# The Topology of Spatial Scenes in $\mathbb{R}^2$

Joshua A. Lewis, Matthew P. Dube, and Max J. Egenhofer

School of Computing and Information Science, University of Maine
5711 Boardman Hall, Orono, ME 04469-5711, USA
joshua.lewis@umit.maine.edu, matthew.dube@umit.maine.edu, max@spatial.maine.edu

**Abstract.** Spatial scenes are abstractions of some geographic reality, focusing on the spatial objects identified and their spatial relations. Such qualitative models of space enable spatial querying, computational comparisons for similarity, and the generation of verbal descriptions. A specific strength of spatial scenes is that they offer a focus on particular types of spatial relations. While past approaches to representing spatial scenes, by recording exhaustively all binary spatial relations, capture accurately how pairs of objects are related to each other, they may fail to distinguish certain spatial properties that are enabled by an ensemble of objects. This paper overcomes such limitations by introducing a model that considers (1) the topology of potentially complexly structured spatial objects, (2) modeling applicable relations by their boundary contacts, and (3) considering exterior partitions and exterior relations. Such qualitative scene descriptions have all ingredients to generate topologically correct graphical renderings or verbal scene descriptions.

**Keywords:** Spatial scenes, topological relations, compound objects, ensembles of spatial regions.

## 1 Introduction

Spatial scenes are abstractions of some geographic reality, focusing on the spatial objects identified and their spatial relations [3]. Objects may also be associated with non-spatial properties [31], yielding comprehensive qualitative representations of some mini-world. Such qualitative models of space enable spatial querying, computational comparisons for similarity, and the generation of verbal descriptions. A specific strength of spatial scenes is that they offer a focus on particular types of spatial relations so that users' analyses may put explicit emphasis on properties that may be of particular importance for the task at hand. For instance, matching a sketch, represented as a spatial scene [17], with the closest configurations in a spatial database may ignore metric relations embodied in the sketch, as sketches are often not to scale, while emphasizing relations about relative directions among the objects drawn. This paper focuses on the *topology of spatial scenes* that are embedded in $\mathbb{R}^2$.

Fig. 1 depicts two simple scenes consisting of an ensemble of *spatial regions*. A spatial region is closed, homogenously 2-dimensional without separations or holes [14]. A scene representation may be thought of as a directed graph, in which each node represents a region and each directed edge, connecting two distinct nodes, captures the spatial relations between the two regions.

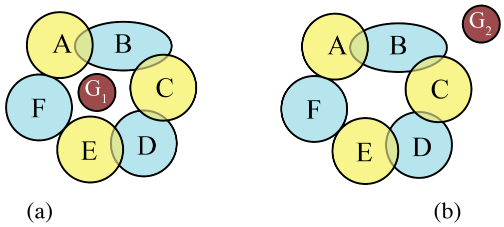



**Fig. 1**: A graphical depiction of a spatial scene in $\mathbb{R}^2$ for which the mere set of binary topological relations is insufficient to distinguish between (a) object $G_1$ being surrounded by the union of A through F and (b) object $G_2$ being outside the union.

Representations of spatial scenes typically resort to capturing exhaustively all binary spatial relations, yielding for a scene of $n$ objects and $m$ types of spatial relations $m*(n^2–n)/2$ relations. In some cases this set of qualitative scene relations may be further reduced by eliminating redundancies that are implied by compositions [18,36] as well as dependencies across relations [20,29,38]. While this approach captures accurately how each object is related to each other object in a spatial scene, it may fail to distinguish certain spatial properties that are enabled by the *ensemble* of objects. The two configurations shown in Fig. 1 share the same pairs of topological relations as modeled by the 4-intersection [14,22] or RCC-8 [34]. While direction relations, such as $G_1$ is south of B, but $G_2$ is north-east of B, would contribute to distinguishing the two configurations, on a purely topological basis the set of binary topological relations would suggest they are topologically equivalent [17,21].

The mere pairwise recording of binary relations may have another shortcoming when multiple objects interact with the same object. The distinction between different types of *meet*, *overlap*, *covers*, and *coveredBy* relations between two regions have been addressed extensively [13,21,26], but these methods may fall short of capturing topological equivalence if multiple, possibly complex boundary-boundary interactions occur. Fig. 2 shows for two configurations how the breakdown of such a complex multi-object relation into all pairs of binary relations is insufficient to distinguish some critical topological information enabled only by the ensemble of the objects.

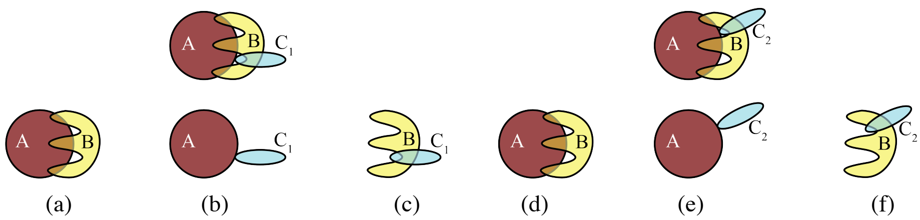



**Fig. 2**: Two configurations that differ by the way C touches A, but their pairs of binary relations show, however, pairwise the same topological relations: (a) and (d) with the same type of *overlap*, (b) and (e) with a simple *meet*, and (c) and (f) with the same type of *overlap*.

This paper develops a formal model that addresses an arbitrarily complex topology of region-based spatial scenes, embedded in $\mathbb{R}^2$. After a summary of related work (Section 2), Section 3 develops a model for complexly structured objects. Section 4 accounts for intersection ambiguities among multiple objects (Fig. 2) by modeling boundary contacts, while Section 5 addresses relations in different exterior partitions. Section 6 demonstrates how both boundary contacts and exterior partitions may need to be used simultaneously for capturing the topology of complex spatial scenes. The paper concludes in Section 7 with a summary and a discussion of future work.

## 2 Topological Relations

Existing methods for modeling spatial relations have been comprehensively compiled in several survey articles [10,11,25]. While early approaches [24] addressed spatial relations in an integrated fashion, the use of tailored methods for different types—topological, direction, and metric—has prevailed during the last decade. Current models for *topological* relations fall primarily into two major categories: (1) those based on connection [34] and (2) those based on intersection [14,15]. For two simple regions, both models yield the same topological relations (Fig. 3).

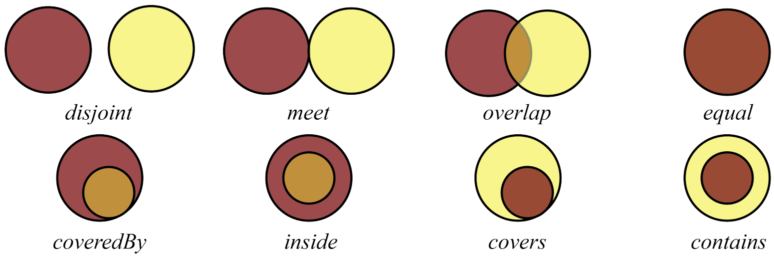



**Fig. 3**: The eight topological relations between two simple regions in $\mathbb{R}^2$ resulting from both the 9-intersection and the region-connection calculus (using the 9-intersection terminology).

Their extensions to capture more details—up to topological equivalence [21]—are particularly relevant for relations with boundary intersections, such as *overlap* [26], *meet*, and *covers*/*coveredBy* [13]. Some alternative models for spatial relations based on interval relations [26], direction [30,39], and 2-D strings [5], have been primarily applied to image retrieval. While RCC as a first-order theory deals with sets of regions, including possible holes, one could assume that this approach would lead directly to capturing the topology of a scene. The idiosyncrasies of the connection-base relations of single regions with multiple parts, however, yield undesired effects, as it captures topologically distinct configurations by the same relation (Fig. 4).


