In fact, we shall assume that

$$\mathrm{Supp}(f_{\mathrm{fin}}) = Z(\mathbf{A}_{\mathrm{fin}})K'\delta K' \tag{8.28}$$

for some $\delta \in M_2(\widehat{\mathbf{Z}})$ and some open compact subgroup $K' \subseteq K_{\mathrm{fin}}$ under which f_{fin} is bi-invariant. This entails no loss of generality, since any function f_{fin} as described above (8.28) is a finite linear combination of functions as in (8.28).

Proposition 8.26. *For f as above, $f \in L^q(G(\mathbf{A}), \overline{\omega})$ for all $q \geq 1$.*

Proof. Let $f_\alpha(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$ denote the right-hand side of (8.26). As a function of $G(\mathbf{R})$, it is bi-K_∞-invariant. Indeed, the right or left action of K_∞ on the space of matrices $M_2(\mathbf{R}) \cong \mathbf{R}^4$ is unitary, so it preserves the norm $a^2+b^2+c^2+d^2$. Therefore it is convenient to integrate using the Cartan decomposition (3.2). For $p = \frac{q}{2} \geq \frac{1}{2}$, we have

$$\int_{\overline{G}(\mathbf{A})} |f(g)|^{2p} dg \ll \int_{\overline{G}(\mathbf{R})} |f_\infty(g)|^{2p} dg \ll \int_1^\infty f_\alpha(\left(\begin{smallmatrix} t^{1/2} & \\ & t^{-1/2} \end{smallmatrix}\right))^{2p} (1-t^{-2}) dt$$

(see e.g. the integration formulas (7.27) and (7.23) of [KL2]). The latter integral is

$$= \int_1^\infty \frac{(1-t^{-2})}{(t+t^{-1}+2)^{p\alpha}} dt = \int_4^\infty \frac{1}{u^{p\alpha}} du < \infty,$$

since $p \geq \frac{1}{2}$ and $\alpha > 2$. □

By the above proposition, $f \in L^1(G(\mathbf{A}), \overline{\omega})$. Therefore it defines an operator $R(f)$ on $L^2(\omega)$, given by the kernel

$$K(g_1, g_2) = \sum_{\gamma \in \overline{G}(\mathbf{Q})} f(g_1^{-1} \gamma g_2).$$

We will work with Arthur's truncated kernel $K^T(g_1, g_2)$, defined as follows. For $T > 0$, let $\tau_T : G(\mathbf{A}) \longrightarrow \{0,1\}$ be the characteristic function of the set of $g \in G(\mathbf{A})$ with height $H(g) > T$. Then

$$\begin{aligned} K^T(g_1, g_2) &= K(g_1, g_2) - \sum_{\delta \in B(\mathbf{Q}) \backslash G(\mathbf{Q})} \sum_{\mu \in \overline{M}(\mathbf{Q})} \left(\int_{N(\mathbf{A})} f(g_1^{-1} \mu n \delta g_2) dn \right) \tau_T(\delta g_2) \\ &= \sum_{\gamma \in \overline{G}(\mathbf{Q})} f(g_1^{-1} \gamma g_2) - \sum_{\delta \in B(\mathbf{Q}) \backslash G(\mathbf{Q})} \sum_{\mu \in \overline{M}(\mathbf{Q})} \left(\int_{N(\mathbf{A})} f(g_1^{-1} \mu n \delta g_2) dn \right) \tau_T(\delta g_2). \end{aligned} \tag{8.29}$$

This is a function on $G(\mathbf{A}) \times G(\mathbf{A})$, but it is not hard to see that it is well-defined on $(B(\mathbf{Q}) \backslash G(\mathbf{A})) \times (G(\mathbf{Q}) \backslash G(\mathbf{A}))$.

Proposition 8.27. *For all $g_1, g_2 \in G(\mathbf{A})$, $K^T(g_1, g_2)$ is absolutely convergent, i.e.*

$$\sum_{\gamma \in \overline{G}(\mathbf{Q})} |f(g_1^{-1} \gamma g_2)| + \sum_{\delta \in B(\mathbf{Q}) \backslash G(\mathbf{Q})} \sum_{\mu \in \overline{M}(\mathbf{Q})} \int_{N(\mathbf{A})} |f(g_1^{-1} \mu n \delta g_2)| dn \tau_T(\delta g_2) < \infty.$$

Furthermore, the above is bounded on compact subsets of $\overline{G}(\mathbf{A}) \times \overline{G}(\mathbf{A})$.

Proof. By Proposition 18.4 of [KL2] and the discussion following its proof, the sum over γ is convergent, and in fact continuous as a function of (g_1, g_2), so the assertions hold for this piece of the function. For the same reason, the sum $\sum_\mu \sum_{n' \in N(\mathbf{Q})} |f(g_1^{-1}\mu n' n \delta g_2)|$ converges to a continuous function of $n \in N(\mathbf{Q})\backslash N(\mathbf{A})$. Therefore it is integrable over the compact set $N(\mathbf{Q})\backslash N(\mathbf{A})$, i.e.

$$\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \sum_\mu \sum_{n' \in N(\mathbf{Q})} |f(g_1^{-1}\mu n' n \delta g_2)| dn = \sum_\mu \int_{N(\mathbf{A})} |f(g_1^{-1}\mu n \delta g_2)| dn < \infty.$$

By Lemma 17.1 of [KL2], $\tau_T(\delta g_2) \neq 0$ for at most one $\delta \in B(\mathbf{Q})\backslash G(\mathbf{Q})$. In fact, since K^T is left $G(\mathbf{Q})$-invariant as a function of g_2, we can assume that g_2 lies in a fixed fundamental domain for $\overline{G}(\mathbf{Q})\backslash \overline{G}(\mathbf{A})$, so the set of δ that contribute to the sum is finite and independent of g_2 ([KL2], Proposition 17.2). The first assertion of the proposition now follows immediately. From the fact that the expression is a finite sum of functions of $(g_1, g_2) \in \overline{G}(\mathbf{Q}) \times \overline{G}(\mathbf{A})$, each of which is a product of a continuous function with a characteristic function, we see that it is bounded on compact subsets. $\square$

Let $K_{[0,\pi)}$ denote the set of matrices $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ with $\theta \in [0, \pi)$. Then

$$F \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k \,\Big|\, x \in [-\tfrac{1}{2}, \tfrac{1}{2}], y > 0, x^2 + y^2 \geq 1, k \in K_{[0,\pi)} \right\}$$

is a fundamental domain in $\mathrm{SL}_2(\mathbf{R})$ for the quotient $\mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R})$. This means that the projection $F \to \mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R})$ is surjective, and injective except on a set of measure zero. The set $Z(\mathbf{R})^+ F$ is then a fundamental domain for $\mathrm{SL}_2(\mathbf{Z})\backslash G(\mathbf{R})^+$, and it follows (from strong approximation for $G(\mathbf{A})$ and the "divorce theorem" on page 101 of [KL2]) that the set

$$\mathfrak{F} = Z(\mathbf{R})^+ F \times K_{\text{fin}}$$

is a fundamental domain in $G(\mathbf{A})$ for $G(\mathbf{Q})\backslash G(\mathbf{A})$. The subset

$$\overline{\mathfrak{F}} = F \times K_{\text{fin}}$$

contains a fundamental domain for $Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})$, and can be used as a domain of integration for the latter quotient ([KL2], Corollary 7.44). Let $L^2(\mathfrak{F}, \omega)$ be the Hilbert space of measurable functions $\phi : \mathfrak{F} \longrightarrow \mathbf{C}$ such that

- $\phi(zg) = \omega(z)\phi(g)$ for all $g \in \mathfrak{F}$ and $z \in Z(\mathbf{A}) \cap \mathfrak{F} = Z(\mathbf{R})^+ \times \widehat{\mathbf{Z}}^*$,
- $\|\phi\|^2_{\overline{\mathfrak{F}}} = \displaystyle\int_{\overline{\mathfrak{F}}} |\phi(g)|^2 dg < \infty$.

Lemma 8.28. *Let $\delta \in M_2(\widehat{\mathbf{Z}})$ be as in (8.28), and define $D \in \mathbf{Z}^+$ by $D\widehat{\mathbf{Z}} = (\det\delta)\widehat{\mathbf{Z}}$. Suppose*

$$f(g_1^{-1}\mu \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) g_2) \neq 0$$

for some $g_1, g_2 \in \mathfrak{F}$, $\mu \in M(\mathbf{Q})$, and $t \in \mathbf{A}$. Then $t_{\text{fin}} \in \frac{1}{D}\widehat{\mathbf{Z}}$ and $\mu \in Z(\mathbf{Q})\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right)$ for integers $a, d > 0$ with $ad = D$.

Proof. Consider the finite part $f_{\mathrm{fin}}(k_1^{-1}\mu\left(\begin{smallmatrix}1 & t_{\mathrm{fin}}\\ 0 & 1\end{smallmatrix}\right)k_2) \neq 0$. By (8.28), there exists $\beta \in \mathbf{A}_{\mathrm{fin}}^*$ such that

$$\det \mu \in \beta^2 D\widehat{\mathbf{Z}}^*.$$

By strong approximation for the ideles ([KL2], Prop. 5.10), $\mathbf{A}_{\mathrm{fin}}^* = \mathbf{Q}^*\widehat{\mathbf{Z}}^*$, so $\beta = r\beta'$ for some $r \in \mathbf{Q}^*$ and $\beta' \in \widehat{\mathbf{Z}}^*$. Therefore

$$r^{-2}\det \mu \in D\widehat{\mathbf{Z}}^* \cap \mathbf{Q}^* = \{D, -D\}.$$

Writing $r^{-1}\mu = \left(\begin{smallmatrix}a & \\ & d\end{smallmatrix}\right) \in M(\mathbf{Q})$, we have $ad = \pm D$. Replacing r by $-r$ if necessary, we can assume that $a > 0$. Now $k_1^{-1}\left(\begin{smallmatrix}a & \\ & d\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & t_{\mathrm{fin}}\\ 0 & 1\end{smallmatrix}\right)k_2 \in \mathrm{Supp}(f_{\mathrm{fin}})$, and since its determinant belongs to $D\widehat{\mathbf{Z}}^*$, we see that its $Z(\mathbf{A}_{\mathrm{fin}})$ component as in (8.28) must belong to $\widehat{\mathbf{Z}}^*$. It follows that $k_1^{-1}\left(\begin{smallmatrix}a & at_{\mathrm{fin}}\\ 0 & d\end{smallmatrix}\right)k_2 \in M_2(\widehat{\mathbf{Z}})$, and hence $\left(\begin{smallmatrix}a & at_{\mathrm{fin}}\\ 0 & d\end{smallmatrix}\right) \in M_2(\widehat{\mathbf{Z}})$. This means that $a, d \in \mathbf{Z}$ and $t_{\mathrm{fin}} \in \frac{1}{a}\widehat{\mathbf{Z}} \subseteq \frac{1}{D}\widehat{\mathbf{Z}}$. Finally, the fact that f_∞ is supported in $G(\mathbf{R})^+$ implies $ad > 0$, so $ad = D$. $\square$

Proposition 8.29. *Let f be as described at the beginning of this section, and suppose in addition that f_∞ is twice continuously differentiable. Let V be the function on $[0,\infty)$ attached to f_∞ as in* (8.2)*. Suppose there exists $\varepsilon > 0$ such that for all $u > 0$,*

$$\begin{cases} V(u), V'(u) \ll (1+u)^{-1-\varepsilon} \\ V''(u) \ll (1+u)^{-3/2-\varepsilon}. \end{cases} \tag{8.30}$$

(The bound on $V(u)$ is already a consequence of (8.27)*.) Then*

$$\|K^T\|^2_{\mathfrak{F}\times G(\mathbf{Q})\backslash G(\mathbf{A})} = \int_{\overline{\mathfrak{F}}}\int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} |K^T(g_1,g_2)|^2 dg_2 dg_1 < \infty, \tag{8.31}$$

or equivalently,

$$\|K^T\|^2_{\mathfrak{F}\times\mathfrak{F}} = \int_{\overline{\mathfrak{F}}}\int_{\overline{\mathfrak{F}}} |K^T(g_1,g_2)|^2 dg_2 dg_1 < \infty.$$

Remark: We do not assume that V is differentiable at the endpoint $u = 0$.

Proof. The proof is somewhat involved and will be given in the next subsection. It basically follows §19 of [KL2]. $\square$

Under the hypotheses of the above proposition, we can define a map $T_{K^T} : L^2(\omega) \to L^2(\mathfrak{F},\omega)$ by

$$T_{K^T}\phi(g_1) = \int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} K^T(g_1,g_2)\phi(g_2)dg_2.$$

Let $r : L^2(\mathfrak{F},\omega) \to L^2(\omega)$ be the map defined by $r\phi(G(\mathbf{Q})g) = \phi(g)$ for a.e. $g \in \mathfrak{F}$. (The set of points $g \in \mathfrak{F}$ for which $\phi(g)$ is not uniquely determined by $G(\mathbf{Q})g$ has measure 0.) Because $\mathfrak{F}$ is a fundamental domain for $G(\mathbf{Q})\backslash G(\mathbf{A})$, the map r is an isomorphism. By identifying the two spaces in this way, we can abuse terminology and refer to T_{K^T} as an operator on $L^2(\omega)$. For future reference, we let $L^2_0(\mathfrak{F},\omega)$ be the preimage of $L^2_0(\omega)$ under r.

Corollary 8.30. *The map $T_{K^T} : L^2(\omega) \longrightarrow L^2(\mathfrak{F},\omega) \cong L^2(\omega)$ is a Hilbert-Schmidt operator. The Hilbert-Schmidt norm $\|T_{K^T}\|^2_{HS}$ is equal to* (8.31).

Proof. This is a consequence of (8.31). (See [RS] Theorem VI.23). □

The next proposition shows that T_{K^T} coincides with $R(f)$ on the cuspidal subspace, and it then follows from Corollary 8.30 that $R_0(f)$ is Hilbert-Schmidt.

