subject. The theorem was missed by Cantor (who was very anxious to prove it). It was proved independently by Bernstein and Schröder. See pages 449–450 of Cantor's *Collected Works* for an interesting exchange of correspondence concerning it between Cantor and Dedekind.

The theorem that needs to be proved may be stated as follows:

THEOREM 7. *Let A and B be sets such that there exists a one-to-one map of A into B and a one-to-one map of B into A. Then there exists a one-to-one correspondence between A and B.*

An unusually clear and readable proof of Theorem 7 appears on page 340 of the third edition of *A Survey of Modern Algebra* by Birkhoff and Mac Lane. For a reader who would like a brisk but rather sophisticated proof, Exercise 3 is recommended (note that Exercise 15 in Section 1.4 must be done as a prelude). In this account still another proof is offered. It is not fundamentally different from the usual proofs; indeed it is unlikely that anyone will ever devise a truly novel proof. Nor can the proof be recommended for its brevity. It does however have the merit of being a rather quick corollary of material that is worth developing for its own sake. Once it has been decided to work up this theory of infinite cycle decompositions, the procedure flows naturally and needs no artifices.

We shall deduce Theorem 7 from another theorem which is essentially just a variant.

THEOREM 8. *Let f be a one-to-one map of a set A into itself. Let C be a subset of A containing $f(A)$. Then there exists a one-to-one correspondence between A and C.*

We give at once the deduction of Theorem 7 from Theorem 8.

Proof of Theorem 7 from Theorem 8: Let $r:A \rightarrow B$ and $s:B \rightarrow A$ be the given functions, and write $f = sr$. Then f is a one-to-one function of A into itself. Let $C = s(B)$. Then $C \supset f(A)$ and we are in a position to apply Theorem 8, getting a one-to-one correspondence between A and C. Since s provides a one-to-one correspondence between B and C, we achieve the desired one-to-one correspondence between A and B.

Very likely, nearly all readers are familiar with the decomposition of a permutation of a finite set into disjoint cycles. We shall review it briefly.

Let f be a permutation of a finite set A, so that f maps A one-to-one onto itself. Elements $a_1, \ldots, a_r$ in A form a *cycle* under f if $f(a_1) = a_2$, $f(a_2) = a_3, \ldots, f(a_{r-1}) = f(a_r)$, $f(a_r) = a_1$. In words: f sends each of $a_1, \ldots, a_r$ into the next and sends a_r back into a_1. The set A splits into disjoint cycles. To see this, start with an arbitrary a in A and apply f to it repeatedly. The sequence obtained returns to a after a finite number of steps and we have thus constructed a cycle. If this cycle does not exhaust A, start with a new element and treat it in the same fashion. The procedure is continued till the disjoint cycles obtained fill up A.

What we now wish to do is to repeat this discussion with A allowed to be infinite; then we shall broaden the context a trifle further by not insisting that f is onto (while maintaining the assumption that it is one-to-one). This calls for the introduction of infinite cycles.

Our notation for an infinite cycle will be

$$(\ldots a_{-2}\, a_{-1}\, a_0\, a_1\, a_2 \ldots)$$

and the understanding is that the function f sends every element into the next one on the right, so that we have $f(a_i) = a_{i+1}$ for all integers i (positive, zero, and negative). More exactly, this will be called a *bilateral* infinite cycle, to be distinguished from the *unilateral* ones that will shortly be introduced.

Now the following is true in the infinite case just as in the finite case: If f is a one-to-one mapping of a set A onto itself, then A splits under f into disjoint cycles. The method of obtaining the decomposition follows the same lines in the infinite case as in the finite case. Start with any $a \in A$, and apply to it repeatedly both f and f^{-1}. The array that is generated can be exhibited as

$$\ldots, f^{-1}(f^{-1}(a)), f^{-1}(a),\ a, f(a), f(f(a)), \ldots$$

If there is ever a repetition in this array, the whole collection of elements boils down to a finite cycle containing a. Otherwise we obtain a bilateral infinite cycle. By repeating this procedure, we insert every element into a cycle, and different cycles are disjoint.

We hasten to reassure any worried reader that no transfinite procedure is called for here. The decomposition can be done all at once, and in fact the discussion serves as a nice example of an equivalence relation. Introduce the notation f^n to mean the result of composing f with itself n times, and extend it to all integers by having f^0 mean the identity function and $f^{-m} = (f^{-1})^m$. Then the equivalence relation we need is definable as follows: $a \sim b$ if $b = f^n(a)$ for some n. The equivalence classes are precisely the cycles discussed above.

We proceed to the final step. By a *unilateral* infinite cycle

$$(a_1, a_2, a_3, \ldots)$$

we mean a subset where $f(a_i) = a_{i+1}$ for $i = 1, 2, 3, \ldots$ and a_1 is not in the range of f. Now suppose that f is a one-to-one mapping of A into itself that need not be onto. We assert that A splits into disjoint cycles, where the concept has been widened to include unilateral infinite cycles.

The discussion is just a slight variant of what we have already done. Starting with an element $a \in A$, we apply f to it repeatedly. Although f is not onto, we shall (in this discussion) venture to use the symbol f^{-1}; it is defined only on the range of f. Apply f^{-1} to a as long as possible. This may go on forever, or may end in a finite number of steps. If we ever encounter a repetition, a finite cycle containing a will emerge. Otherwise we get an infinite cycle, which may be either bilateral or unilateral. The decomposition is again describable by an equivalence relation.

With all this accomplished we are ready for the proof of Theorem 8.

Proof of Theorem 8: Since Theorem 8 was stated several pages back, we recapitulate it. We are given a set A, a one-to-one map f of A into itself, and a set C lying between A and $f(A)$. We are to devise a one-to-one map of A onto C.

Break A into cycles under f. We shall use the following notation: $A = D \cup E$, where D combines all the finite cycles and bilateral infinite cycles, and E combines all the unilateral infinite cycles. Note that f maps D onto itself. (In fact, D is the largest subset of A carried onto itself by f. See in this connection Exercise 17 in Section 1.4.) In order to discuss E, let us list what might be some of the unilateral infinite cycles:

$$(a_1, a_2, a_3, \ldots)$$
$$(b_1, b_2, b_3, \ldots)$$
$$(c_1, c_2, c_3, \ldots)$$

We can now envisage $f(A)$ very explicitly, for $f(A)$ consists of D together with all of E except the initial elements of the unilateral cycles. In symbols,

$$f(A) = D \cup \{a_2, a_3, \ldots\} \cup \{b_2, b_3, \ldots\} \cup \{c_2, c_3, \ldots\} \cup \ldots$$

while $a_1, b_1, c_1, \ldots$ do not belong to $f(A)$.

What can C look like? The answer is that C adds to $f(A)$ some of the missing initial elements of unilateral cycles. For instance, C might adjoin to A the elements a_1 and c_1, but might not contain b_1. We can now invent the required one-to-one correspondence (say g) between A and C. On D take g to be any one-to-one map of D onto itself; for instance $g = f$ or $g =$ the identity will do. On the unilateral cycles which appear completely in C, take g to be the identity. Finally, on the unilateral cycles which remain incomplete in C, take $g = f$. Then g maps A one-to-one onto C. This proves Theorem 8, and thereby also Theorem 7.

Remark: The decomposition into cycles obtained here is a special case of Exercise 5 in Section 1.5.

We restate Theorem 7.