**Fig. 4**: Two scenes with a set of regions, both of which RCC classifies as *partially overlapping* (PO).

## 2.1 Models of Complex Spatial Features

*Complex regions* mainly refer to regions with separations of the interior (multiple components) and of the exterior (holes). Worboys and Bofakos [44] introduced the concept of a *generic area* based on a labeled-tree representation. Areal objects with separations, holes, and islands nested to any finite level are modeled by describing each level with a set of nodes. These nodes are topologically equivalent to the unit disk, pairwise disjoint or with a finite intersection, and may spatially contain their child nodes. The root node represents the entire region object. Clementini *et al.* [8] define *composite regions* as closed, non-empty, 2-dimensional subsets of $\mathbb{R}^2$. Each component is a simple region (with no holes). The components' interiors are pairwise disjoint and their boundaries are either pairwise disjoint or intersect at a finite set of points. A *complex region,* however, has components that may be either simple regions or regions with holes [7].

The OpenGIS Consortium (OGC) has informally described geometric features, called *simple features*, in the OGC Abstract Specification [32] and in the Geography Markup Language GML [33] an XML encoding for the transport and storage of geographic information. Among these features, *MultiPolygons* refer to complex regions that may have multiple components and holes. A complex region [37] comprises one or several regular sets, called *faces* that may have holes and other faces as islands in the holes. A hole within a face can at most touch the boundary of the face or of another hole at a single point. Each face is atomic and cannot be further decomposed. A complex region can comprise multiple faces, which can either be *disjoint*, *meet* at one or many single boundary points, or lie *inside* of the hole of another face and possibly share one or more single boundary points with the hole.

Ontological aspects of parts [12,42] and holes [4] have been studied extensively, with a specific emphasis on part-whole relations as well as mereology—the theory of parthood and relations—and mereotopology [41]—a first-order theory that blends mereological and topological concepts. These reflections and formalizations have been linked mostly to the region-connection calculus [34].

## 2.2 Models for Topological Relations with Holes or Separations

Among the many variations of models for topological relations are six previous approaches that address explicitly some aspects of a qualitative spatial-relation system for holed or separated regions. We briefly summarize their key features and analyze what aspects of holes and separations they cover, including their existing support for similarity and composition reasoning.

- The region-connection calculus RCC-8 [34] specifies eight relations that also apply to regions that may have holes and/or separations. This elegant yet coarse model suffers from a representation shortcoming, since it does not distinguish between significantly different topological configurations as it makes no distinction for holed regions whether an interaction takes place from the outside or the inside. Since the RCC-8 composition table [35] applies to an all-encompassing region, the particularities of holes and parts that constrain inferences at times are not accounted for, so that only an upper bound for the inference possibilities is available at times, which renders some of these composition inferences imprecise, because impossible derivations are included in the set of inferred possibilities.

- The use of the vanilla 9-intersection for complex regions [37] yields 33 relations, some of which expose the same issues as RCC-8. It further considers only holes that are fully contained, but none that are on the fringe (i.e., tangential). No neighborhood graphs or compositions have been derived.
- The Topological-Relations-for-Composite-Regions (TRCR) model [7,8] captures primarily relations of regions with separations, but indicates TRCR's extension to holed regions. TRCE uses a coarse underlying relation model derived from disjunctions of 9-intersection relations. But these disjunctions are not jointly exhaustive and pairwise disjoint, so that, for instance, equal must be derived from A *in* B and B *in* A. Therefore, the set of relations has no explicit identity relation. Here also no neighborhood graphs or compositions have been derived.
- Egenhofer *et al*. [19] treat topological relations of holed regions exhaustively, allowing for tangential holes, holes that fill a region completely, and multiple holes, but at the expense of a tedious enumeration of all details. While they address the transition to hole-free regions through examples, no generic method for determining a conceptual neighborhood graph or the composition of relations is given.
- Vasardani's model for relations with holed regions is highly expressive, identifying 23 relations between a hole-free to a holed region [23] and 152 between two single-holed regions, together with neighborhood graphs, compositions, and a summary model that applies to multi-holed regions [43]. It, however, deals only with fully contained holes and does not address regions with separations.
- Tryfona and Egenhofer [40] treat relations under the transition from regions with separations to generalized regions by filling in the gaps between parts with a (not necessarily convex) hull. These separations exclude touching components and have no account for holes.

## 3 Relation-Based Compound Spatial Objects

In order to capture complexly structured spatial objects, we use the *relation-based compound spatial object model*, which creates compound objects from geometric combinations—unions and set differences—of basic spatial object types that fulfill explicitly specified constraints about topological relations [16]. Although the basic object types may be spatial regions [14], simple lines or points [15], within the scope of this paper only compound spatial objects made up of regions are considered.

The topological-relation constraints between object parts are critical to construct the intended topology of the objects. For two regions $R_1$ and $R_2$, with the constraint $R_1$ *contains* $R_2$, the closure of the set difference $\overline{R_1 \backslash R_2}$ forms a (closed) region with a single hole (Fig. 5a). On the other hand, if $R_3$ *disjoint* $R_4$ then $R_3 \cup R_4$ forms a (closed) separated region (Fig. 5b); the construct $\overline{(R_5 \backslash R_6)} \cup R_7$, with the constraint that $R_5$ *contains* $R_6$ and $R_5$ is *disjoint* from $R_7$, yields a holed region with a separation in the region's exterior (Fig. 5c); while the construct $\overline{(R_8 \backslash R_9)} \cup R_{10}$, with the constraints $R_8$

*contains* $R_9$ and $R_9$ *contains* $R_{10}$, provides a holed region with a separation in the region's hole (Fig. 5d).

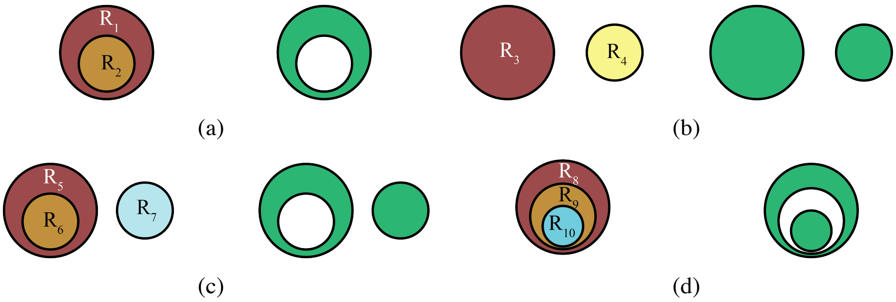



**Fig. 5**: Construction of compound spatial objects: (a) a holed region from $\overline{R_1 \backslash R_2}$ with $R_1$ *contains* $R_2$; (b) a separated region from $R_3 \cup R_4$ with $R_3$ *disjoint* $R_4$; (c) a holed region with an exterior separation from $\overline{(R_5 \backslash R_6)} \cup R_7$ with $R_5$ contains $R_6$ and $R_7$ disjoint $R_5$; and (d) a holed region with a separation in the hole from $\overline{(R_8 \backslash R_9)} \cup R_{10}$ with $R_8$ *contains* $R_9$ and $R_9$ *contains* $R_{10}$.