Proposition 8.31. *Suppose the hypotheses of Proposition 8.29 are satisfied. Then $T_{K^T}|_{L^2_0(\omega)} = R(f)|_{L^2_0(\omega)} = R_0(f)$. As a result, the operator $R_0(f)$ is Hilbert-Schmidt.*

Proof. Let $\phi \in L^2_0(\omega)$ and $g_1 \in \mathfrak{F}$. Then

$$R(f)\phi(g_1) = \int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} K(g_1, g_2)\phi(g_2)dg_2,$$

where the integral converges absolutely since $f \in L^1(\overline{\omega})$ (cf. (10.7) of [KL2]). Thus by the linearity of integration, $T_{K^T}\phi(g_1)$ is equal to

$$R(f)\phi(g_1) - \int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \sum_{\delta\in B(\mathbf{Q})\backslash G(\mathbf{Q})} \sum_{\mu\in\overline{M}(\mathbf{Q})} \left(\int_{N(\mathbf{A})} f(g_1^{-1}\mu n\delta g_2)\phi(g_2)dn\right)\tau_T(\delta g_2)dg_2.$$

It suffices to show that the second term vanishes. At the end of the proof, we will verify that it is absolutely convergent, so we can rearrange the sums and integrals. Granting this for the moment, by the left $G(\mathbf{Q})$-invariance of ϕ, the second term is

$$\int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \sum_{\delta\in B(\mathbf{Q})\backslash G(\mathbf{Q})} \sum_{\mu\in\overline{M}(\mathbf{Q})} \int_{N(\mathbf{A})} f(g_1^{-1}\mu n\delta g_2)dn\,\phi(\delta g_2)\tau_T(\delta g_2)dg_2 \tag{8.32}$$

$$= \int_{\overline{B}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \sum_{\mu\in\overline{M}(\mathbf{Q})} \int_{N(\mathbf{A})} f(g_1^{-1}\mu n g_2)dn\,\phi(g_2)\tau_T(g_2)dg_2$$

$$= \int_{\overline{B}(\mathbf{Q})N(\mathbf{A})\backslash\overline{G}(\mathbf{A})} \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \sum_{\mu\in\overline{M}(\mathbf{Q})} \int_{N(\mathbf{A})} f(g_1^{-1}\mu n n' g_2)dn\,\phi(n'g_2)\tau_T(n'g_2)dn'dg_2$$

$$= \int_{\overline{B}(\mathbf{Q})N(\mathbf{A})\backslash\overline{G}(\mathbf{A})} \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \sum_{\mu\in\overline{M}(\mathbf{Q})} \int_{N(\mathbf{A})} f(g_1^{-1}\mu n g_2)dn\,\phi(n'g_2)\tau_T(g_2)dn'dg_2$$

$$= \int_{\overline{B}(\mathbf{Q})N(\mathbf{A})\backslash\overline{G}(\mathbf{A})} \sum_{\mu\in\overline{M}(\mathbf{Q})} \int_{N(\mathbf{A})} f(g_1^{-1}\mu n g_2)dn \left(\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi(n'g_2)dn'\right)\tau_T(g_2)dg_2.$$

This vanishes because ϕ is cuspidal, and hence $T_{K^T}\phi = R(f)\phi$ as needed.

It remains to prove the absolute convergence. By Proposition 8.27,

$$\Phi_{g_1}(g_2) \stackrel{\text{def}}{=} \sum_{\delta\in B(\mathbf{Q})\backslash G(\mathbf{Q})} \sum_{\mu\in\overline{M}(\mathbf{Q})} \int_{N(\mathbf{A})} |f(g_1^{-1}\mu n\delta g_2)|dn\tau_T(\delta g_2)$$

is convergent, and bounded on compact sets. We will show that it is square-integrable (and in fact bounded) as a function of $g_2 \in \overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})$. Partition the fundamental domain $\mathfrak{F}$ as

$$\mathfrak{F} = \mathfrak{F}_T \cup \widetilde{\mathfrak{F}}_T, \tag{8.33}$$

where $\mathfrak{F}_T = \{g \in \mathfrak{F} \,|\, H(g) > T\}$, and $\widetilde{\mathfrak{F}}_T$ is its complement. Correspondingly, we set $\overline{\mathfrak{F}} = \overline{\mathfrak{F}}_T \cup \overline{\widetilde{\mathfrak{F}}}_T$. Clearly $\overline{\widetilde{\mathfrak{F}}}_T$ is compact. In particular, Φ_{g_1} is square integrable over $\overline{\widetilde{\mathfrak{F}}}_T$.

For $g_2 \in \mathfrak{F}_T$, $\tau_T(\delta g_2) \neq 0$ only for $\delta = 1$ (see e.g. [KL2], Lemma 17.1), so

$$\Phi_{g_1}(g_2) = \sum_{\mu \in \overline{M}(\mathbf{Q})} \int_{N(\mathbf{A})} |f(g_1^{-1}\mu n g_2)|dn \qquad (g_2 \in \mathfrak{F}_T).$$

For $i = 1, 2$, write $g_i = \left(\begin{smallmatrix} 1 & x_i \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y_i^{1/2} & \\ & y_i^{-1/2} \end{smallmatrix}\right) r_i \times k_i$ for $x_i \in \mathbf{R}, y_i > 0, r_i \in K_\infty$, and $k_i \in K_{\mathrm{fin}}$. By Lemma 8.28,

$$\Phi_{g_1}(g_2) = \sum_{\substack{a|D, a>0,\\ ad=D}} \int_{N(\mathbf{R})} |f_\infty(g_{1\infty}^{-1}\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) n g_{2\infty})|dn \int_{N(\frac{1}{D}\widehat{\mathbf{Z}})} |f_{\mathrm{fin}}(k_1^{-1}\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) n k_2)|dn,$$

where $N(\frac{1}{D}\widehat{\mathbf{Z}}) = \{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) | \, t \in \frac{1}{D}\widehat{\mathbf{Z}}\}$. The finite part is obviously bounded by $\mathrm{meas}(\frac{1}{D}\widehat{\mathbf{Z}}) = D$. For the infinite part, we refer ahead to the bound (8.38) in the next section (the proof there for f works just as well for $|f|$), by which for any given $\varepsilon > 0$,

$$\Phi_{g_1}(g_2) \ll_\varepsilon \sum_{ad=D} \frac{(\frac{dy_1y_2}{a})^{\frac{1}{2}}}{(\frac{ay_2}{dy_1} + \frac{dy_1}{ay_2} - 1)^{\frac{1}{2}+\varepsilon}} \ll \frac{(y_1y_2)^{\frac{1}{2}}}{(\frac{y_2}{y_1} + \frac{y_1}{y_2} - 1)^{\frac{1}{2}+\varepsilon}} \ll_{g_1} 1 \qquad (y_2 > T).$$

It follows that Φ_{g_1} is square-integrable on the finite measure space $\overline{\mathfrak{F}}_T$, and hence

$$\Phi_{g_1} \in L^2(\overline{\mathfrak{F}}), \text{ or equivalently, } \Phi_{g_1} \in L^2(\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})).$$

Therefore by Cauchy-Schwarz, for any $\phi \in L^2(\omega)$,

$$\int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \Phi_{g_1}(g)|\phi(g)|dg \leq \Big(\int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} \Phi_{g_1}(g)^2 dg\Big)^{1/2}\Big(\int_{\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A})} |\phi(g)|^2 dg\Big)^{1/2} < \infty.$$

This proves that (8.32) is absolutely convergent. □

Corollary 8.32 (Theorem 6.4)**.** *Suppose $f = f_\infty \times f_{\mathrm{fin}} \in C_c^m(G(\mathbf{A}), \overline{\omega})$ for $m \geq 2$ and f_∞ bi-K_∞-invariant. Then $R_0(f)$ is a Hilbert-Schmidt operator.*

Proof. Because $m \geq 2$, V is twice differentiable on the open interval $(0, \infty)$ (Proposition 8.9). Since it also has compact support, it trivially satisfies (8.30). Hence the result follows from Proposition 8.31. □

Corollary 8.33. *Suppose h satisfies* (8.1) *with $A > 3$ and $B > 4$. Let $f = f_\infty \times f_{\text{fin}}$ for f_∞ corresponding to h, and f_{fin} as described below* (8.27)*. Then $R_0(f)$ is a Hilbert-Schmidt operator.*

Proof. By Proposition 8.16, V satisfies (8.30). Therefore the result follows by Proposition 8.31. □

Proposition 8.34. *Let $r_\ell = \frac{1}{(1+|t|)^\ell}$. Then in the notation of Proposition 8.25,* $\mathrm{Spec}_1^a(r_\ell) < \infty$ *if $\ell > 9$.*

Proof. (See also the remark after Proposition 7.5.) Given $\ell > 9$, fix any $A > 3$, and let $h(t) = \frac{1}{(4A^2+t^2)^{\ell/2}}$. (The purpose of $4A^2$ is to ensure that h is holomorphic on $|\operatorname{Im}(t)| < A$.) Let $f = f_\infty \times f^1$ for f_∞ corresponding to $h_0(t) = (4A^2+t^2)^{-(\ell-1)/4}$ and f^1 the identity Hecke operator on $G(\mathbf{A}_{\text{fin}})$ (corresponding to $\mathtt{n} = 1$). Then $R_0(f)\varphi_{u_j} = h_0(t_j)\varphi_{u_j}$ for all Maass cusp forms u_j. It is not hard to show that h_0 satisfies (8.1) with $B = \frac{\ell-1}{2} > 4$. By equation (7.9),

$$\mathrm{Spec}_1^a(r_\ell) \ll \mathrm{Spec}_1^a(h) \ll \sum_{u_j\in\mathcal{F}} \frac{(1+|t_j|)}{|4A^2+t_j^2|^{\ell/2}} \ll \sum_{u_j} \frac{1}{|4A^2+t_j^2|^{(\ell-1)/2}}$$

$$= \sum_{u_j} |h_0(t_j)|^2 = \|R_0(f)\|_{HS}^2 < \infty.$$

The last step follows from Corollary 8.33. □

8.5 Proof of Proposition 8.29

Here we assume that (8.30) holds. Set

$$K_1(g_1,g_2) = \sum_{\gamma\notin\overline{B}(\mathbf{Q})} f(g_1^{-1}\gamma g_2), \tag{8.34}$$

and

$$K_2^T(g_1,g_2) = \sum_{\gamma\in\overline{B}(\mathbf{Q})} f(g_1^{-1}\gamma g_2) - \sum_{\delta\in B(\mathbf{Q})\backslash G(\mathbf{Q})} \sum_{\mu\in\overline{M}(\mathbf{Q})} \left(\int_{N(\mathbf{A})} f(g_1^{-1}\mu n\delta g_2)\,dn\right)\tau_T(\delta g_2). \tag{8.35}$$

Then

$$K^T(g_1,g_2) = K_1(g_1,g_2) + K_2^T(g_1,g_2).$$

We will show that each of these terms is square integrable over $\overline{\mathfrak{F}}\times\overline{\mathfrak{F}}$.

For $g_i \in \mathrm{SL}_2(\mathbf{R}) \times K_{\text{fin}} \subseteq G(\mathbf{A})$, we write

$$g_i = \begin{pmatrix} 1 & x_i \\ & 1 \end{pmatrix} \begin{pmatrix} y_i^{1/2} & \\ & y_i^{-1/2} \end{pmatrix} r_i \times k_i, \tag{8.36}$$

where $x_i \in \mathbf{R}, y_i > 0, r_i \in K_\infty, k_i \in K_{\text{fin}}$. Note that if $g_i \in \overline{\mathfrak{F}}$, then $x_i \in [-\frac{1}{2},\frac{1}{2}]$ and $y_i \geq \frac{\sqrt{3}}{2}$.

Lemma 8.35. *Given $\alpha > 2$ as in* (8.26)*, then with notation as above, for $g_1, g_2 \in \mathfrak{F}$ we have*

$$\sum_{\gamma \notin \overline{B}(\mathbf{Q})} |f(g_1^{-1}\gamma g_2)| \ll_\alpha \frac{1}{y_1^{\alpha/2-1} y_2^{\alpha/2-1}} + \frac{1}{y_1^{\alpha/2} y_2^{\alpha/2-1}}.$$

Proof. In view of (18.7) of [KL2], the result holds by (18.3) and (18.4) of [KL2] Lemma 18.3 with $C_1 = \{g_1\}$, $C_2 = \{g_2\}$, $L_1 = (y_2/y_1)^{1/2}$, $L_2 = (y_1/y_2)^{1/2}$, and $L_3 = (y_1 y_2)^{1/2}$. □

Proposition 8.36.

$$\|K_1\|_{\mathfrak{F}\times\mathfrak{F}} < \infty.$$

Proof. The square $\|K_1\|^2_{\mathfrak{F}\times\mathfrak{F}}$ of the L^2-norm is

$$\int_{\mathfrak{F}} \int_{\mathfrak{F}} \left| \sum_{\gamma \notin \overline{B}(\mathbf{Q})} f(g_1^{-1}\gamma g_2) \right|^2 dg_1 dg_2 \le \int_{\mathfrak{F}} \int_{\mathfrak{F}} \left(\sum_{\gamma \notin \overline{B}(\mathbf{Q})} |f(g_1^{-1}\gamma g_2)| \right)^2 dg_1 dg_2.$$

By the above lemma, the latter expression is

$$\ll \int_{\frac{\sqrt{3}}{2}}^{\infty} \int_{-1/2}^{1/2} \int_{\frac{\sqrt{3}}{2}}^{\infty} \int_{-1/2}^{1/2} \left(\frac{1}{y_1^{\alpha/2-1} y_2^{\alpha/2-1}} + \frac{1}{y_1^{\alpha/2} y_2^{\alpha/2-1}} \right)^2 \frac{dx_1 dy_1 dx_2 dy_2}{y_1^2\, y_2^2}$$

$$\ll \int_{\frac{\sqrt{3}}{2}}^{\infty} \int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{dy_1 dy_2}{y_1^{\alpha}\, y_2^{\alpha}} < \infty. \qquad \square$$

It remains to treat $K_2^T(g_1, g_2)$, for which $\gamma \in B(\mathbf{Q})$. When $g_1, g_2 \in \mathfrak{F}$, we can assume that $\det\gamma > 0$, since otherwise f_∞ vanishes. Thus, for $\mu \in M(\mathbf{Q})^+$ we define

$$F_{\mu,g_1,g_2}(t) = f(g_1^{-1}\mu \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} g_2) \qquad (t \in \mathbf{A}).$$

Given $g_1, g_2 \in \mathfrak{F}$, we will require bounds for the Fourier transform

$$\widehat{F}_{\mu,g_1,g_2}(r) = \int_{\mathbf{A}} F_{\mu,g_1,g_2}(t)\theta(rt)dt \le D\widehat{F}_{\mu,g_1,g_2,\infty}(r_\infty). \tag{8.37}$$

Here we have bounded the finite part by D as in the proof of Proposition 8.31, and $\widehat{F}_{\mu,g_1,g_2,\infty}(r_\infty) = \int_{\mathbf{R}} f_\infty(g_{1\infty}^{-1}\mu\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)g_{2\infty})e(-r_\infty t)dt$ is the archimedean part.