The advantage of such constructions of compound spatial objects is that the objects' topology enables immediately a specification of the topological relation involving compound spatial objects. There is no need to continue the search for more and more complicated spatial relations for complex objects (e.g., relations between a simple region and a 2-holed region), as the relation-based compound-object model lends itself immediately to specifying topological relations based on the components topological relations. Relations other than *disjoint* and *contains*, such as *meet* or *covers*, may participate in the construction of compound spatial objects as well, but if they yield interior or exterior separations, the methods of partitioning (Section 5) may be necessary.

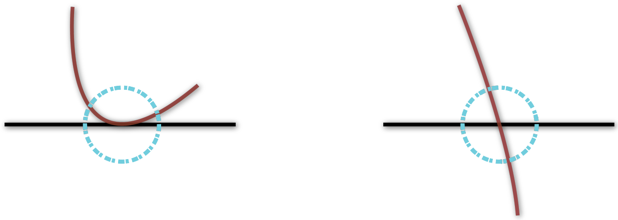
## 4 Boundary Contacts

To fully capture the topology of a spatial scene, one needs to consider the impact of the boundary intersections that the topological relations *meet*, *overlap*, *covers*, and *coveredBy* possess. These boundary intersections are a key ingredient for the reconstruction of a spatial scene from its scene description.

The method chosen to construct this scenario is based on following the boundary components of each object in the scene on a course through a Venn diagram view of the space, in a clockwise direction, combined with the semantic differences of two line-line relations—*touches* and *crosses* [6,21,28]—that are congruent under the 9-intersection.

**Definition 1:** Let *A* and *B* be two lines. *A* and *B* *touch* if the local intersection configuration is such that a consistent clockwise traversal visits the same line twice in a row (Fig. 6a).

**Definition 2:** Let *A* and *B* be two lines. *A* and *B* *cross* if the local intersection configuration is such that a consistent clockwise traversal visits the lines in an alternating sequence (Fig. 6b).



(a) (b)

**Fig. 6**: Line intersections: (a) two lines in a touch configuration, where the order of contact in any cyclic path surrounding the intersection is of the form (black, red, red, black), and (b) two lines in a cross configuration where the order of contact in any cyclic path surrounding the intersection is of the form (black, red, black, red).

**Theorem 1:** Let $A$ and $B$ be two lines such that $A$ *touches* $B$. This configuration cannot be transformed into a single $A$ *crosses* $B$ configuration without first going through another line-line relation with a different 9-intersection matrix.

*Proof:* Both *touches* and *crosses* have the same 9-intersection matrix, but their local intersection structure is different in $\mathbb{R}^2$ because of co-dimension constraints. To get from a single *touch* to a single *cross* without passing through another relation requires fixing the intersection of the two lines topologically, that is, to maintain that, the *intersection zone*—the part of $B°$ originally in common with $A°$—moves only along the interior of the line. If it moves to the boundary of either line, the 9-intersection matrix of the configuration changes. If it moves off of either line in any direction, one of two things must occur: (1) the intersection zone will move to the exterior and cease to be a *touches* or *crosses* configuration, thus going through at least another 9-intersection symbol, or (2) the intersection zone will be moved to the other side of the line, fixing the 9-intersection symbol, but producing two *crosses* configurations in the process. To remove either of these instances would require the boundary of $A$ to enter the interior of $B$ or the boundary of $B$ to enter the interior of $A$, both of which producing a separate 9-intersection matrix. ■

Theorem 1 shows that the line-line relations *touch* and a *cross* (though congruent under the 9-intersection) are different. Herring [28] also demonstrates the differences between these two configurations. To deal with boundaries of regions (bounded by Jordan Curves), the process differentiates occurrences of *touches* from *crosses*. Both *touch* and *cross* can exist in either 0-dimensional or 1-dimensional forms, leaving four possible types of *interactions* between any two boundaries: 0-*touch* (denoted as $t_0$), 0-*cross* (denoted as $c_0$), 1-*touch* (denoted as $t_1$), and 1-*cross* (denoted as $c_1$). Any region's boundary can be partitioned with respect to the other objects in a space based on these four units and a Venn diagram. The development of this method provides a shorthand notation for the process, which we refer to as the *o notation*.

$$\partial A_{comp}: o_S(dimension, T, C) \tag{1}$$

Symbol $A_{comp}$ is a boundary component of a region $A$, $S$ is the collection of regions of which that boundary component is currently *outside* (symbolized by the letter $o$), *dimension* is the qualitative length of the interaction (either 0 or 1), $T$ is the collection of region boundaries that are undergoing a *touch* interaction, and $C$ is the collection of regions undergoing a *cross* interaction. This notation accounts for all but the boundedness of the planar-disk classifying invariant [21], which is addressed in Section 5.

**Theorem 2:** Let $A_1$ and $A_2$ be two regions such that their boundaries *cross* (Fig. 7). Traverse $\partial A_1$ as it approaches $A_2$ from its exterior. The set $C$ must be a subset of $S$.

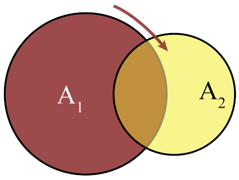



**Fig. 7**: Two regions whose boundaries cross as a $c_0$ intersection, as the cross goes from outside of $A_2$ to inside (symbolized by the arrow) and is of 0-dimension, yielding the $o$ notation $\partial A_1 : o_{\{A_2\}}(0, \emptyset, A_2)$.

*Proof:* Since $\partial A_1$ is in the exterior of $A_2$ as it approaches the intersection, $A_2 \in S$. Since $\partial A_1$ impacts $A_2$, $A_2 \in C$. ■

The same observation in Theorem 2 would also apply to a *touch* configuration upon the sets $T$ and $S$. In the case of *crosses*, the next $o$ notation (Eqn. 1) has its set $S$ become the set $S \backslash C$, as the boundary crossed from the exterior to the interior of the regions in the set $C$. This observation is referred to as the *set difference rule*. In the case of *touches*, there are no corresponding changes in $S$, because the boundary line in the exterior never entered into the interior.

**Theorem 3:** Let $A_1$ and $A_2$ be two regions such that their boundaries *cross* (Fig. 8). Traverse $\partial A_1$ as it approaches $A_2$ from its interior. $C \cap S = \emptyset$.

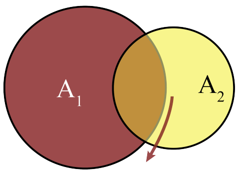



**Fig. 8**: Two regions whose boundaries cross in a $c_0$ intersection, as the cross goes from inside of $A_2$ to outside (symbolized by the arrow), is of 0-dimension, and the position is outside of nothing in the scene yielding the $o$ notation $\partial \mathrm{A}_1 : o_{\{\emptyset\}}(0, \emptyset, \mathrm{A}_2)$.

*Proof:* Since $\partial A_1$ is in the interior of $A_2$ as it approaches the intersection, $A_2 \notin S$. Since $\partial A_1$ impacts $A_2$, $A_2 \in C$. ■

The same observation in Theorem 3 also applies to a *touch* configuration upon the sets $T$ and $S$. In the case of *crosses*, the next $o$ notation (Eqn. 1) has its set $S$ become the set $S \cup C$, as the boundary entered the exterior of those regions in the set $C$. This observation is referred to as the *union rule*. In the case of *touches*, there are no corresponding changes in $S$, because the boundary line in the interior never entered the exterior.

**Theorem 4:** Given the $o$ notation (Eqn. 1) for an arbitrary spatial scene. $T \cap C = \emptyset$.

*Proof:* Since *touch* and *cross* have been shown to be two different configurations (Theorem 1), $T$ and $C$ can have no common elements. ■

There are scenarios in which a spatial scene would produce sets $T$ and/or $C$ such that either or both of these sets would not be subsets of, or pairwise disjoint from, the corresponding set $S$. Let a group of objects constitute a spatial scene such that the set $C$ from the $o$ notation (Eqn. 1) is neither a subset of, nor pairwise disjoint from, $S$. This spatial scene contains an intersection of boundaries such that some go from interior to exterior, while others go from exterior to interior at the same location. The elements that are in common follow the set difference rule, while the elements not in common follow the union rule.

To completely traverse a boundary, the $o$ notation is appended to itself until the entire boundary is traversed. Since the representation starts and ends with the same $o$ notation, the redundant ending is dropped (Fig. 9).

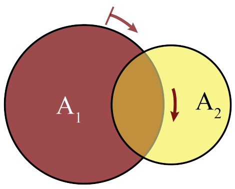



**Fig. 9**: Two regions whose boundaries cross in two $c_0$ intersections, yielding the $o$ notation $\partial A_1: o_{\{A_2\}}(0,\emptyset,A_2)o_{\{\emptyset\}}(0,\emptyset,A_2)$. In this convention, the last encountered state always cycles back to the initial outside condition, with the next impact being the first cross/touch.

We now consider two scenarios where multiple boundaries come together at a single point (Fig. 10a) or with interrupting intersection (Fig. 10b).


(a) (b)

**Fig. 10**: Two scenarios where multiple boundaries come together: (1) Three objects—$A_1$ *overlap* $A_2$, $A_1$ *meet* $A_3$, *and* $A_2$ *meet* $A_3$—to exhibit the boundary notation, and (b) The sequence for $A_3$ is $\partial A_3: o_{\{A_1,A_2\}}(1,\emptyset,A_1)o_{\{A_1,A_2\}}(0,A_2,A_1)\ o_{\{A_1,A_2\}}(1,\emptyset,A_1)o_{\{A_2\}}(0,\emptyset,A_1)$. In this configuration, the change to the subscript based on the cross does not happen until the 1-dimensional cross is finished in its entirety.

For Fig. 10a, we get three sequences (Eqs. 2-4), starting each boundary traversal in the common exterior in a clockwise direction. The union rule and set difference rule apply for the *crosses* configurations, while both rules do not apply for the *touches* configurations.

$$\partial A_1: o_{\{A_2,A_3\}}(0,\emptyset,A_2)o_{\{A_3\}}(0,A_3,A_2)o_{\{A_2,A_3\}}(1,A_3,\emptyset) \quad (2)$$

$$\partial A_2: o_{\{A_1,A_3\}}(1,A_3,\emptyset)o_{\{A_1,A_3\}}(0,A_3,A_1)o_{\{A_3\}}(0,\emptyset,A_1) \quad (3)$$

$$\partial A_3: o_{\{A_1,A_2\}}(1, A_1, \emptyset) o_{\{A_1,A_2\}}(0, \{A_1, A_2\}, \emptyset) o_{\{A_1,A_2\}}(1, A_2, \emptyset) \tag{4}$$

For Fig. 10b, we get three sequences (Eqs. 5-7), starting each boundary traversal in the common exterior in a clockwise direction. This particular instance demonstrates how a shorter intersection that interrupts a longer intersection is handled for the case of *crosses*. The boundary traversal of $A_1$ (Eqn. 5) reveals that the union rule ($S \rightarrow S \cup C$) is not followed until the completion of the *cross*, despite the interruption of the *touch*. Similarly, the boundary traverse of $A_3$ (Eqn. 7) reveals that the set difference rule ($S \rightarrow S \backslash C$) is not followed until the completion of the *cross*, despite the interruption of the *touch*. The notation thus ensures that the *cross* has not been fully completed, distinguishing this scenario from other scenarios where a 0-*touch* and 0-*cross* would coincide between two 1-*cross* impacts. For the scenario of *touches* being interrupted, this cannot be maintained by the mere knowledge of boundary interactions, as it requires the consideration of the partitioning structure of the objects upon the space (Section 5).

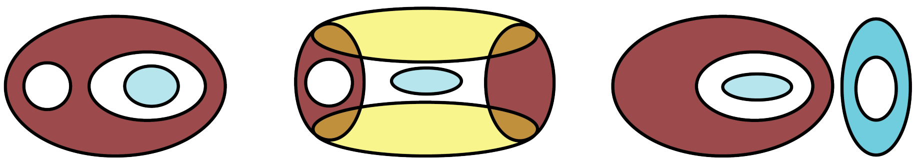
$$\partial A_1: o_{\{A_2,A_3\}}(0, \emptyset, A_3) o_{\{A_2\}}(1, \emptyset, A_3) o_{\{A_2\}}(0, A_2, A_3) o_{\{A_2\}}(1, \emptyset, A_3) \tag{5}$$

$$\partial A_2: o_{\{A_1,A_3\}}(0, \{A_1, A_3\}, \emptyset) \tag{6}$$

$$\partial A_3: o_{\{A_1,A_2\}}(1, \emptyset, A_1) o_{\{A_1,A_2\}}(0, A_2, A_1) o_{\{A_1,A_2\}}(1, \emptyset, A_1) o_{\{A_2\}}(0, \emptyset, A_1) \tag{7}$$

## 5 Exterior Partitions

The model presented thus far for describing a topological scene captures the relations between explicitly defined objects. However, a complete representation requires identifying relationships between objects that lie within separated components of the common exterior. These separated components are classified as *exterior partitions*, and a mechanism is provided for identifying them within a scene. This identification is necessary since existing methods, such as the 9-intersection, capture object-to-object relations, but not the relations between an object and a group of objects (Fig. 11).



(a) (b) (c)

**Fig. 11**: Examples of topologically distinct configurations: (a) a holed region, (b) a union of objects forms a hole that contains other objects, and (c) a holed region with a separation.

Intuitively, the exterior of an object is pictured as resting entirely outside of that object. It is commonly understood that the exterior of a set A within an embedding space $X$ is A's complement; that is, $X \backslash \overline{A}$, or succinctly, $A^-$. In the simplest cases, exteriors are connected, but it need not be so.

An exterior partition is synonymous with a holed region within an object (Fig. 11a) or a hole formed from the union of multiple objects (Fig. 11b). In these instances the common exterior is not *path-connected*. This statement is equivalent to saying that the exterior is separated, or partitioned. Similarly, it can be seen that an object with a satellite (Fig. 11c) also lacks this property of path connectedness. In order to be self-contained we review the basic definitions of continuous, path, and path-connected [1].

**Definition 3:** Let $X$ and $Y$ be topological spaces. A function $f: X \rightarrow Y$ is *continuous* if $f^{-1}(V)$ is open in $X$ for every open set $V \subset Y$.

**Definition 4:** Let $Y$ be a topological space, and let $[0,1] \subset R$ have the standard topology. A *path* in $Y$ is a continuous function $p: [0,1] \rightarrow Y$. The path *begins at* $p(0)$ and *ends at* $p(1)$.

**Definition 5:** Let $A$ be a set in a topological space $X$. A is *path-connected* if for every $a_i, a_j \in A$, there exists a path $p \subset A$ that begins at $a_i$ and ends at $a_j$.

If an object's interior is separated (Fig. 11c), it can safely be assumed that the separated area is surrounded by something else—generally the common exterior, but possibly other objects. In the same sense an exterior partition is subject to a similar surrounded relation—a singular object, or the union of multiple objects, can partition it (Fig. 11b). In the latter instance, the objects collectively surrounding a partition are also path-connected. A multigraph highlights such a collective surrounding (Fig. 12).