Lemma 8.37. *Let $g_1, g_2 \in \mathrm{SL}_2(\mathbf{R})$ be of the form of* (8.36) *(but of course with no $G(\mathbf{A}_{\mathrm{fin}})$ component), and let $\mu = \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) \in M(\mathbf{Q})^+$. Suppose V satisfies* (8.30) *for $u > 0$. Then*

$$\widehat{F}_{\mu,g_1,g_2,\infty}(0) \ll \sqrt{\frac{dy_1 y_2}{a}} \left(\frac{ay_2}{dy_1} + \frac{dy_1}{ay_2} - 1 \right)^{-\frac{1}{2}-\varepsilon}. \tag{8.38}$$

If, in addition, $ay_2 \neq dy_1$, then for real $r \neq 0$ we have

$$\widehat{F}_{\mu,g_1,g_2,\infty}(r) \ll r^{-2}\sqrt{\frac{a}{dy_1y_2}}. \tag{8.39}$$

If V satisfies (8.30) *also at the endpoint $u = 0$, then* (8.39) *holds even when $ay_2 = dy_1$.*

Proof. We have

$$\begin{aligned} f_\infty(g_{1\infty}^{-1}\mu\begin{pmatrix}1 & t\\ & 1\end{pmatrix}g_{2\infty}) &= f_\infty(\begin{pmatrix} y_1^{-1/2}y_2^{1/2}a & y_1^{-1/2}y_2^{-1/2}(-dx_1+ax_2+at)\\ 0 & y_1^{1/2}y_2^{-1/2}d\end{pmatrix}) \\ &= V(\frac{ay_2}{dy_1}+\frac{dy_1}{ay_2}+\frac{(-dx_1+ax_2+at)^2}{ady_1y_2}-2). \end{aligned} \tag{8.40}$$

The Fourier transform is thus given by

$$\begin{aligned} \widehat{F}_{\mu,g_1,g_2,\infty}(r) &= \int_{\mathbf{R}} V(\frac{ay_2}{dy_1}+\frac{dy_1}{ay_2}+\frac{(-dx_1+ax_2+at)^2}{ady_1y_2}-2)e(-rt)dt \\ &= \int_{\mathbf{R}} V(\frac{ay_2}{dy_1}+\frac{dy_1}{ay_2}+\frac{at^2}{dy_1y_2}-2)e(-r(t-x_2+\frac{dx_1}{a}))dt \\ &= e(r(x_2-\frac{dx_1}{a}))\int_{\mathbf{R}} V(\frac{ay_2}{dy_1}+\frac{dy_1}{ay_2}+\frac{at^2}{dy_1y_2}-2)e(-rt)dt \\ &= e(r(x_2-\frac{dx_1}{a}))\sqrt{\frac{dy_1y_2}{a}}\int_{\mathbf{R}} V(A_{y_1,y_2}+t^2)e(-r\sqrt{\tfrac{dy_1y_2}{a}}t)dt, \end{aligned} \tag{8.41}$$

where $A_{y_1,y_2} = \frac{ay_2}{dy_1}+\frac{dy_1}{ay_2}-2 \geq 0$.

If $r = 0$, the estimate $V(u) \ll (1+u)^{-1-\varepsilon}$ (of (8.30)) implies that $\widehat{F}_{\mu,g_1,g_2,\infty}(0)$ is

$$\begin{aligned} \ll \sqrt{\frac{dy_1y_2}{a}}\int_{\mathbf{R}}\frac{dt}{(1+A_{y_1,y_2}+t^2)^{1+\varepsilon}} &= \frac{\sqrt{\frac{dy_1y_2}{a}}}{(1+A_{y_1,y_2})^{1+\varepsilon}}\int_{\mathbf{R}}\frac{dt}{(1+\frac{t^2}{1+A_{y_1,y_2}})^{1+\varepsilon}} \\ &= \sqrt{\frac{dy_1y_2}{a}}\frac{(1+A_{y_1,y_2})^{1/2}}{(1+A_{y_1,y_2})^{1+\varepsilon}}\int_{\mathbf{R}}\frac{dt}{(1+t^2)^{1+\varepsilon}}. \end{aligned}$$

The estimate (8.38) follows. If $r \neq 0$, then by Proposition 8.8, (8.41) is

$$\ll \sqrt{\frac{dy_1y_2}{a}}\left(r\sqrt{\frac{dy_1y_2}{a}}\right)^{-2}\int_{\mathbf{R}}\left|\frac{d^2}{dt^2}V(A_{y_1,y_2}+t^2)\right|dt.$$

In order to prove (8.39), it suffices to show that the above integral is bounded independently of a, d, y_1, y_2. Using the bounds (8.30), we have

$$\int_{\mathbf{R}}\left|\frac{d^2}{dt^2}V(A_{y_1,y_2}+t^2)\right|dt = \int_{\mathbf{R}}|2V'(A_{y_1,y_2}+t^2)+4t^2V''(A_{y_1,y_2}+t^2)|dt$$

$$\ll \int_{\mathbf{R}} 2(1+A_{y_1,y_2}+t^2)^{-1-\varepsilon}dt + \int_{\mathbf{R}} 4t^2(1+A_{y_1,y_2}+t^2)^{-3/2-\varepsilon}dt$$
$$\leq \int_{\mathbf{R}} 2(1+t^2)^{-1-\varepsilon}dt + \int_{\mathbf{R}} 4t^2(1+t^2)^{-3/2-\varepsilon}dt < \infty,$$

as needed. Notice that when $ay_2 \neq dy_1$, $A_{y_1,y_2} > 0$ and the above involves only derivatives of V on the open interval $(0,\infty)$. □

Recall the partition $\mathfrak{F} = \mathfrak{F}_T \cup \widetilde{\mathfrak{F}}_T$ from (8.33). It will be convenient to decompose the norm of K_2^T accordingly as

$$\|K_2^T\|_{\mathfrak{F}\times\mathfrak{F}} = (\|K_2^T\|^2_{\mathfrak{F}\times\mathfrak{F}_T} + \|K_2^T\|^2_{\mathfrak{F}\times\widetilde{\mathfrak{F}}_T})^{1/2},$$

and consider each piece separately.

Proposition 8.38. *Under the hypotheses of Proposition 8.29,*

$$\|K_2^T\|_{\mathfrak{F}\times\mathfrak{F}_T} < \infty.$$

Proof. Suppose $g_2 \in \mathfrak{F}_T$. Then $\tau_T(\delta g_2) \neq 0$ only if $\delta = 1$ (cf. Lemma 17.1 of [KL2]). Hence

$$K_2^T(g_1,g_2) = \sum_{\gamma\in\overline{B}(\mathbf{Q})} f(g_1^{-1}\gamma g_2) - \sum_{\mu\in\overline{M}(\mathbf{Q})} \int_{N(\mathbf{A})} f(g_1^{-1}\mu n g_2)dn$$
$$= \sum_{\mu\in\overline{M}(\mathbf{Q})} \left(\sum_{\eta\in N(\mathbf{Q})} f(g_1^{-1}\mu\eta g_2) - \int_{N(\mathbf{A})} f(g_1^{-1}\mu n g_2)dn \right).$$

The rearrangement is justified by Proposition 8.27.

We would like to apply the Poisson summation formula to the sum over η. To justify this, note that by (8.40) and (8.30), $F_{\mu,g_1,g_2,\infty}(t) \ll t^{-2}$, while $F_{\mu,g_1,g_2,\mathrm{fin}}$ is a Schwartz-Bruhat function on $\mathbf{A}_{\mathrm{fin}}$ (see the proof of Proposition 19.10 of [KL2]). On the other hand, write $\mu = \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right)$, take g_1, g_2 in the form of (8.36), and suppose $ay_2 \neq dy_1$. Then by the above lemma, $\widehat{F}_{\mu,g_1,g_2,\infty}(t) \ll t^{-2}$ for $t \neq 0$. Hence by [KL2] Theorem 8.17, the adelic Poisson summation formula can be applied to the global function F_{μ,g_1,g_2}.

Therefore for fixed g_1, using Lemma 8.28 we see that for almost all $g_2 \in \mathfrak{F}_T$,

$$K_2^T(g_1,g_2) = \sum_{\substack{\mu=\mathrm{diag}(a,d)\\ ad=D, a>0}} \sum_{t\in\mathbf{Q}^*} \widehat{F}_{\mu,g_1,g_2}(t).$$

(Poisson sumation may fail to hold on the set of g_2 with $y_2 \in \{\frac{dy_1}{a} | ad = D\}$, but this is of measure 0.) Because $F_{\mu,g_1,g_2,\mathrm{fin}}$ is a Schwartz-Bruhat function, $\widehat{F}_{\mu,g_1,g_2,\mathrm{fin}}$ is as well (cf. [KL2], Proposition 8.13). Therefore its support is contained in $\frac{1}{M}\widehat{\mathbf{Z}}$ for some integer $M > 0$. By (8.37) and (8.39),

$$\sum_{\substack{\mu=\mathrm{diag}(a,d)\\ ad=D, a>0}} \sum_{t\in\mathbf{Q}^*} \widehat{F}_{\mu,g_1,g_2}(t) \ll \sum_{\substack{\mu=\mathrm{diag}(a,d)\\ ad=D, a>0}} \sum_{t\in\frac{1}{M}\mathbf{Z}-\{0\}} t^{-2}\sqrt{\frac{a}{dy_1y_2}} \ll \frac{1}{\sqrt{y_1y_2}} \quad (8.42)$$

for a.e. $g_2 \in \mathfrak{F}_T$. Therefore

$$\|K_2^T\|^2_{\overline{\mathfrak{F}}\times\overline{\mathfrak{F}}_T} = \int_{\overline{\mathfrak{F}}}\int_{\overline{\mathfrak{F}}_T} |K_2^T(g_1,g_2)|^2 dg_2 dg_1 \ll \int_{\frac{\sqrt{3}}{2}}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{T}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx_2dy_2dx_1dy_1}{y_1^3y_2^3},$$

which is clearly finite. □

Lemma 8.39. *Let $C = C_\infty \times C_{\mathrm{fin}}$ be a compact subset of $\overline{G}(\mathbf{A})$. Then for any $\mu \in M(\mathbf{Q})$,*

$$\int_C \int_{\overline{\mathfrak{F}}} |\widehat{F}_{\mu,g_1,g_2}(0)|^2 dg_1 dg_2 < \infty.$$

Proof. The above integral factorizes as

$$\int_{C_\infty}\int_F |\widehat{F}_{\mu,g_1,g_2,\infty}(0)|^2 dg_{1\infty}dg_{2\infty} \int_{C_{\mathrm{fin}}}\int_{K_{\mathrm{fin}}} |\widehat{F}_{\mu,k,g_2,\mathrm{fin}}(0)|^2 dk\, dg_{2\mathrm{fin}}, \quad (8.43)$$

where F is the archimedean part of $\overline{\mathfrak{F}}$, defined on page 108. Observe that by (8.28), $f_{\mathrm{fin}}(k_1^{-1}\mu\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)g_{2\mathrm{fin}}) \neq 0$ only if

$$\begin{pmatrix}1&t\\0&1\end{pmatrix} \in Z(\mathbf{A}_{\mathrm{fin}})\mu^{-1}K_{\mathrm{fin}}\delta K_{\mathrm{fin}}C_{\mathrm{fin}}^{-1}.$$

Since $\mu^{-1}K_{\mathrm{fin}}\delta K_{\mathrm{fin}}C_{\mathrm{fin}}^{-1}$ is compact, taking the determinant of both sides we see that the $Z(\mathbf{A}_{\mathrm{fin}})$-part of $\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)$ is also restricted to a compact set, i.e.

$$\begin{pmatrix}1&t\\0&1\end{pmatrix} \in Z_0\,\mu^{-1}K_{\mathrm{fin}}\delta K_{\mathrm{fin}}C_{\mathrm{fin}}^{-1}.$$

for some compact subset $Z_0 \subseteq Z(\mathbf{A}_{\mathrm{fin}})$. The above set is compact, so it follows that t is restricted to some compact subset $B \subseteq \mathbf{A}_{\mathrm{fin}}$, and hence

$$|\widehat{F}_{\mu,k_1,g_2,\mathrm{fin}}(0)| \le \int_B |f_{\mathrm{fin}}(k_1^{-1}\mu\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)g_2)|dt \le \mathrm{meas}(B).$$

Therefore, the non-archimedean double integral in (8.43) is finite.