Each object depicted in the scene is represented as a vertex. Edges connect objects that intersect through the *meet* or *overlap* relations, with one edge representing each distinct intersection between the objects. Additional relations are not germane to the construction of edges and can be safely disregarded.

In such a graph, objects with explicitly defined holes are represented as self-loops. To generalize, however, unique loops of arbitrary length can possibly define a surrounds relation—encapsulating an exterior partition or another object. By identifying all such loops the search for exterior partitions can begin. In order to define the object topology for any partitioned segment of the exterior—and any object possibly residing within—the exterior partition must be represented explicitly as a set itself.

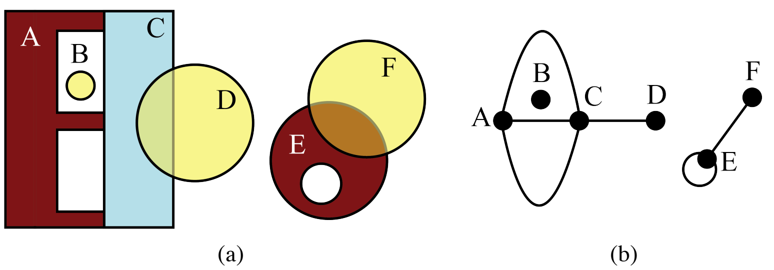
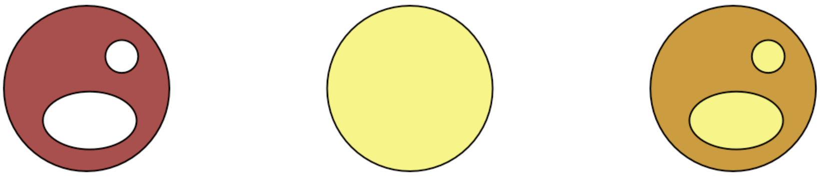



**Fig. 12**: A spatial scene represented (a) graphically and (b) by its corresponding graph.

The first step in this process is to represent a holed object A without the exterior partition; in essence an operation is provided to fill the hole, creating a new object B.

The concept of a region with its holes filled (Fig. 13) was introduced under the term of a *generalized region* [19], and has been redefined with the term *topological hull* [2] for a 3-dimensional digital embedding (i.e., $\mathbb{Z}^3$). To account for $\mathbb{R}^2$, we give a new definition of the topological hull.



(a) (b) (c)

**Fig. 13**: (a) An object in a scene, (b) the object's topological hull, and (c) the object overlaid by its topological hull.

**Definition 6:** Let $A$ be a closed, path-connected set in $\mathbb{R}^2$ with co-dimension 0 within the standard topology, and let $B$ be the smallest closed set homeomorphic to an $n$-disk such that $A \subseteq B$. $B$ is called the *topological hull of* $A$, denoted as $[A]$.

The interior of the originating object, $A$, is a subset of the interior of the topological hull $B$. Importantly, $B^\circ$ covers the region coincident to the interior of the partition if it exists, in addition to $A^\circ$.

**Theorem 5:** If $B$ is the topological hull of $A$, then $A^\circ \subseteq B^\circ$.

*Proof:* Since $A \subseteq B$ and $B$ is closed, $A^\circ \subseteq \bar{B}$. Since $B$ is homeomorphic to an $n$-disk and the standard topology is enforced, $A^\circ \subseteq B^\circ$. ■

In a similar fashion, the boundary of the topological hull is a subset of the originating object's boundary as the originating object also bounds any exterior partitions within itself.

**Theorem 6:** If $B$ is the topological hull of $A$, then $\partial A \supseteq \partial B$.

*Proof:* Since $A$ is closed, and $B$ is the smallest closed n-disk such that $A \subseteq B$, the boundary of $B$ is part of the boundary of $A$. Furthermore, no part of the boundary of $B$ can be outside of the boundary of $A$, however parts of the boundary of $A$ may actually sit within $B^\circ$, therefore the improper subset is allowed. ■

Using such a construction all of the topological hulls of an object $A$ are represented, even if the object has separations, by taking the union of all topological hulls of $A$. The new object is the *aggregate topological hull* of $A$.

**Definition 7:** Let $A$ be a closed set in $\mathbb{R}^2$ with co-dimension 0 within the standard topology. Consider the collection of path-connected subsets $P$ such that $\bigcup_{p\in P} p = A^\circ$ and $|P|$ is minimized. The *aggregate topological hull* $[A]$ is the set $B = \bigcup_{p\in P}[p]$.

Since Definition 6 is a special case of Definition 7 (namely when $A$ has a single path-connected $p$), the same notation is used for the topological hull $[A]$ and the aggregate topological hull $[A]$.

The simplest scenario consists of an object $A$ with no separations. If $A$ is path-connected then the topological hull of $A$ equals the aggregate topological hull of A (Fig. 14a).

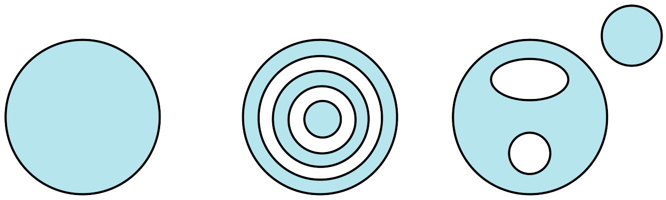



**Fig. 14**: (a) The topological hull of $A$ equals the aggregate topological hull of $A$, (b) all separations in $A$ are nested, and (c) $A$ has unnested separations; the aggregate topological hull is not path-connected.

**Theorem 7:** If $B$ is the topological hull of $A$ and $C$ is the aggregate topological hull of $A$, then $B = C$.

*Proof:* Since $B$ is the topological hull of $A$, $A$ is path-connected. The collection of sets $P$ that minimize the cardinality of the set $P$ has exactly one member, $A$. By definition 5, the aggregate topological hull is constructed by considering the topological hull of each member set. Since $A$ is the only member set, $C = B$, a symmetric relation. Since $B$ is path-connected, $C$ must similarly be path-connected. ■

Exterior partitions and additional components of $A$ can also be nested within $A$ (Fig. 14b), similar to the construction of a Russian *matryoshka* doll. In this analogy, the outermost shell contains a number of smaller shells within itself, each transitively contained within the first. Unlike a *matryoshka* doll, however, multiple divisions can exist at the same level.

**Theorem 8:** If $A$ is a set and $C$ its aggregate topological hull such that all but one of A's separations are nested within another (Fig. 14b), then $C$ is path-connected.

*Proof:* Consider the separation $j$ that is not nested within any other separation. Since $j$ cannot be nested within any other separation and is the only one that cannot be nested, $j$ must surround all of the other separations. Now consider $C_j$ and all immediately nested $C_k$. $C_k \subset C_j$. Any other $C_*$ nested in a $C_k$, follows this relationship by the transitivity of subsets. This procedure exhausts all of separations. ■

The next case is that of an object $A$ with multiple unnested separations (Fig. 14c). In such a case the aggregate topological hull of $A$ is not path-connected.

**Theorem 9:** If $A$ has more than one separation that cannot be nested, $C$ is not path-connected.

*Proof:* By Theorems 7 and 8, every set has at least one part that cannot be nested. Theorem 3 covers a path-connected $A$. Theorem 4 covers a completely nested $A$. This theorem is the final case: multiple unnested. Consider one of the unnested sets $j$ and all other sets nested within it. By Theorem 8, the $C_{j*}$ for this set is path-connected. Consider another one of the unnested sets $k$ and all other sets nested within it. By Theorem 8, the $C_{k*}$ for this set is path-connected. For $C$ to be path-connected would imply that $\exists C_{j*}, C_{k*} \ni C_{j*}^{\circ} \cap C_{k*}^{\circ} = \neg\emptyset$. But $C_j$ and $C_k$ share nothing under definition 5. Therefore, $C$ cannot be path-connected. ■

By utilizing the properties of an aggregate topological hull, exterior separations within an object $A$ can be identified. If the set difference between the aggregate topological hull $C$ and the originating object $A$ is nonempty, the resulting set coincides with the existence of any exterior partitions contained within.

**Theorem 10:** If $C$ is the aggregate topological hull of $A$, $C$ is path-connected, and $C \backslash A = \neg\emptyset$, then $A$ separates $A^{-}$ into separated components.

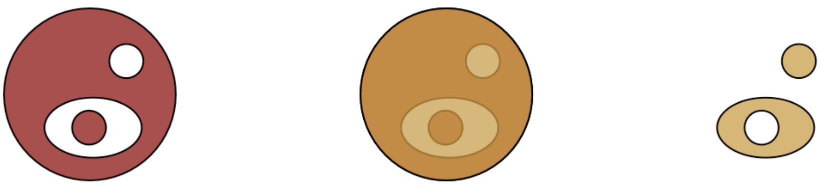
*Proof:* Assume not. There is thus a member of $C$ that is in the aggregate topological hull of $A$ and not in $A$ that is path-connected to every other member of $A^{-}$. By Theorems 7–9, $A$ is either a nested construction or itself path-connected. Consider the surrounding set $j$ in the nested construction or the only set $A$ in the path-connected construction. The boundary of $C_j$ is a Jordan Curve, separating its exterior from its interior. The non-empty intersection suggests that at least one element is not in common with $A$, yet the boundary of $C$ must be contained within the boundary of $A$ (by Theorem 6), and the interior of $A$ must be contained within the interior of $C$ (by Theorem 5). There is thus a point in the interior of $C$ that is not in the closure of $A$, but still bounded by the Jordan Curve. This point is excludable from the exterior, a contradiction of the assumption. ■

Conversely there is an additional case, where $A$ and the aggregate topological hull of $A$ completely coincide. In this instance $A$ does not contain an exterior partition.

**Theorem 11:** If $C$ is the aggregate topological hull of $A$, and $A$=$C$, then $A^{-}$ is path-connected.

*Proof:* Since $C$ and $A$ have the same membership, $A$ is either an $n$-disk, or by Theorem 9, would be a combination of $n$-disks such that none were nested. In this case, the exterior remains path-connected. ■

After proving the existence of one or more partitioned regions of the exterior, it is necessary to represent them explicitly. By Theorem 10, if $C \backslash A = \neg\emptyset$, $A$ partitions some region of the exterior. Closure performed on this set produces an object $H$, which coincides with the partitioned regions of the exterior within $A$ (Fig. 15), and can similarly contain nested separations, that is, satellites that are not path-connected, or have no separations.

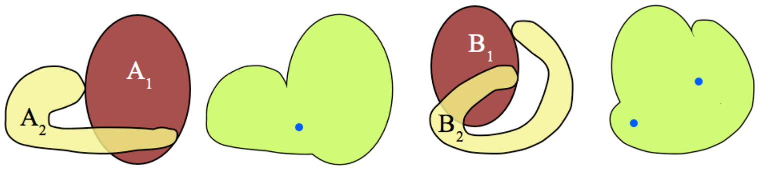


(a) (b) (c)

**Fig. 15**: (a) An object with holes and a separated region, (b) the topological hull overlaid over the object, and (c) the exterior partitions within the object.

It is, therefore, necessary to take the topological hull of $H$, and recursively any resulting objects, until the scene has been exhaustively explored. In this manner the topology between objects, the exterior partitions that they form, and any objects contained within can be sufficiently represented.

Bounded and unbounded boundary components [21], which are not captured explicitly in the *o* notation, are distinguished with the aggregate topological hull. The two scenes in Fig. 16a and 16c have the same *o* notation, but they differ in the number of boundary intersections that are bounded and unbounded. Scene A has one bounded boundary intersection, while scene B has two. Unbounded intersections are in the boundary of the aggregate topological hull, while bounded intersections are in its interior. For bounded intersections, this difference is captured by the intersection of the aggregate topological hull's interior with the intersection of the two regions' boundaries, resulting in this example in sets of cardinality one (Fig. 16b) and two (Fig. 16d).




$[A_1 \cup A_2]^\circ \cap (\partial A_1 \cap \partial A_2)$ $[B_1 \cup B_2]^\circ \cap (\partial B_1 \cap \partial B_2)$

(a) (b) (c) (d)

**Fig. 16**: Two spatial scenes with different bounded and unbounded boundary intersections, which are captured through the aggregate topological hull.

## 6 Integration of *o* Notation and Topological Hull

This section addresses how the combined use of the *o notation* and the *topological hull* enables the modeling of a complex spatial scene. While so far only scenes were considered that require only one of the two in order to determine completely the topology of a spatial scene, we consider here cases that need both simultaneously.

We start with the two scenes, A and B, in Fig. 17 that have the same *o* notation (Eqs. 8a-c for 16a, and replacing B for A to obtain the *o* notation for 16b). The two scenes are topologically different, however, as scene A has one path-connected exterior separation formed between regions $A_1$ and $A_2$, while scene B has two (between $B_1$ and $B_2$).


(a) (b)

**Fig. 17**: Two topologically different spatial scenes with the same boundary intersections.

$$\partial \mathrm{A}_1: o_{\{A_2 A_3\}}(0, \emptyset, \mathrm{A}_3) o_{\{A_2\}}(1, \mathrm{A}_2, \emptyset) o_{\{A_2\}}(0, \mathrm{A}_2, \mathrm{A}_3) o_{\{A_2 A_3\}}(1, \mathrm{A}_2, \emptyset) \quad (8a)$$

$$\partial \mathrm{A}_2: o_{\{A_1 A_3\}}(0, \mathrm{A}_1, \mathrm{A}_3) o_{\{A_1\}}(1, \emptyset, \mathrm{A}_1) o_{\{A_1\}}(0, \emptyset, \mathrm{A}_3) o_{\{A_1 A_3\}}(1, \mathrm{A}_1, \emptyset) \quad (8b)$$

$$\partial \mathrm{A}_3: o_{\{A_1 A_2\}}(0, \emptyset, \mathrm{A}_2) o_{\{A_1\}}(1, \emptyset, \{\mathrm{A}_1, \mathrm{A}_2\}) o_{\{A_2\}}(0, \emptyset, \mathrm{A}_1) \quad (8c)$$

The $o$ notation alone cannot capture the difference for this construction, but the topological hull enables a distinction. Fig. 18 shows for both scenes the step-wise construction of the topological hulls (Fig. 18a and 18d), the unions of the regions (Fig. 18b and 18e), and their holes as the differences (Fig. 18c and 18f).