For the infinite part, without loss of generality we can assume that $C_\infty \subseteq \mathrm{SL}_2(\mathbf{R})$, so it consists of elements $\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}\sqrt{y}&\\&\sqrt{y}^{-1}\end{smallmatrix}\right)k_\infty$ with $-L \le x \le L$ and $0 < T_1 \le y \le T_2$ for some constants L, T_1, T_2. By (8.38),

$$|\widehat{F}_{\mu,g_1,g_2,\infty}(0)|^2 \ll_\varepsilon \frac{dy_1y_2}{a(\frac{dy_1}{ay_2}+\frac{ay_2}{dy_1}-1)^{1+\varepsilon}} \ll \frac{y_1y_2}{(\frac{dy_1}{ay_2}+\frac{ay_2}{dy_1})^{1+\varepsilon}},$$

where the latter bound holds by the fact that $\frac{dy_1}{ay_2}+\frac{ay_2}{dy_1} \ge 2$. Hence

$$\int_{C_\infty}\int_F |\widehat{F}_{g_1,g_2,\mu,\infty}(0)|^2 dg_{1\infty}dg_{2\infty}$$

$$\ll \int_{-L}^{L}\int_{-1/2}^{1/2}\int_{T_1}^{T_2}\int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{y_1y_2}{(\frac{dy_1}{ay_2}+\frac{ay_2}{dy_1})^{1+\varepsilon}}\frac{dy_1dy_2}{y_1^2y_2^2}dx_1dx_2$$

$$\ll \int_{T_1}^{T_2}\left(\int_{\frac{\sqrt{3}}{2}}^{\infty}\frac{1}{(\frac{dy_1}{ay_2}+\frac{ay_2}{dy_1})^{1+\varepsilon}}\frac{dy_1}{y_1}\right)\frac{dy_2}{y_2} = \int_{T_1}^{T_2}\left(\int_{\frac{d\sqrt{3}}{2ay_2}}^{\infty}\frac{1}{(y+y^{-1})^{1+\varepsilon}}\frac{dy}{y}\right)\frac{dy_2}{y_2}.$$

The inner integral is absolutely convergent, and defines a continuous function of y_2. Therefore the outer integral converges as well. □

Lemma 8.40. *Given $\delta \in B(\mathbf{Q})\backslash G(\mathbf{Q})$, there exists a finite subset $A_\delta \subseteq \overline{M}(\mathbf{Q})$ such that $\widehat{F}_{\mu,g_1,\delta g_2}(0)$ is identically 0 as a function of $(g_1,g_2)\in\mathfrak{F}\times\mathfrak{F}$ for all $\mu\in\overline{M}(\mathbf{Q})$ which are not in A_δ.*

Proof. Write $g_{i\,\mathrm{fin}} = k_i \in K_{\mathrm{fin}}$. The lemma follows by looking at the finite part

$$\widehat{F}_{\mu,g_1,\delta g_2,\mathrm{fin}}(0) = \int_{\mathbf{A}_{\mathrm{fin}}} f_{\mathrm{fin}}(k_1^{-1}\left(\begin{smallmatrix} a & \\ & d\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t\\ 0 & 1\end{smallmatrix}\right)\delta k_2)dt.$$

By the Bruhat decomposition $G(\mathbf{Q}) = B(\mathbf{Q})\cup B(\mathbf{Q})\left(\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}\right)N(\mathbf{Q})$, we can take

$$\delta \in \{1\}\cup\{\left(\begin{smallmatrix} 0 & 1\\ 1 & r\end{smallmatrix}\right) | r\in\mathbf{Q}\}.$$

When $\delta = 1$, the assertion follows from Lemma 8.28. Hence, we may suppose that $\delta = \left(\begin{smallmatrix} 0 & 1\\ 1 & r\end{smallmatrix}\right)$. Suppose

$$f_{\mathrm{fin}}(k_1^{-1}\left(\begin{smallmatrix} a & \\ & d\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1\\ 1 & r\end{smallmatrix}\right)k_2) \neq 0.$$

Taking the determinant and arguing as in the proof of Proposition 8.28, we can assume that $a>0$, $ad=\pm D$, and

$$\begin{pmatrix} a & \\ & d\end{pmatrix}\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & r\end{pmatrix} = \begin{pmatrix} at & a+(at)r\\ d & dr\end{pmatrix}\in M_2(\widehat{\mathbf{Z}}).$$

In particular, $d\in\mathbf{Z}$ and $at\in\widehat{\mathbf{Z}}$. From the fact that the upper right-hand entry also belongs to $\widehat{\mathbf{Z}}$, it then follows that

$$\frac{D}{d} = \pm a \in (r\widehat{\mathbf{Z}}+\widehat{\mathbf{Z}})\cap\mathbf{Q} = \frac{1}{\beta}\mathbf{Z}$$

if $r=\frac{\alpha}{\beta}$ for $\alpha,\beta\in\mathbf{Z}$ relatively prime. It follows that $d|\beta D$. In particular, the set of such d is finite. □

Proposition 8.41.

$$\|K_2^T\|_{\mathfrak{F}\times\tilde{\mathfrak{F}}_T} < \infty.$$

Proof. By definition, for $g_1,g_2\in\mathfrak{F}$, we have

$$K_2^T(g_1,g_2) = \sum_{\mu\in\overline{M}(\mathbf{Q})}\sum_{t\in\mathbf{Q}} F_{\mu,g_1,g_2}(t) - \sum_{\delta\in B(\mathbf{Q})\backslash G(\mathbf{Q})}\sum_{\mu\in\overline{M}(\mathbf{Q})}\widehat{F}_{\mu,g_1,\delta g_2}(0)\tau_T(\delta g_2).$$

As in the proof of Proposition 8.38, for fixed g_1, we can apply Poisson summation to the sum over t for a.e. g_2, so $K_2^T(g_1,g_2)$ is equal almost everywhere to the sum of the following three functions:

1. $\displaystyle\sum_{\substack{\mu=\mathrm{diag}(a,d),\\ ad=D,a>0}} \sum_{t\in\frac{1}{M}\mathbf{Z}-\{0\}} \widehat{F}_{\mu,g_1,g_2}(t)$

2. $\displaystyle\sum_{\substack{\mu=\mathrm{diag}(a,d),\\ ad=D,a>0}} \widehat{F}_{\mu,g_1,g_2}(0)$

3. $\displaystyle -\sum_{\delta\in B(\mathbf{Q})\backslash G(\mathbf{Q})} \sum_{\mu\in A_\delta} \widehat{F}_{\mu,g_1,\delta g_2}(0)\tau_T(\delta g_2),$

where A_δ is the finite subset of $\overline{M}(\mathbf{Q})$ given by Lemma 8.40. By Minkowski's inequality, it suffices to show that each of these three functions is square integrable over $(g_1, g_2) \in \overline{\mathfrak{F}} \times \overline{\overline{\mathfrak{F}}}_T$.

By (8.42), the integral of the square of first function over $\overline{\mathfrak{F}} \times \overline{\overline{\mathfrak{F}}}_T$ is

$$\ll \int_{\frac{\sqrt{3}}{2}}^{T} \int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{dy_1}{y_1^3}\frac{dy_2}{y_2^3} < \infty.$$

The square integrability of each summand of the second function was proven in Lemma 8.39 above, and it follows by Minkowski's inequality that the second function itself is square integrable over the given set. For the third function, by [KL2] Proposition 17.2, there are only finitely many $\delta \in B(\mathbf{Q})\backslash G(\mathbf{Q})$ such that $\tau_T(\delta g) \neq 0$ for some $g \in \widetilde{\mathfrak{F}}_T$. Therefore it suffices to show that $\|\widehat{F}_{\mu,g_1,\delta g_2}(0)\tau_T(\delta g_2)\|_{\mathfrak{F}\times\widetilde{\mathfrak{F}}_T}$ is finite for fixed δ and μ. We have

$$\left\|\widehat{F}_{\mu,g_1,\delta g_2}(0)\tau_T(\delta g_2)\right\|^2_{\mathfrak{F}\times\widetilde{\mathfrak{F}}_T} \leq \int_{\overline{\mathfrak{F}}}\int_{\overline{\overline{\mathfrak{F}}}_T} |\widehat{F}_{\mu,g_1,\delta g_2}(0)|^2 dg_2 dg_1$$

$$= \int_{\overline{\mathfrak{F}}}\int_{\delta\overline{\overline{\mathfrak{F}}}_T} |\widehat{F}_{\mu,g_1,g_2}(0)|^2 dg_2 dg_1,$$

which is finite by Lemma 8.39, since $\delta\overline{\overline{\mathfrak{F}}}_T$ is compact and factorizable. □

Proof of Proposition 8.29. Since $K^T = K_1 + K_2^T$, it suffices by Minkowski's inequality to show that the latter two functions are square integrable over $\overline{\mathfrak{F}}\times\overline{\mathfrak{F}}$. By Proposition 8.36, $\|K_1\|_{\mathfrak{F}\times\mathfrak{F}} < \infty$. By Proposition 8.38 and Proposition 8.41,

$$\|K_2^T\|^2_{\mathfrak{F}\times\mathfrak{F}} = \|K_2^T\|^2_{\mathfrak{F}\times\mathfrak{F}_T} + \|K_2^T\|^2_{\mathfrak{F}\times\widetilde{\mathfrak{F}}_T} < \infty.$$

This completes the proof. □

9 Kloosterman sums

Fix a modulus $N \in \mathbf{Z}^+$, and let χ be a Dirichlet character modulo N of conductor $\mathfrak{c}_\chi$. We have defined the following generalized Kloosterman sum for any $c \in N\mathbf{Z}^+$ and nonzero $\mathtt{n} \in \mathbf{Z}$:

$$S_\chi(a, b; \mathtt{n}; c) = \sum_{\substack{x, x' \in \mathbf{Z}/c\mathbf{Z}, \\ xx' = \mathtt{n}}} \overline{\chi(x)} e(\frac{ax + bx'}{c}). \tag{9.1}$$

Although $\gcd(\mathtt{n}, N) = 1$ elsewhere in this paper, we make no such restriction in this section. Note that when $\mathtt{n} > 1$, x need not be invertible in $\mathbf{Z}/c\mathbf{Z}$. Furthermore, χ is *not* generally a Dirichlet character modulo c, and should be viewed simply as a multiplicative function on $\mathbf{Z}/c\mathbf{Z}$. In particular it can happen that $\chi(x) \neq 0$ when $(x, c) > 1$.

In the special case where $\mathtt{n} = 1$, we obtain the usual twisted Kloosterman sum with character χ defined by

$$S_\chi(a, b; c) = \sum_{x \in (\mathbf{Z}/c\mathbf{Z})^*} \overline{\chi(x)} e(\frac{ax + b\overline{x}}{c}), \tag{9.2}$$

where $x\overline{x} \equiv 1 \bmod c$. If χ is the principal character modulo N, then we simply write $S(a, b; c)$, which is the classical Kloosterman sum.

Suppose $\mathtt{n} = \mathtt{n}_1\mathtt{n}_2$ where $(\mathtt{n}_1, c) = 1$. Then replacing x' by $\overline{\mathtt{n}_1}x'$, we have

$$S_\chi(a, b; \mathtt{n}; c) = S_\chi(a, b\mathtt{n}_1; \mathtt{n}_2; c). \tag{9.3}$$

In particular, if $(\mathtt{n}, c) = 1$ we have

$$S_\chi(a, b; \mathtt{n}; c) = S_\chi(a, b\mathtt{n}; c).$$

This holds in other situations as well; see (9.24) below. In his Ph.D. thesis, J. Andersson discusses the generalized Kloosterman sums (9.1), which were apparently first defined by Bykovsky, Kuznetsov and Vinogradov ([A], [BKV]). He gives elementary proofs of the following identities, special cases of which were given by [BKV] and Selberg [Sel1].

Proposition 9.1. *If either* $(N, \mathtt{n}) = 1$ *or* $(N, b) = 1$*, then*

$$S_\chi(a, b; \mathtt{n}; c) = \sum_{d | (\mathtt{n}, b, c)} \overline{\chi(d)}\, d\, S_\chi(a, \tfrac{b\mathtt{n}}{d^2}; \tfrac{c}{d}). \tag{9.4}$$

The identity also holds if χ *is taken to be the principal character modulo* c *(resp.* c/d*) on the left (resp. right). In the case where* χ *is principal, we have*

$$S(a_1, a_2; a_3; c) = S(a_{\sigma(1)}, a_{\sigma(2)}; a_{\sigma(3)}; c) \tag{9.5}$$

for any permutation $\sigma \in S_3$.

See [A] for the proofs. In his proof of (9.4) (Theorem 1 on page 109 of [A]), some hypothesis on N (such as $(N,\mathtt{n})=1$ or $(N,b)=1$) is used implicitly in order for χ to be well-defined modulo c/d in $S_\chi(a,\frac{b\mathtt{n}}{d^2},\frac{c}{d})$.

The purpose of this section is to prove the following Weil bound for the sum (9.1).

Theorem 9.2. *For integers $c\in N\mathbf{Z}$ and $a,b,\mathtt{n}\in\mathbf{Z}$ with $c,\mathtt{n}$ nonzero, we have the bounds*

$$|S_\chi(a,b;\mathtt{n};c)|\le\tau(\mathtt{n})\,\tau(c)\,(a\mathtt{n},b\mathtt{n},c)^{1/2}\,c^{1/2}\,\mathfrak{c}_\chi^{1/2}$$

and

$$|S_\chi(a,b;\mathtt{n};c)|\le\tau(\mathtt{n})\,\tau(c)\,(a\mathtt{n},b\mathtt{n},c)^{1/2}\,c^{1/2}\,\mathfrak{c}_\chi^{1/4}\prod_{p|\mathfrak{c}_\chi}p^{1/4}$$

for the divisor function τ.

Remark: Bruggeman and Miatello produce a bound when $\mathtt{n}=1$, which is valid over any totally real field (cf. Section 2.4 of [BM]). They use the trivial bound at primes $p|N$, which results in the estimate

$$|S_\chi(a,b;c)|=O(c^{\frac12+\varepsilon}\prod_{p|N}p^{c_p/2}).$$

This is somewhat weaker than the estimates in Theorem 9.2, whose full strength was required in the proof of Proposition 7.12.

9.1 A bound for twisted Kloosterman sums

The proof of Theorem 9.2 follows three steps: express (9.1) as a product of local factors, relate the local factors to twisted Kloosterman sums (9.2), and apply a Weil bound to the latter. The present section establishes the Weil bound needed for the last step. The classical Kloosterman sums satisfy the Weil/Salié bound

$$|S(a,b;c)|\le\tau(c)(a,b,c)^{1/2}c^{1/2}\tag{9.6}$$

(cf. [IK], Corollary 11.12). It should be noted that the above bound does *not* hold for $S_\chi(a,b;c)$. See Example 9.9 below. In general, one must account for the conductor of χ as well.