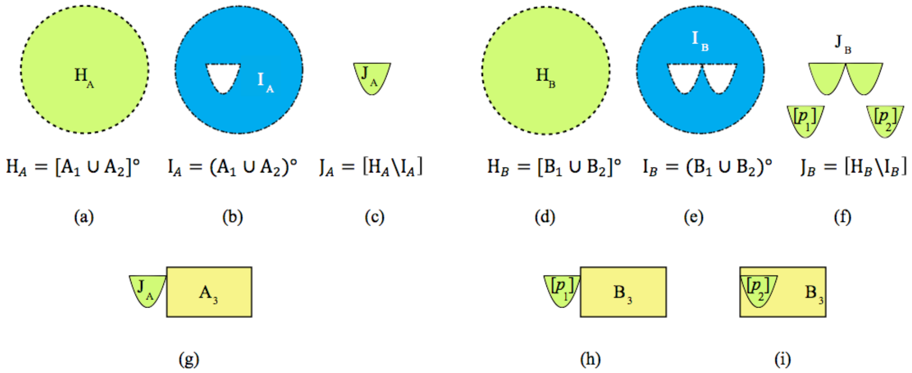



**Fig. 18**: Two complex spatial scenes that share the same boundary intersections, but are topologically different captured by using the aggregate topological hull.

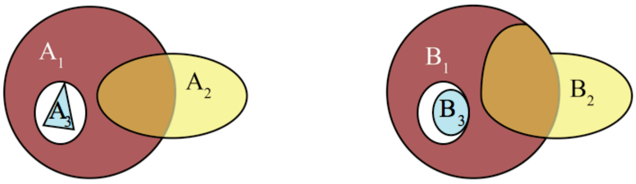
Since scene A has a single, bounded, path-connected exterior component, a single relation captures the exterior partition (Eqn. 9a). On the other hand, scene B has two bounded, path-connected exterior components. Therefore, an exterior partition assessment needs to be applied to both (Eqs. 9b and 9c), revealing the difference between scenes A and B.

$$\mathrm{J}_A \; meet \; \mathrm{A}_3 \quad (9a)$$

$$[p_1] \; meet \; \mathrm{B}_3 \quad (9b)$$

$$[p_2] \; coveredBy \; \mathrm{B}_3 \quad (9c)$$

The second scenario that considers $o$ notation *and* topological hulls features two scenes (Fig. 19) with the same aggregate topological hulls, yet different $o$ notations. In this case, the $o$ notation *per se* is insufficient to capture the scenes' topological differences; therefore, both are needed.




(a) (b)

**Fig. 19**: Two spatial scenes, each consisting of three objects, for which the *o* notation *and* the topological hull are needed to completely distinguish them from each other.

While the aggregate topological hulls $H_A = [A_1 \cup A_2]$ and $H_B = [B_1 \cup B_2]$ *contain*, respectively, $A_3$ and $B_3$, they cannot capture the differences in which $A_1$ and $B_1$ *surround* $A_3$ and $B_3$, respectively—namely in a 0-dimensional or a 1-dimensional boundary intersection. Likewise, the different types of *overlaps* of $A_1$ and $A_2$ and $B_1$ and $B_2$ are also only evident from their *o* notations. To differentiate these two properties, the boundary components need to be analyzed in each scene over the three objects involved, plus their respective holes $A_1H = H_A{}^{\circ} \backslash (A_1 \cup A_2)^{\circ}$ and $B_1H = H_B{}^{\circ} \backslash (B_1 \cup B)^{\circ}$. The differences appear then, for instance, the *o* notations of $A_1$ vs. $B_1$ (Eqs. 10a and 10c) and the *o* notations of $A_3$ and $B_3$ (Eqs. 10b and 10d).

$$\partial A_1: o_{\{A_1H, A_2, A_3\}}(0, \emptyset, A_2) o_{\{A_1H, A_3\}}(0, \emptyset, A_2) \tag{10a}$$

$$\partial A_3: o_{\{A_2\}}(0, A_1H, \emptyset) \tag{10b}$$

$$\partial B_1: o_{\{B_1H, B_2, B_3\}}(1, \emptyset, B_2) o_{\{B_1H, B_3\}}(0, \emptyset, B_2) \tag{10c}$$

$$\partial B_3: o_{\{B_2\}}(1, B_1H, \emptyset) \tag{10d}$$

## 7 Conclusions and Future Work

The topology of a scene as an ensemble of regions has been formalized. To this end, the boundary intersections between objects with the relations *meet*, *overlap*, *covers*, *coveredBy*, and *entwined* were further explored. These boundary intersections take the form *crosses* or *touches*, in either a 0-dimensional or 1-dimensional flavor. By recording these boundary impacts about each object in clockwise orientation the objects of the spatial scene can be related, even where there exist separations or more than two objects. The result of this approach is a symbolic representation of the topological scene, comprised of these intersection strings, called the *o* notation.

Furthermore, an operation is presented for creating the smallest, path-connected 2-disk of an object, in essence filling any holes within the *topological hull*. This enables the capture of relations between an object and a group of objects and also allows for the differentiation of separated regions of the common exterior. Such separations can be holes within individual objects, or holes formed by the union of multiple objects. Similarly, the *aggregate topological hull* enables nested or separated groups of objects to be considered. The set difference between a given topological hull and its originating object(s) captures surrounding or nested objects.

The methods to record boundary contacts and exterior partitions extend the established methods for capturing the topological relation between *two* simple regions [21] to an ensemble of spatial regions, where regions may have holes or separations, and the ensemble may create exterior separations (essentially holes in unions of regions).

These methods for representing a spatial scene can be expanded upon in the future through new lines of inquiry. First, the topological hull operation should be extended to allow for the inclusion of lines, the torus in $\mathbb{R}^3$, and any objects embedded in $\mathbb{R}^n$ generally, possibly relating to the topological hull definition in $\mathbb{Z}^3$ [2]. Neither lines in $\mathbb{R}^2$ nor regions and volumes in $\mathbb{R}^3$ are captured in the current model. The generalization of boundary contacts and exterior partitions for such settings is a challenge. Second, the description of a spatial scene involving objects such as handled or otherwise-spiked regions should be considered. A handled or spiked region fits within the compound-spatial object model by considering not only regions but also lines (e.g., for a region R5 and a line L1, their union $\mathrm{R5} \cup \mathrm{L1}$—when combined with the constraint R5 *doubleMeet* L1 [15]—yields a simple region with a handle, which creates an exterior partition.

Finally, the concepts of qualitative direction and distance could be added to the description of the spatial scene in order to further tighten the descriptive power of a scene by accounting for the continuous transformations objects represented within may undergo while preserving their relatedness to their neighbors, as well as accounting for their relative direction to one another using any number of qualitative methods for capturing such information.

Some may want to consider using first-order logic, in combination with either the 9-intersection relations or the RCC language an alternative approach. But one could also just use first-order predicate calculus altogether to capture the semantics of spatial scenes. Such comparative assessments may or may not have value in the future.

The method *per se*, even only for regions in $\mathbb{R}^2$, features two theoretical and two applied challenges: (1) a proof that the combination of boundary contacts and exterior partitions is fully sufficient to capture completely the topology of an ensemble of regions; (2) a normalization of the boundary components, as the *o* notation repeats for the same boundary interaction the same information along each participating region boundary; (3) the automatic generation of a graphical depiction of the topology of a spatial scene from its scene topology description; and (4) the automatic generation of a verbal description of the topology of a spatial scene from its scene topology description, at various levels of detail.

**Acknowledgments.** This work was partially supported by NSF Grants IIS-1016740 (PI: Max Egenhofer) and DGE-0504494 (PI: Kate Beard). COSIT reviewers provided valuable feedback.