Theorem 9.3. *Let p be any prime. Suppose $c=p^\ell$ and χ is a Dirichlet character of conductor $\mathfrak{c}_\chi=p^\gamma$ for $\gamma\le\ell$. Then for any integers a,b,*

$$|S_\chi(a,b;c)|\le\tau(c)\,(a,b,c)^{1/2}\,c^{1/2}\,\mathfrak{c}_\chi^{1/2}\tag{9.7}$$

and

$$|S_\chi(a,b;c)|\le\tau(c)\,(a,b,c)^{1/2}\,c^{1/2}\,\mathfrak{c}_\chi^{1/4}p^{1/4}.\tag{9.8}$$

Remarks: The proof will occupy the remainder of this section. The methods are standard and in large part elementary, but because there seems to be no proof in the literature, we will include the details. The general case (for c not necessarily a prime power) is contained in Theorem 9.2, whose proof will follow later.

The case $\ell = 1$ is the most difficult, but it is well-known.

Proposition 9.4. *Suppose $c = p$ is prime. Then $|S_\chi(a, b; p)| \leq 2(a, b, p)^{1/2} p^{1/2}$.*

Proof. When $p = 2$, the proposition is trivial. If p is odd and $p \nmid ab$, this was proven by Weil for principal χ, and extended to non-principal χ by Chowla ([We], [Ch]; see also [Co]). These sources deal only with the case $b = 1$, but the general case follows easily by a change of variables.

If $p|a$ and $p \nmid b$ (or vice versa), then $S_\chi(a, b; p)$ is a character sum precisely of the kind discussed in Section 5.8. In this case, if χ is the principal character modulo p, the value of the sum is -1. If χ is non-principal, then $|S_\chi(a, b; p)| = p^{1/2}$ ([Hua], Theorem 7.4.4).

Lastly, if $p|a$ and $p|b$, then by the triangle inequality, $|S_\chi(a, b; p)| \leq p = (a, b, p)^{1/2} p^{1/2}$. □

The case $p = c^\ell$ with $\ell \geq 2$ is elementary, as first shown for the case of principal χ by Salié [Sal], whose work was later refined by Estermann [Es]. We will follow the presentation in Section 12.3 of [IK]. It requires a knowledge of the number of solutions to certain quadratic congruences, given as Lemma 9.6 below. Although this is standard, we include the proof because of its central importance in what follows.

Lemma 9.5. *Let $n, D > 0$, with $p \nmid D$. Let M be the number of solutions of*

$$x^2 \equiv D \textit{ mod } p^n. \tag{9.9}$$

Then

$$M = \begin{cases} 1 & \textit{if } p = 2, n = 1 \\ 0 & \textit{if } p = 2, n = 2, D \equiv 3 \textit{ mod } 4 \\ 2 & \textit{if } p = 2, n = 2, D \equiv 1 \textit{ mod } 4 \\ 0 & \textit{if } p = 2, n > 2, D \not\equiv 1 \textit{ mod } 8 \\ 4 & \textit{if } p = 2, n > 2, D \equiv 1 \textit{ mod } 8 \\ 1 + (\frac{D}{p}) & \textit{if } p > 2. \end{cases}$$

Proof. See e.g. [Land], Theorem 87. □

Lemma 9.6. *Let a be an integer and $p \nmid a$ a prime. Consider the congruence*

$$ax^2 + Bx + c \equiv 0 \textit{ mod } p^n \tag{9.10}$$

for $n > 0$. Write $\Delta = B^2 - 4ac = p^\delta \Delta'$, where $p \nmid \Delta'$. Let M denote the number of solutions to (9.10)*. Then if $p \neq 2$,*

$$M = \begin{cases} p^{\lfloor \frac{n}{2} \rfloor} & \text{if } \delta \geq n, \\ 2p^{\delta'} & \text{if } \delta = 2\delta' < n \text{ and } (\frac{\Delta'}{p}) = 1, \\ 0 & \text{otherwise, i.e. } \delta < n \text{ and } (\delta \text{ is odd or } (\frac{\Delta'}{p}) = -1). \end{cases}$$

When $\delta > 0$, all solutions are prime to p if $p \nmid B$, and divisible by p otherwise. Suppose $\delta = 0$ and $(\frac{\Delta}{p}) = 1$. Then both solutions are prime to p if $p \nmid c$ (in particular if $p|B$), but if $p|c$ then exactly one of the two solutions is divisible by p.

If $p = 2$ and B is even, then $\delta \geq 2$ and

$$M = \begin{cases} 2^{\lfloor \frac{n}{2} \rfloor} & \text{if } \delta \geq n+2, \\ 2^{\min(n-\delta+1,2)} 2^{\delta'-1} & \text{if } 2 \leq \delta = 2\delta' < n+2 \text{ and } \Delta' \equiv 1 \text{ mod } 2^{\min(n-\delta+2,3)} \\ 0 & \text{otherwise.} \end{cases}$$

By (9.10)*, all solutions have the same parity as c. Furthermore, when $\delta > 2$, all solutions are odd if $4 \nmid B$, and even otherwise. When $\delta = 2$, all solutions are even if $4 \nmid B$, and odd otherwise.*

If $p = 2$ and B is odd, then $M = \begin{cases} 2 & \text{if } \Delta \equiv 1 \text{ mod } 8 \\ 0 & \text{otherwise.} \end{cases}$

Note that $\Delta \equiv 1$ mod 8 if and only if c is even. In this case, exactly one of the two solutions is even.

Proof. First, suppose $p \neq 2$. Then (9.10) is equivalent to

$$(2ax + B)^2 \equiv \Delta \text{ mod } p^n. \tag{9.11}$$

If $\delta \geq n$, the solutions of (9.11) are given by $2ax + B \equiv 0 \text{ mod } p^{\lceil \frac{n}{2} \rceil}$. There is a unique solution x modulo $p^{\lceil \frac{n}{2} \rceil}$, so there are $p^{n - \lceil \frac{n}{2} \rceil} = p^{\lfloor \frac{n}{2} \rfloor}$ solutions modulo p^n. The solutions x are coprime to p if and only if $p \nmid B$.

Suppose $\delta < n$. If δ is odd, it is easy to see that (9.11) has no solution. Suppose δ is even and write $\delta = 2\delta'$. Then the solutions of the congruence are given by $2ax + B \equiv p^{\delta'} X \text{ mod } p^n$, where

$$X^2 \equiv \Delta' \text{ mod } p^{n-\delta}.$$

By Lemma 9.5, this congruence has solutions (necessarily two) if and only if Δ' is a quadratic residue modulo p. So if $(\frac{\Delta'}{p}) = -1$, (9.11) has no solution. Otherwise, the solutions of (9.11) are given by

$$2ax + B \equiv p^{\delta'}(X + p^{n-\delta}\alpha) \text{ mod } p^n,$$

where α ranges through $(\mathbf{Z}/p^{\delta'}\mathbf{Z})$. Therefore, in this case the number of solutions is $2p^{\delta'}$ (two choices for X, and $p^{\delta'}$ choices for α). If $\delta > 0$, then a solution x is divisible by p if and only if $p|B$. Now suppose $\delta = 0$. Then $x \equiv (2a)^{-1}(X - B) \bmod p^n$, where $X^2 \equiv \Delta \bmod p^n$. If $p \nmid c$ (which is the case if $p|B$), then $(X - B)$, and hence x, is prime to p since

$$(X - B)(X + B) \equiv X^2 - B^2 \equiv -4ac \bmod p.$$

When $p|c$ (so that $p \nmid B$), then exactly one of the solutions is divisible by p, as can be seen by considering $ax^2 + Bx + c \equiv x(ax + B) \equiv 0 \bmod p$.

Now consider $p = 2$. The congruence (9.10) is equivalent to

$$(2ax + B)^2 \equiv \Delta \bmod 2^{n+2}. \tag{9.12}$$

Suppose $B = 2B'$ is even. Then (9.12) has a solution only if $\delta \geq 2$. In that case, the congruence is equivalent to

$$(ax + B')^2 \equiv 2^{\delta-2}\Delta' \bmod 2^n. \tag{9.13}$$

If $\delta - 2 \geq n$, then $ax + B' \equiv 0 \bmod 2^{\lceil \frac{n}{2} \rceil}$. So as before, the congruence (9.13) has $2^{\lfloor \frac{n}{2} \rfloor}$ solutions. These solutions x are prime to 2 if and only if $2 \nmid B'$, or equivalently, $4 \nmid B$. Now suppose $\delta - 2 < n$. Then (9.13) is possible only if $\delta = 2\delta'$ is even, in which case we can write $ax + B' \equiv 2^{\delta'-1}X \bmod 2^n$, where $X^2 \equiv \Delta' \bmod 2^{n-\delta+2}$. By Lemma 9.5, such X exists if and only if $\Delta' \equiv 1 \bmod 2^{\min(n-\delta+2,3)}$, and the number of solutions X modulo $2^{n-\delta+2}$ is $2^{\min(n-\delta+1,2)}$. In this case we can take

$$ax + B' \equiv 2^{\delta'-1}(X + 2^{n-\delta+2}\alpha) \bmod 2^n$$

for any $\alpha \in \mathbf{Z}/2^{\delta'-1}\mathbf{Z}$. Therefore (9.13) has $2^{\min(n-\delta+1,2)}2^{\delta'-1}$ solutions. If $\delta > 2$, then we see that $2|x \iff 2|B' \iff 4|B$. If $\delta = 2$, then $\Delta' = (B')^2 - ac$ is odd, and we see that

$$B' \text{ even} \implies c \text{ odd} \implies x \text{ odd},$$

$$B' \text{ odd} \implies c \text{ even} \implies x \text{ even}.$$

(The fact that x and c have the same parity when $2|B$ is immediate from (9.10)).

Lastly, suppose $p = 2$ and B is odd. Then (9.12) is solvable only if $\delta = 0$. In that case, the congruence $X^2 \equiv \Delta \bmod 2^{n+2}$ is solvable if and only if $\Delta \equiv 1 \bmod 8$. The solutions to the latter congruence can be denoted X, $X + 2^{n+1}$, $-X$, $-X + 2^{n+1}$. Therefore $2ax + B \equiv \pm X \bmod 2^{n+1}$. This means $x \equiv (\frac{\pm X - B}{2})a^{-1} \bmod 2^n$ has exactly two solutions. Because $x^2 + x \equiv 0 \bmod 2$, we see that one solution is odd and one is even. □

Proposition 9.7. *Let $c = p^{2\alpha}$ with $\alpha \geq 1$. Let χ be a Dirichlet character modulo c, of conductor p^γ ($\gamma \leq 2\alpha$). Suppose $(a, c) = 1$ and $c \nmid b$. Then:*

1. *If p is odd, then $|S_\chi(a, b; c)| \leq 2p^{3\alpha/2}$.*
2. *If $p = 2$, then $|S_\chi(a, b; c)| \leq 4p^{3\alpha/2}$.*

3. *If p is odd and $\gamma \leq 2\alpha - 1$, then $|S_\chi(a,b;c)| \leq 2p^\alpha$. If further (i) $p|b$ or (ii) $p \nmid b$ and ab is not a quadratic residue mod p, then $S_\chi(a,b;c) = 0$.*

4. *If $p = 2$ and $\gamma \leq 2\alpha - 2$, then $|S_\chi(a,b;c)| \leq 2^{\min(\alpha-1,2)} p^\alpha \leq 4p^\alpha$.*

Proof. We apply Lemma 12.2 of [IK] with $f(y) = y$ and $g(y) = \frac{ay^2+b}{y}$, which gives

$$S_\chi(a,b;c) = p^\alpha \sum_{\substack{y \in (\mathbf{Z}/p^\alpha \mathbf{Z})^* \\ h(y) \equiv 0 \bmod p^\alpha}} \overline{\chi(y)} e(\frac{ay + b\overline{y}}{c}), \tag{9.14}$$

the summand being independent of the choice of representative for y, where

$$h(y) = a - by^{-2} + By^{-1} \tag{9.15}$$

for B determined by

$$\overline{\chi(1 + zp^\alpha)} = e(\frac{Bz}{p^\alpha}). \tag{9.16}$$

This immediately gives $|S_\chi(a,b;c)| \leq p^\alpha M$, where M is the number of solutions to

$$ay^2 + By - b \equiv 0 \bmod p^\alpha, \quad (y,p) = 1. \tag{9.17}$$

By Lemma 9.6, $M \leq 2\gcd(2,p)p^{\alpha/2}$. This proves 1 and 2.

Now suppose p is odd and $\gamma \leq 2\alpha - 1$. If $\gamma \leq \alpha$, then $B = 0$ by (9.16). If $\alpha < \gamma \leq 2\alpha - 1$, then taking $z = p^{\gamma-\alpha}$ in (9.16) we have $e(\frac{B}{p^{2\alpha-\gamma}}) = 1$. Hence $\frac{B}{p^{2\alpha-\gamma}} \in \mathbf{Z}$. So we see that $p|B$ whenever $\gamma \leq 2\alpha - 1$. Therefore by Lemma 9.6, (9.17) has no solutions y which are prime to p, unless $p \nmid b$ and $4ab$ (and hence ab) is a quadratic residue modulo p. In the latter case, there are exactly two such solutions, so that $|S_\chi(a,b;c)| \leq 2p^\alpha$. This proves 3.

Next, assume $p = 2$ and $\gamma \leq 2\alpha - 2$. If $\gamma \leq \alpha$, then $B = 0$ by (9.16). If $\alpha < \gamma \leq 2\alpha - 2$, then taking $z = p^{\gamma-\alpha}$ in (9.16) gives $e(\frac{B}{p^{2\alpha-\gamma}}) = 1$. Hence $\frac{B}{p^{2\alpha-\gamma}} \in \mathbf{Z}$, so that $4|B$ whenever $\gamma \leq 2\alpha - 2$. By Lemma 9.6, (9.17) has solutions y only if b is odd (and so $\delta = 2$ in the notation of the lemma). The number of solutions is at most $2^{\min(\alpha-1,2)}$. Assertion 4 follows. □

Proposition 9.8. *Let $c = p^{2\alpha+1}$ with $\alpha \geq 1$. Let χ be a Dirichlet character modulo c, of conductor p^γ ($\gamma \leq 2\alpha + 1$). Suppose $(a,c) = 1$ and $c \nmid b$. Then:*

1. *If p is odd, then $|S_\chi(a,b;c)| \leq 2p^{3\alpha/2+1}$.*

2. *If $p = 2$, then $|S_\chi(a,b;c)| \leq 4p^{3\alpha/2+1}$.*

3. *If p is odd and $\gamma \leq 2\alpha$, then $|S_\chi(a,b;c)| \leq 2p^{\alpha+1/2}$. Furthermore, if (i) $p|b$ or (ii) $p \nmid b$ and ab is a quadratic residue modulo p, then $S_\chi(a,b;c) = 0$.*

4. *If $p = 2$ and $\gamma \leq 2\alpha - 1$, then $|S_\chi(a,b;c)| \leq 2^{\min(3,\alpha)} p^\alpha \leq 8p^\alpha$.*

Proof. We apply Lemma 12.3 of [IK] with $f(y) = y$ and $g(y) = \frac{ay^2+b}{y}$, which gives

$$S_\chi(a, b; c) = p^\alpha \sum_{\substack{y \in (\mathbf{Z}/p^\alpha \mathbf{Z})^*, \\ h(y) \equiv 0 \bmod p^\alpha}} \overline{\chi(y)} e(\frac{ay + b\overline{y}}{c}) G_p(y). \tag{9.18}$$

Here $h(y)$ is given by (9.15) as before, but this time B is defined by

$$\overline{\chi(1 + zp^\alpha)} = e\left(\frac{Bz}{p^{\alpha+1}} + (p-1)\frac{Bz^2}{2p}\right), \tag{9.19}$$

and $G_p(y)$ is the Gauss sum

$$G_p(y) = \sum_{z \bmod p} e(\frac{d(y)z^2 + h(y)p^{-\alpha}z}{p}) \tag{9.20}$$

for

$$d(y) = by^{-3} + (p-1)\frac{B}{2}y^{-2}. \tag{9.21}$$

Because $|G_p(y)| \leq p$, we have

$$|S_\chi(a, b; c)| \leq p^{\alpha+1} M, \tag{9.22}$$

where M is the number of solutions to (9.17). As before, $M \leq 2\gcd(2, p)p^{\alpha/2}$, so 1 and 2 follow.

Suppose p is odd and $\gamma \leq 2\alpha$. If $\gamma \leq \alpha$, then $B = 0$ by (9.19). If $\alpha < \gamma \leq 2\alpha$, then setting $z = p^{\gamma-\alpha}$ in (9.19) gives

$$1 = e\left(\frac{B}{p^{2\alpha+1-\gamma}} + \frac{(p-1)}{2}Bp^{2(\gamma-\alpha)-1}\right) = e\left(\frac{B}{p^{2\alpha+1-\gamma}}\right).$$

Thus $p|B$ whenever $\gamma \leq 2\alpha$. As in the previous proof, the congruence (9.17) has solutions (necessarily two in number) only if $p \nmid b$ and ab is a quadratic residue modulo p. Because $p|B$, $d(y) \equiv by^{-3} \not\equiv 0 \bmod p$. Hence by (12.37) of [IK], $|G_p(y)| = p^{1/2}$. It now follows that $|S_\chi(a, b; c)| \leq 2p^\alpha p^{1/2}$, which proves 3.

Now suppose $p = 2$ and $\gamma \leq 2\alpha - 1$. If $\gamma \leq \alpha$, then then $B = 0$ by (9.19). If $\alpha < \gamma \leq 2\alpha - 1$, then setting $z = p^{\gamma-\alpha}$ in (9.19) gives

$$1 = e\left(\frac{B}{p^{2\alpha+1-\gamma}} + \frac{Bp^{2(\gamma-\alpha)}}{p^2}\right) = e\left(\frac{B}{p^{2\alpha+1-\gamma}}\right).$$

Because $2\alpha+1-\gamma \geq 2$, we see that $4|B$. As in the previous proof, the number of solutions to (9.17) is $M \leq 2^{\min(\alpha-1,2)} \leq 4$. Assertion 4 now follows immediately by (9.22). □

Example 9.9. *Let p be an odd prime, and let χ be a primitive Dirichlet character of modulus p^3. Then there exist $a, b \in (\mathbf{Z}/p^3\mathbf{Z})^*$ such that*

$$S_\chi(a, b; p^3) = p^2.$$

In particular, if $c = p^3$ for $p \geq 17$,

$$|S_\chi(a, b; c)| > \tau(c)(a, b, c)^{1/2}c^{1/2}.$$

Proof. We apply the above proposition with $\alpha = 1$. If, in (9.19), $p|B$, then

$$\overline{\chi}(1+zp) = e(\frac{Bz}{p^2}),$$

which implies that $\overline{\chi}(1+zp^2) = 1$, and hence $\mathfrak{c}_\chi | p^2$. Thus assuming χ is primitive, $p \nmid B$. Take $a = \frac{p-1}{2}B$ and $b = -\frac{p-1}{2}B$, and consider $S_\chi(a, b; p^3)$ for χ primitive. In the notation of the previous proof,

$$h(y) = \tfrac{p-1}{2}B + \tfrac{p-1}{2}By^{-2} + By^{-1} \equiv 0 \bmod p$$

$$\iff y^2 - 2y + 1 \equiv 0 \bmod p \iff y \equiv 1 \bmod p.$$

Therefore since $a + b = 0$, (9.18) gives

$$S_\chi(a, b; p^3) = pG_p(1).$$

In the notation of (9.21), we have

$$d(1) = -\tfrac{p-1}{2}B + \tfrac{p-1}{2}B = 0.$$

Since $h(1) = 0$ as well, we have $G_p(1) = p$. Thus $S_\chi(a, b; p^3) = p^2$. $\square$

Proposition 9.10. *Suppose $c = p^\ell$ and $\mathfrak{c}_\chi = p^\gamma$ for $\gamma \le \ell$. If $(a, c) = 1$ and $c|b$, then* $|S_\chi(a, b; c)| = \begin{cases} p^{\ell/2} & \text{if } \gamma = \ell, \\ 0 & \text{otherwise.} \end{cases}$

Proof. When $c|b$, $S_\chi(a, b; c) = \sum_{d \in (\mathbf{Z}/c\mathbf{Z})^*} \overline{\chi(d)} e(\frac{ad}{c})$ is a Gauss sum. If $\gamma < \ell$, then the Gauss sum vanishes ([Hua], Theorem 7.4.2). If $\gamma = \ell$, then the absolute value of the Gauss sum is $p^{\ell/2}$. $\square$

Corollary 9.11. *Suppose $c = p^\ell$ for $\ell \ge 1$, $\mathfrak{c}_\chi = p^\gamma$ for $\gamma \le \ell$, and $(a, c) = 1$. Then:*

- $|S_\chi(a, b; c)| \le 2\gcd(2, p)^2\, c^{1/2} p^{1/4} \mathfrak{c}_\chi^{1/4}$,
- $|S_\chi(a, b; c)| \le \tau(c) c^{1/2} p^{1/4} \mathfrak{c}_\chi^{1/4}$,
- $|S_\chi(a, b; c)| \le 2\gcd(2, p)^2\, c^{1/2} \mathfrak{c}_\chi^{1/2}$,
- $|S_\chi(a, b; c)| \le \tau(c) c^{1/2} \mathfrak{c}_\chi^{1/2}$.

Proof. This follows directly from what we have proven above. We just need to examine each case. In view of Proposition 9.10, we can assume that $c \nmid b$. First, suppose p is odd and $\ell = 2\alpha$ is even. If $\gamma = 2\alpha$, then by Proposition 9.7 (1),

$$|S_\chi(a, b; c)| \le 2p^\alpha p^{\alpha/2} = 2c^{1/2}\mathfrak{c}_\chi^{1/4}.$$

If $\gamma \leq 2\alpha - 1$, then by Proposition 9.7 (3), we have $|S_\chi(a,b;c)| \leq 2c^{1/2}$. Now consider ℓ odd. If $\ell = 1$, then the bounds hold by Proposition 9.4. Suppose $\ell = 2\alpha+1$ for $\alpha \geq 1$. If $\gamma = 2\alpha+1$, then by Proposition 9.8 (1),

$$|S_\chi(a,b;c)| \leq 2p^{\alpha+\frac{1}{2}}p^{\frac{\alpha}{2}+\frac{1}{2}} = 2p^{1/4}c^{1/2}\mathfrak{c}_\chi^{1/4} \leq 2c^{1/2}\mathfrak{c}_\chi^{1/2}.$$

The last step holds since $\gamma \geq 1$. If $\gamma \leq 2\alpha$, then $|S_\chi(a,b;c)| \leq 2p^{\alpha+\frac{1}{2}} = 2c^{1/2}$ by Proposition 9.8 (3). This establishes the bounds when p is odd.

Now consider the case $p = 2$, and suppose $\ell = 2\alpha$ is even. When $\alpha = 1$, the bounds are trivial because $S_\chi(a,b;c)$ is a sum over $(\mathbf{Z}/4\mathbf{Z})^*$ and is hence bounded by 2. So we can assume that $\alpha > 1$. If $\gamma = 2\alpha$, then by Proposition 9.7 (2),

$$|S_\chi(a,b;c)| \leq 4p^\alpha p^{\alpha/2} = 4c^{1/2}\mathfrak{c}_\chi^{1/4} \leq (2\alpha+1)c^{1/2}\mathfrak{c}_\chi^{1/4} = \tau(c)c^{1/2}\mathfrak{c}_\chi^{1/4}.$$

If $\gamma = 2\alpha - 1$, then

$$|S_\chi(a,b;c)| \leq 4p^\alpha p^{\alpha/2} = 4c^{1/2}p^{(\gamma+1)/4} \leq \tau(c)p^{1/4}c^{1/2}\mathfrak{c}_\chi^{1/4} \leq \tau(c)c^{1/2}\mathfrak{c}_\chi^{1/2}.$$

The last step holds because $\gamma \geq 1$. If $\gamma \leq 2\alpha - 2$, then by Proposition 9.7 (4),

$$|S_\chi(a,b;c)| \leq 4p^\alpha = 4c^{1/2} \leq (2\alpha+1)c^{1/2} = \tau(c)c^{1/2},$$

since $\alpha > 1$. Now consider ℓ odd. If $\ell = 1$, then the bounds are obvious since the summation only has one term. Suppose $\ell = 2\alpha+1$ with $\alpha \geq 1$. If $\gamma = 2\alpha+1$, then by Proposition 9.8 (2),

$$|S_\chi(a,b;c)| \leq 4p^{\alpha+\frac{1}{2}}p^{\frac{\alpha}{2}+\frac{1}{2}} = 4p^{1/4}c^{1/2}\mathfrak{c}_\chi^{1/4} \leq 4c^{1/2}\mathfrak{c}_\chi^{1/2} \leq \tau(c)c^{1/2}\mathfrak{c}_\chi^{1/2},$$

since $\gamma \geq 1$ and $\tau(c) = 2\alpha+2 \geq 4$. If $\gamma = 2\alpha$, then by Proposition 9.8 (2),

$$|S_\chi(a,b;c)| \leq 4p^{\alpha+\frac{1}{2}}p^{\frac{\alpha}{2}+\frac{1}{2}} = 4p^{1/4}c^{1/2}p^{1/4}\mathfrak{c}_\chi^{1/4}.$$

If $\alpha \geq 2$, then $4p^{1/4} \leq 2\alpha+2 = \tau(c)$, and the first two inequalities follow. That the remaining ones also hold follows from $p^{\frac{1}{2}+\frac{\alpha}{2}} \leq p^\alpha = \mathfrak{c}_\chi^{1/2}$. If $\alpha = 1$, then $S_\chi(a,b;c)$ is a sum over $(\mathbf{Z}/8\mathbf{Z})^*$, so it is bounded by 4, and the inequalities clearly hold in this case as well. Finally, if $\gamma \leq 2\alpha - 1$, then by Proposition 9.8 (4),

$$|S_\chi(a,b;c)| \leq 2^{\min(3,\alpha)}p^\alpha \leq (2\alpha+2)p^{\alpha+\frac{1}{2}} = \tau(c)c^{1/2}. \qquad \square$$

Proposition 9.12. *The results of Propositions 9.7, 9.8 and Corollary 9.11 hold if we exchange the roles of a and b.*

Proof. This follows from the fact that $S_\chi(a,b;c) = S_{\overline{\chi}}(b,a;c)$. □

We now have all of the pieces in place to prove Theorem 9.3.

Proof of Theorem 9.3. Suppose $c = p^\ell$, and $\mathfrak{c}_\chi = p^\gamma$ for $\gamma \le \ell$. We need to show first that for any a, b,

$$|S_\chi(a, b; c)| \le \tau(c)\,(a, b, c)^{1/2} c^{1/2} \mathfrak{c}_\chi^{1/2}.$$

If $c|a$ and $c|b$, this is trivial. Suppose $(a, b, c) = p^{a_p}$ for $a_p = \mathrm{ord}_p(a)$, and write $a' = p^{-a_p} a$ and $b' = p^{-a_p} b$. Then by Corollary 9.11,

$$\begin{aligned} |S_\chi(a, b, c)| = p^{a_p} |S_\chi(a', b', p^{\ell - a_p})| &\le \tau(c) p^{a_p} p^{(\ell - a_p)/2} \mathfrak{c}_\chi^{1/2} \\ &= \tau(c)(a, b, c)^{1/2} c^{1/2} \mathfrak{c}_\chi^{1/2}. \end{aligned}$$

If $(a, b, c) = p^{\mathrm{ord}_p(b)}$, the inequality can be proven in the same way after applying Proposition 9.12.

The second assertion, that

$$|S_\chi(a, b; c)| \le \tau(c)\,(a, b, c)^{1/2} c^{1/2} \mathfrak{c}_\chi^{1/4} p^{1/4},$$

follows in the same manner. □

9.2 Factorization

Now we turn our attention back to the generalized Kloosterman sum $S_\chi(a, b; \mathtt{n}; c)$ (9.1), expressing it as a product of local factors. These factors will in turn be expressed in terms of the sums $S_\chi(a, b; c)$ studied in the previous section.

Let χ be any multiplicative function $\mathbf{Z}/c\mathbf{Z} \longrightarrow \mathbf{C}$. Suppose $c = qr$ with $(q, r) = 1$. Then using

$$\mathbf{Z}/c\mathbf{Z} = (\mathbf{Z}/q\mathbf{Z}) \times (\mathbf{Z}/r\mathbf{Z}),$$

we see that χ has a canonical factorization $\chi(x) = \chi_q(x)\chi_r(x)$, where χ_q and χ_r are multiplicative functions on $\mathbf{Z}/q\mathbf{Z}$ and $\mathbf{Z}/r\mathbf{Z}$ respectively. If χ is a Dirichlet character modulo N, viewed as a function on $\mathbf{Z}/c\mathbf{Z}$, and if $(r, N) = 1$, then $\chi_r = \mathbf{1}$ is the constant function 1 on $\mathbf{Z}/r\mathbf{Z}$ (not to be confused with the principal character modulo r).

Proposition 9.13. *Suppose χ is a multiplicative function modulo N, and $q, r \in \mathbf{Z}^+$ with $(q, r) = 1$ and $qr \in N\mathbf{Z}$. Write $\chi(x) = \chi_q(x)\chi_r(x)$ as above. Then*

$$S_\chi(a, b; \mathtt{n}; qr) = S_{\chi_q}(a\overline{r}, b\overline{r}; \mathtt{n}; q) S_{\chi_r}(a\overline{q}, b\overline{q}; \mathtt{n}; r),$$

where $\overline{r}r \equiv 1$ mod q and $\overline{q}q \equiv 1$ mod r.

Proof. By the Chinese reminder theorem, $x = r\overline{r}t + q\overline{q}d$ runs through a complete residue system mod qr when t and d run through complete residue systems mod q and r respectively.

For fixed $x = r\overline{r}t + q\overline{q}d$, an integer x' satisfies $xx' \equiv \mathtt{n}$ mod qr if and only if

$$tx' \equiv \mathtt{n} \bmod q \quad \text{and} \quad dx' \equiv \mathtt{n} \bmod r.$$

Again by the Chinese reminder theorem, the set of all such x' is parametrized by $x' = r\overline{r}t' + q\overline{q}d'$, as t' and d' run through all solutions of $tt' \equiv \mathtt{n} \bmod q$ and $dd' \equiv \mathtt{n} \bmod r$, respectively.

Therefore

$$S_\chi(a, b; \mathtt{n}; qr) = \sum_{\substack{tt' \equiv \mathtt{n} \\ \bmod q}} \sum_{\substack{dd' \equiv \mathtt{n} \\ \bmod r}} \overline{\chi(r\overline{r}t + q\overline{q}d)} e\Big(\frac{a(r\overline{r}t + q\overline{q}d) + b(r\overline{r}t' + q\overline{q}d')}{qr}\Big)$$

$$= \Big(\sum_{\substack{tt' \equiv \mathtt{n} \\ \bmod q}} \overline{\chi_q(t)} e(\frac{a\overline{r}t + b\overline{r}t'}{q}) \Big) \Big(\sum_{\substack{dd' \equiv \mathtt{n} \\ \bmod r}} \overline{\chi_r(d)} e(\frac{a\overline{q}d + b\overline{q}d'}{r}) \Big). \qquad \square$$

For $p|c$, write $c = p^{c_p} c^{(p)}$ and $\mathtt{n} = p^{\mathtt{n}_p} \mathtt{n}^{(p)}$, where $p \nmid c^{(p)} \mathtt{n}^{(p)}$. Then by successive applications of the proposition and (9.3), we obtain the following.

Corollary 9.14. *With notation as above,*

$$S_\chi(a, b; \mathtt{n}; c) = \prod_{p|c} S_{\chi_p}(a\overline{c^{(p)}}, b\overline{c^{(p)}}\mathtt{n}^{(p)}; p^{\mathtt{n}_p}; p^{c_p}). \tag{9.23}$$

If $p|N$, then χ_p is the Dirichlet character mod p^{c_p} defined as in (5.35)*, so that $\chi_p(d) = 0$ if $p|d$. If $p \nmid N$, then $\chi_p = \mathbf{1}$ is the constant function* 1 *on $\mathbf{Z}/p^{c_p}\mathbf{Z}$.*

Each local factor in (9.23) can be expressed in terms of the familiar twisted Kloosterman sums (9.2), as the next proposition shows.

Proposition 9.15. *Fix integers $k \geq 0$ and $\ell \geq 1$, and let χ_p be a Dirichlet character modulo p^ℓ. Then*

$$S_{\chi_p}(a, b; p^k; p^\ell) = S_{\chi_p}(a, bp^k; p^\ell), \tag{9.24}$$

If instead of a Dirichlet character, $\chi_p = \mathbf{1}$ is the constant function 1 *on $\mathbf{Z}/p^\ell\mathbf{Z}$, then when $k < \ell$,*

$$S_{\mathbf{1}}(a, b; p^k; p^\ell) = \begin{cases} p^k \displaystyle\sum_{i=\max(0,k-a_p)}^{\min(b_p,k)} S(\frac{a}{p^{k-i}}, \frac{b}{p^i}; p^{\ell-k}) & \text{if } k \leq a_p + b_p \\ 0 & \text{otherwise,} \end{cases} \tag{9.25}$$

where as usual $a_p = \operatorname{ord}_p(a)$ and $b_p = \operatorname{ord}_p(b)$. For the $k \geq \ell$ case, the sum is evaluated in (9.27) *below. It vanishes unless $\ell \leq a_p + b_p + 1$.*

Proof. The left-hand side of (9.24) is a sum over $xx' = p^k$ in $(\mathbf{Z}/p^\ell\mathbf{Z})$. If $p|x$, then $\chi_p(x) = 0$. Therefore we can take $x \in (\mathbf{Z}/p^\ell\mathbf{Z})^*$, and $x' = \overline{x}p^k$. Eq. (9.24) follows.

For the case $\chi_p = \mathbf{1}$, suppose first that $k < \ell$. Group the sum in $S_{\mathbf{1}}(a, b; p^k; p^\ell)$ according to $i = \operatorname{ord}_p(x) \leq k$. Suppose

$$xx' \equiv p^k \bmod p^\ell. \tag{9.26}$$

Then $x = p^i t$ and $x' = p^{k-i}\bar{t}$ for some $t\bar{t} \equiv 1 \bmod p^{\ell-k}$. For given t, $\bar{t}$, we have solutions $x = p^i(t + p^{\ell-k}d)$ and $x' = p^{k-i}(t' + p^{\ell-k}d')$. As d, d' and t range through $\mathbf{Z}/p^{k-i}\mathbf{Z}$, $\mathbf{Z}/p^i\mathbf{Z}$ and $(\mathbf{Z}/p^{\ell-i}\mathbf{Z})^*$ respectively, x and x' give all incongruent solutions to (9.26) modulo p^ℓ. Thus

$$S_{\mathbf{1}}(a,b;p^k;p^\ell) = \sum_{i=0}^{k} \sum_{t\in(\mathbf{Z}/p^{\ell-k}\mathbf{Z})^*} \sum_{d=1}^{p^{k-i}} \sum_{d'=1}^{p^i} e(\frac{ap^i(t+p^{\ell-k}d)+bp^{k-i}(\bar{t}+p^{\ell-k}d')}{p^\ell})$$

$$= \sum_{i=0}^{k} \sum_{t\in(\mathbf{Z}/p^{\ell-k}\mathbf{Z})^*} e(\frac{ap^i t + bp^{k-i}\bar{t}}{p^\ell}) \sum_{d=1}^{p^{k-i}} e(\frac{ad}{p^{k-i}}) \sum_{d'=1}^{p^i} e(\frac{bd'}{p^i}).$$

The i^{th} summand is non-zero only if $p^{k-i}|a$ and $p^i|b$. In this situation, write $a = p^{k-i}a'$ and $b = p^i b'$. Then the above is

$$= p^k \sum_{\substack{0\le i\le k,\\ k-a_p\le i\le b_p}} \sum_{\substack{t\bar{t}\equiv 1\\ \bmod p^{\ell-k}}} e(\frac{a't+b'\bar{t}}{p^{\ell-k}}) = p^k \sum_{\substack{0\le i\le k,\\ k-a_p\le i\le b_p}} S(a',b';p^{\ell-k}).$$

This proves (9.25).

Now suppose $k \geq \ell$. Then $xx' \equiv 0 \bmod p^\ell$, and we write $x = p^i t$, $x' = p^{\ell-i}t'$, for $t \in (\mathbf{Z}/p^{\ell-i}\mathbf{Z})^*$ and $t' \in \mathbf{Z}/p^i\mathbf{Z}$. Thus

$$S_{\mathbf{1}}(a,b;p^k;p^\ell) = \sum_{i=0}^{\ell} \sum_{t} \sum_{t'} e(\frac{ap^i t + bp^{\ell-i}t'}{p^\ell})$$

$$= \sum_{i=0}^{\ell} \sum_{t\in(\mathbf{Z}/p^{\ell-i}\mathbf{Z})^*} e(\frac{at}{p^{\ell-i}}) \sum_{t'\in\mathbf{Z}/p^i\mathbf{Z}} e(\frac{bt'}{p^i}). \tag{9.27}$$

The sum over t can be evaluated explicitly using

$$\sum_{t\in(\mathbf{Z}/p^r\mathbf{Z})^*} e(\frac{at}{p^r}) = \sum_{t=1}^{p^r} e(\frac{at}{p^r}) - \sum_{t=1}^{p^{r-1}} e(\frac{apt}{p^r}) = \begin{cases} p^r - p^{r-1} & \text{if } 0 < r \le a_p \\ -p^{r-1} & \text{if } r = a_p + 1 \\ 0 & \text{if } r > a_p + 1, \end{cases}$$

and the sum over t' is p^i or 0 according to whether or not $i \le b_p$. In particular, the i^{th} term of (9.27) vanishes unless $\ell - i \le a_p + 1$ and $i \le b_p$, i.e. $\ell - a_p - 1 \le i \le b_p$. Thus the whole expression vanishes unless $\ell \le a_p + b_p + 1$. □

9.3 Proof of Theorem 9.2

We will bound each term of (9.23). Suppose $p|N$. Then by Proposition 9.15,

$$S_{\chi_p}(a\overline{c^{(p)}}, b\overline{c^{(p)}}\mathtt{n}^{(p)}; p^{\mathtt{n}_p}; p^{c_p}) = S_{\chi_p}(a\overline{c^{(p)}}, b\overline{c^{(p)}}\mathtt{n}; p^{c_p}).$$

Applying Theorem 9.3 to the latter sum,

$$\begin{aligned}|S_{\chi_p}(a\overline{c^{(p)}}, b\overline{c^{(p)}}\mathtt{n}^{(p)}; p^{\mathtt{n}_p}; p^{c_p})| &\le \tau(p^{c_p})\,(a, b\mathtt{n}, p^{c_p})^{1/2} p^{c_p/2} \mathfrak{c}_{\chi_p}^{1/2} \\ &\le \tau(p^{c_p})\,(a\mathtt{n}, b\mathtt{n}, p^{c_p})^{1/2} p^{c_p/2} \mathfrak{c}_{\chi_p}^{1/2}. \qquad (9.28)\end{aligned}$$

Now suppose $p|c$ but $p \nmid N$. Then $\chi_p = \mathbf{1}$, and if $\mathtt{n}_p < c_p$, by (9.25) we have

$$S_{\mathbf{1}}(a\overline{c^{(p)}}, b\overline{c^{(p)}}\mathtt{n}^{(p)}; p^{\mathtt{n}_p}; p^{c_p}) = \sum_{i=\max(0,\mathtt{n}_p-a_p)}^{\min(\mathtt{n}_p,b_p)} p^{\mathtt{n}_p} S(\frac{a\overline{c^{(p)}}}{p^{\mathtt{n}_p-i}}, \frac{b\overline{c^{(p)}\mathtt{n}^{(p)}}}{p^i}; p^{c_p-\mathtt{n}_p})$$

Therefore applying the Weil bound $|S(a, b; c)| \le \tau(c)(a, b, c)^{1/2} c^{1/2}$ to each term in the sum, we find (still assuming $\mathtt{n}_p < c_p$)

$$\begin{aligned}|S_{\mathbf{1}}(a\overline{c^{(p)}}, b\overline{c^{(p)}}\mathtt{n}^{(p)}; p^{\mathtt{n}_p}; p^{c_p})| &\le \sum_i \tau(p^{c_p-\mathtt{n}_p}) p^{\mathtt{n}_p} (\frac{a}{p^{\mathtt{n}_p-i}}, \frac{b}{p^i}, p^{c_p-\mathtt{n}_p})^{1/2} p^{c_p/2-\mathtt{n}_p/2} \\ &\le (\mathtt{n}_p+1)(c_p+1)(a\mathtt{n}, b\mathtt{n}, p^{c_p})^{1/2} p^{c_p/2}, \qquad (9.29)\end{aligned}$$

since the sum has at most $(\mathtt{n}_p + 1)$ terms. If $\mathtt{n}_p \ge c_p$, the bound (9.29) also holds, since from (9.27),

$$\begin{aligned}|S_{\mathbf{1}}(a\overline{c^{(p)}}, b\overline{c^{(p)}}\mathtt{n}^{(p)}; p^{\mathtt{n}_p}; p^{c_p})| &\le \sum_{i=0}^{c_p} p^{c_p-i} p^i \\ &\le (\mathtt{n}_p+1) p^{c_p} = \tau(p^{\mathtt{n}_p})(a\mathtt{n}, b\mathtt{n}, p^{c_p})^{1/2} p^{c_p/2}.\end{aligned}$$

Multiplying the local bounds (9.28) and (9.29) together, by (9.23) we have

$$|S_\chi(a, b; \mathtt{n}; c)| \le \tau(\mathtt{n})\tau(c)(a\mathtt{n}, b\mathtt{n}, c)^{1/2} c^{1/2} \mathfrak{c}_\chi^{1/2},$$

which proves the first inequality in Theorem 9.2. The proof of the second inequality is identical, using the second inequality of Theorem 9.3 for (9.28) in the case that $p|\mathfrak{c}_\chi$, and using the classical Weil bound (9.6) in place of (9.28) in the case that $p|N$ but χ_p is principal, i.e. $p \nmid \mathfrak{c}_\chi$.

10 Equidistribution of Hecke eigenvalues

The Hecke eigenvalues attached to cusp forms have many interesting statistical properties. On one hand, there is the "horizontal" Sato-Tate problem of fixing a newform $u(z)$ and determining the distribution of the Hecke eigenvalues at all primes away from the level. If u is non-dihedral, then conjecturally the normalized eigenvalues ν_p^u are equidistributed relative to the Sato-Tate measure

$$d\mu_\infty(x) = \begin{cases} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\,dx & \text{if } -2 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases} \tag{10.1}$$

This problem is very deep, being tied to the analytic properties of the symmetric power L-functions of u. It has now been proven if u is holomorphic of weight $\mathtt{k} \ge 2$ by Barnet-Lamb, Geraghty, Harris, and Taylor, [BLGHT].

Another point of view is the "vertical" problem of fixing the prime p and determining the distribution of the eigenvalues of T_p on a parametric family of cusp forms, as the parameter (level, weight) tends to infinity. This question has been addressed independently by several authors: for Maass forms by Bruggeman [Brug] and Sarnak [Sar2], and for holomorphic forms by Serre [Ser] and Conrey/Duke/Farmer [CDF]. Strikingly, the relevant measure in this case is not the Sato-Tate measure, but the p-adic measure

$$d\mu_p(x) = \frac{p+1}{(p^{1/2}+p^{-1/2})^2 - x^2}\,d\mu_\infty(x).$$

Serre's article discusses many interesting applications of this result. Effective versions have been given by Murty and Sinha ([MS]) and Lau and Wang ([LW]).

In the holomorphic case, one obtains a different vertical result using Petersson's trace formula in which each Hecke eigenvalue has an analytic weight coming from Fourier coefficients and the Petersson norm of the cusp form. When weighted in this way, the eigenvalues of T_p become equidistributed relative to the Sato-Tate measure itself (independent of p), as the level $N \to \infty$ ([LiC], [KL3]).

In this section we treat the case of Maass forms from the latter perspective, using the Kuznetsov trace formula. We will prove that for a fixed prime $p \nmid N$, the eigenvalues of T_p on the Maass eigenforms, when given the weights that arise naturally in the Kuznetsov formula, become equidistributed relative to the Sato-Tate measure as the level goes to infinity. An interesting feature is that the weights depend on the choice of f_∞ (or equivalently, its Selberg transform $h(t)$), while the measure is independent of this choice.

Fix an integer $m > 0$ and a function $h(t)$ as in Theorem 8.1. We will apply the Kuznetsov formula with $m_1 = m_2 = m$. Fix a prime $p \nmid N$ and an exponent $\ell \ge 0$. For a Maass eigenform $u \in \mathcal{F}$, define the normalized Hecke eigenvalue

$$\nu_{p^\ell}^u = \omega'(p)^{\ell/2}\lambda_{p^\ell}(u) \in \mathbf{R}.$$

The value is real because $\omega'(p)^{\ell/2}T_{p^\ell}$ is self-adjoint, and it is bounded in absolute value by a number depending only on p^ℓ (see p. 73). For all $\ell \geq 0$,

$$\nu^u_{p^\ell} = X_\ell(\nu^u_p),$$

where

$$X_\ell(2\cos\theta) = \frac{\sin((\ell+1)\theta)}{\sin\theta} = e^{i\ell\theta} + e^{i(\ell-2)\theta} + \cdots + e^{-i\ell\theta}$$

is the Chebyshev polynomial of degree ℓ (see e.g. Proposition 29.8 of [KL2]).

Now for each $u_j \in \mathcal{F}$, define a weight

$$w_{u_j} = \frac{|a_m(u_j)|^2}{\|u_j\|^2}\frac{h(t_j)}{\cosh(\pi t_j)}, \tag{10.2}$$

where t_j is the spectral parameter of u_j. Note that at this point, w_{u_j} may be a complex number. However, in the equidistribution result below (Theorem 10.2), we shall impose an extra hypothesis to ensure that w_{u_j} is a nonnegative real number for all j.

Proposition 10.1. *With $h(t)$ as in Theorem 8.1, we have*

$$\sum_{u\in\mathcal{F}} X_\ell(\nu^u_p)w_u = \begin{cases} J\psi(N) + O(N^{\frac12+\varepsilon}) & \text{if } \ell = 2\ell' \text{ with } 0 \leq \ell' \leq \operatorname{ord}_p(m) \\ O(N^{\frac12+\varepsilon}) & \text{otherwise} \end{cases}$$

as $N \to \infty$, where

$$J = \frac{1}{\pi^2}\int_{-\infty}^{\infty} h(t)\tanh(\pi t)\,t\,dt = \frac{4}{\pi}V(0) = \frac{4}{\pi}f_\infty(1). \tag{10.3}$$

Here, V and f_∞ are the functions attached to h in (8.15) *and* (8.3) *respectively, and the equalities in* (10.3) *are from* (3.17).

Remark: This demonstrates the existence of cusp forms with nonvanishing m^{th} Fourier coefficient for all sufficiently large N.

Proof. Taking $m_1 = m_2 = m$ and $\mathtt{n} = p^\ell$ in Theorem 7.14, the cuspidal term is

$$\sum_{u\in\mathcal{F}} \lambda_{p^\ell}(u)w_u = \overline{\omega'(p)}^{\ell/2}\sum_u X_\ell(\nu^u_p)w_u, \tag{10.4}$$

the sum converging absolutely. This is equal to the first geometric term

$$T(m,m,p^\ell)\psi(N)\overline{\omega'(p^{\ell/2})}\frac{1}{\pi^2}\int_{\mathbf{R}} h(t)\tanh(\pi t)\,t\,dt = T(m,m,p^\ell)\psi(N)\overline{\omega'(p^{\ell/2})}J$$

plus the remaining geometric terms and minus the continuous term. By Proposition 7.8 and Proposition 7.12, the latter terms are both $O(N^{1/2+\varepsilon})$.[4] It is easy to see that $T(m,m,p^\ell) = 1$ if and only if $\ell = 2\ell'$ for some $0 \leq \ell' \leq \operatorname{ord}_p(m)$. Multiplying through by $\omega'(p)^{\ell/2}$, the result follows. □

[4]These bounds were proven for $h \in PW^{12}(\mathbf{C})^{\text{even}}$, but they hold as well for h as in Theorem 8.1 so long as $A > \frac14$ and $B > 2$, as shown in the proof of Proposition 8.24.

Theorem 10.2. *Fix a prime p and let $m > 0$ be an integer. For each $n = 1, 2, \ldots,$*

- *let N_n be a positive integer coprime to p, such that $\lim_{n\to\infty} N_n = \infty$*
- *let ω_n' be a Dirichlet character modulo N_n*
- *let $\mathcal{F}_n$ be an orthogonal basis for $L_0^2(N_n, \omega_n')$ consisting of Maass eigenforms.*

Define weights w_u as in (10.2)*. Suppose $h(t)$ is chosen as in Theorem 8.1 so that J in* (10.3) *is nonzero, and $h(t_j) \geq 0$ for all spectral parameters t_j. (The latter condition will be discussed afterwards.) For each n, define the multiset*

$$S_n = \{\nu_p^u \,|\, u \in \mathcal{F}_n\}.$$

Then the sequence $\{S_n\}$ is w_u-equidistributed with respect to the measure

$$d\mu(x) = \sum_{\ell'=0}^{\mathrm{ord}_p(m)} X_{2\ell'}(x)\, d\mu_\infty(x), \tag{10.5}$$

where $d\mu_\infty(x)$ is the Sato-Tate measure (10.1)*. This means that for any continuous function f on $\mathbf{R}$, we have*

$$\lim_{n\to\infty} \frac{\sum_{u\in\mathcal{F}_n} f(\nu_p^u) w_u}{\sum_{u\in\mathcal{F}_n} w_u} = \int_{\mathbf{R}} f(x) d\mu(x). \tag{10.6}$$

Remarks: (1) If we choose m so that $p \nmid m$, then $d\mu = d\mu_\infty$ is the Sato-Tate measure itself. In this case, the measure is independent of p, m and h.

(2) The theorem illustrates in particular the fact that the normalized Hecke eigenvalues ν_p^u are dense in the interval $[-2, 2]$. Thus the Ramanujan Conjecture, if true, is optimal. In the other direction, the theorem provides evidence for the conjecture, by virtue of the fact that the measure is supported on $[-2, 2]$. Any counterexamples to the Ramanujan conjecture are sparse enough to be undetectable in (10.6).

Proof. Setting $\ell = 0$ in Proposition 10.1 gives

$$\sum_{u\in\mathcal{F}} w_u = J\psi(N) + o(N). \tag{10.7}$$

In particular, the denominator in (10.6) is nonzero when n is sufficiently large. We may assume that this is the case for all n. By (10.7), for all $\ell \geq 0$ we have

$$\begin{aligned} \lim_{n\to\infty} \frac{\sum_{u\in\mathcal{F}_n} X_\ell(\nu_p^u) w_u}{\sum_{u\in\mathcal{F}_n} w_u} &= \begin{cases} 1 & \text{if } \ell = 2\ell', \text{ with } 0 \leq \ell' \leq \mathrm{ord}_p(m) \\ 0 & \text{otherwise} \end{cases} \\ &= \int_{\mathbf{R}} X_\ell(x) d\mu(x). \end{aligned}$$

The latter equality holds because the polynomials $X_\ell(x)$ are orthonormal with respect to the Sato-Tate measure (see e.g. [KL2], Proposition 29.7). By linearity, (10.6) holds for all polynomials. Let $I \supseteq [-2,2]$ be a compact interval containing ν_p^u for all u. (According to the Ramanujan conjecture, we can take $I = [-2,2]$, but we do not assume this here. See [Ro] Proposition 2.9 for an elementary proof of the existence of I.) As one can show, both sides of (10.6) define continuous linear functionals on $C(I)$, relative to the sup-norm topology. Since the set of polynomials is dense, it follows that (10.6) holds for all continuous functions, as required.

In more detail, suppose f is any continuous function on I. Given $\varepsilon > 0$, let P be a polynomial approximating f to within ε on the interval I. Then for any n,

$$\begin{aligned}\left|\frac{\sum_{u\in\mathcal{F}_n} f(\nu_p^u)w_u}{\sum_{u\in\mathcal{F}_n} w_u} - \int_{\mathbf{R}} f(x)d\mu(x)\right| &\leq \left|\frac{\sum_{u\in\mathcal{F}_n}(f(\nu_p^u)-P(\nu_p^u))w_u}{\sum_{u\in\mathcal{F}_n} w_u}\right| \\ + \left|\frac{\sum_{u\in\mathcal{F}_n} P(\nu_p^u)w_u}{\sum_{u\in\mathcal{F}_n} w_u} - \int_{\mathbf{R}} P(x)d\mu(x)\right| &+ \left|\int_{\mathbf{R}}(P(x)-f(x))d\mu(x)\right| \\ \leq \varepsilon + \left|\frac{\sum_{u\in\mathcal{F}_n} P(\nu_p^u)w_u}{\sum_{u\in\mathcal{F}_n} w_u} - \int_{\mathbf{R}} P(x)d\mu(x)\right| &+ \varepsilon\int_{\mathbf{R}} d\mu(x). \end{aligned} \tag{10.8}$$

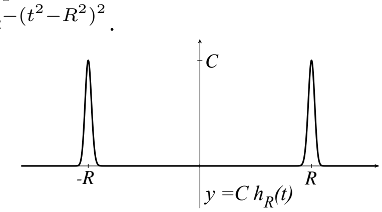
In the first term of (10.8), we have used the fact that $w_u \geq 0$ for all u, which holds because of the hypotheses imposed on h and the fact that $\cosh(\pi t_j) \geq 0$ for all t_j. The latter assertion is clear when $t_j \in \mathbf{R}$ by the definition of cosh. The hypothetical exceptional parameters are of the form $t_j = ix_j$ for $x_j \in (-\frac{1}{2}, \frac{1}{2})$, so that $\cosh(\pi t_j) = \cosh(i\pi x_j) = \cos(\pi x_j) \geq 0$ as well.

As shown in the first part of the proof, the middle term of (10.8) has the limit 0 as $n \to \infty$. Therefore

$$\limsup_{n\to\infty}\left|\frac{\sum_{u\in\mathcal{F}_n} f(\nu_p^u)w_u}{\sum_{u\in\mathcal{F}_n} w_u} - \int_{\mathbf{R}} f(x)d\mu(x)\right| \leq \varepsilon(1+\int_{-2}^{2} d\mu(x)).$$

Letting $\varepsilon \to 0$, we obtain (10.6) as needed. □

In the theorem, we assumed that $h(t_j) \geq 0$ for all spectral parameters t_j. Since $t_j \in \mathbf{R} \cup i(-\frac{1}{2}, \frac{1}{2})$, the condition holds if h is nonnegative on the real and imaginary axes. Examples of allowable h include the Gaussian $h(t) = e^{-t^2}$ and the function $h_R(t) = e^{-(t^2-R^2)^2}$.



The latter detects just those Maass forms with spectral parameter close to $\pm R$. When we apply the theorem to h_R, the fact that the result is independent of R

shows that the equidistribution holds even when we restrict to a small piece of the spectrum.

Other functions h satisfying the hypotheses of the theorem may be constructed as follows. Let h be the Selberg transform of a function $f_\infty = F^* * F$, where $F \in C_c^m(G^+//K_\infty)$ for $m \geq 12$. Let $f^1 : G(\mathbf{A}_{\text{fin}}) \to \mathbf{C}$ be the identity Hecke operator, corresponding to $\mathtt{n} = 1$. Then if u is a Maass cusp form with spectral parameter t, by Proposition 4.8 we have

$$h(t) = \frac{\langle R(f)\varphi_u, \varphi_u \rangle}{\langle \varphi_u, \varphi_u \rangle} = \frac{\big\langle R(F \times f^1)\varphi_u, R(F \times f^1)\varphi_u \big\rangle}{\|u\|^2} \geq 0.$$

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Notation index

Subject index