## References


1. Adams, C., Franzosa, R.: *Introduction to Topology: Pure and Applied*. Pearson Prentice Hall (2008)
2. Aktouf, Z., Bertrand, G., Perroton, L.: A Three-Dimensional Holes Closing Algorithm. Pattern Recognition Letters 23(5), 523-531 (2002)
3. Bruns, T., Egenhofer, M.: Similarity of Spatial Scenes. In: Kraak, M., Molenaar, M. (eds.): Seventh International Symposium on Spatial Data Handling, pp. 31–42 (1996)
4. Cassati R., Varzi, A.: Holes and Other Superficialities. MIT Press (1994)
5. Chang, S.K., Shi, Q., Yan, C.: Iconic Indexing by 2-D Strings. IEEE Transactions on Pattern Analysis and Machine Intelligence 9(3), 413-428 (1987)
6. Clementini, E., Di Felice, P.: Topological Invariants for Lines. IEEE Transaction on Knowledge and Data Engingeering 10(1), 38-54 (1998)
7. Clementini, E., Di Felice, P.: A Model for Representing Topological Relationships between Complex Geometric Features in Spatial Databases. Information Sciences 90(1-4), 121-136 (1996)
8. Clementini, E., Di Felice, P., Califano, G.: Composite Regions in Topological Queries. Information Systems 20(7), 579-594 (1995)
9. Cohn, A., Bennett, B., Gooday, J., Gotts, N.: Qualitative Spatial Representation and Reasoning with the Region Connection Calculus. GeoInformatica 1(3), 275-316 (1997)
10. Cohn, A., Hazarika, S.: Qualitative Spatial Representation and Reasoning: An Overview. Fundamenta Informaticae 46(1-2), 2-32 (2001)
11. Cohn, A., Renz, J.: Qualitative Spatial Representation and Reasoning. In: van Hermelen, F., Lifschitz, V., Porter, B. (Eds.) Handbook of Knowledge Representation, pp. 551-596 (2008)
12. Cohn, A., Varzi, A.: Mereotopological Connection. Journal of Philosophical Logic 32(4), 357-390 (2003)
13. Egenhofer, M.: A Model for Detailed Binary Topological Relationships. Geomatica 47(3), 261–273 (1993)
14. Egenhofer, M., Franzosa, R.: Point-Set Topological Spatial Relations. International Journal of Geographical Information Systems 5(2), 161-174 (1991)
15. Egenhofer, M., Herring, J.: Categorizing Binary Topological Relationships Between Regions, Lines, and Points in Geographic Databases, Department of Surveying Engineering, University of Maine, Orono, ME (1991)
16. Egenhofer, M.: A Reference System for Topological Relations between Compound Spatial Objects. In: Heuser, C., Pernul, G. (Eds.) Advances in Conceptual Modeling—Challenging Perspectives, ER 2009 Workshops, Lecture Notes in Computer Science Vol. 5833, pp. 307-316 (2009)
17. Egenhofer, M.: Query Processing in Spatial-Query-by-Sketch. Journal of Visual Languages and Computing 8(4), 403-424 (1997)
18. Egenhofer, M.: Deriving the Composition of Binary Topological Relations. Journal of Visual Languages and Computing 5(2), 133-149 (1994)
19. Egenhofer, M., Clementini, E., Di Felice, P.: Topological Relations between Regions with Holes. International Journal of Geographical Information Systems 8(2), 129-142 (1994)
20. Egenhofer, M., Dube, M.: Topological Relations from Metric Refinements. In: Wolfson, O., Agrawal, D., Lu, C.-T. (Eds.) 17th ACM SIGSPATIAL International Symposium on Advances in Geographic Information Systems, ACM-GIS 2009, pp. 158-167 (2009)
21. Egenhofer, M., Franzosa, R.: On the Equivalence of Topological Relations. International Journal of Geographical Information Systems 9(2), 133-152 (1995)
22. Egenhofer, M., Herring, J.: A Mathematical Framework for the Definition of Topological Relationships. In: Brassel, K., Kishimoto, H. (Eds.) Fourth International Symposium on Spatial Data Handling, pp. 803-813 (1990).

23. Egenhofer, M., Vasardani, M.: Spatial Reasoning with a Hole. In: Winter, S., Duckham, M., Kulik, L., Kuipers, B. (Eds.) Spatial Information Theory–8th International Conference, COSIT 2007, Lecture Notes in Computer Science Vol. 4736, pp. 303-320 (2007)
24. Freeman, J.: The Modeling of Spatial Relations. Computer Graphics and Image Processing 4(2), 156-171 (1975)
25. Galton, A.: Spatial and Temporal Knowledge Representation. Earth Science Informatics 2(3), 169-187 (2009)
26. Galton, A.: Modes of Overlap. Journal of Visual Languages and Computing 9(1), 61-79 (1998)
27. Guesgen, H.: Spatial Reasoning Based on Allen's Temporal Logic. Technical Report, International Computer Science Institute, Berkeley, CA (1989)
28. Herring, J.: The Mathematical Modeling of Spatial and Non-Spatial Information in Geographic Information Systems. In: Mark, D., Frank, A. (Eds.) Cognitive and Linguistic Aspects of Geographic Space, pp. 313-350, Kluwer Academic (1991).
29. Li, S., Cohn, A.: 2009, Reasoning with Topological and Directional Spatial Information. Computational Intelligence 28(4), 579-616 (2012)
30. Nabil, M., Shephard, J., Ngu, A.: 2D Projection Interval Relationships: A Symbolic Representation of Spatial Relationships. In: Egenhofer, M., Herring, J. (Eds.) Fourth International Symposium on Advances in Spatial Databases, Lecture Notes in Computer Science Vol. 951, pp. 292-309 (1995)
31. Nedas, K., Egenhofer, M.: Spatial-Scene Similarity Queries. Transactions in GIS 12(6), 661–681 (2008)
32. OGC: OpenGIS Geography Markup Language (GML) Encoding Standard, http://www.opengeospatial.org/standards/gml (2005)
33. OGC: OGC Abstract Specifications OpenGIS Consortium (OGC), http://www.opengis.org/techno/specs.htm (1999).
34. Randell, D., Cui, Z., Cohn, A.: A Spatial Logic Based on Regions and Connection. In: Third International Conference on Knowledge Representation and Reasoning, pp. 165-176 (1992)
35. Randell, D., Cohn, A., Cui, Z.: Computing Transitivity Tables: A Challenge for Automated Theorem Provers. 11th International Conference on Automated Deduction (CADE-92), pp. 786-790 (1992)
36. Rodríguez, M.A., Egenhofer, M., Blaser, A.: Query Pre-processing of Topological Constraints: Comparing a Composition-Based with Neighborhood-Based Approach. In: Hadzilacos, T., Manolopoulos, Y., Roddick, J., Theodoridis, Y. (Eds.) Advances in Spatial and Temporal Databases, 8th International Symposium, SSTD 2003, Santorini Island, Greece, LNCS, vol. 2750, pp. 362-379
37. Schneider, M., Behr, T.: 2006, Topological Relationships between Complex Spatial Objects. ACM Transactions on Database Systems 31(1): 39-81.
38. Sharma, J.: Integrated Topology- and Direction-Reasoning in GIS. in: Craglia, M., Onsrud, H. (Eds.) Second ESF-NSF Summer Institute in Geographic Information. Taylor & Francis, pp. 435-447 (1999)
39. Sistla, A., Yu, C.: Reasoning about Qualitative Spatial Relationships. Journal of Automated Reasoning 25(4), 291-328 (2000)
40. Tryfona, N., Egenhofer, M.: Consistency Among Parts and Aggregates: A Computational Model. Transactions in GIS 1(3), 189-206 (1996)
41. Varzi, A.: Parts, Wholes, and Part-Whole Relations: The Prospects of Mereotopology. Data & Knowledge Engineering 20(3), 259-286 (1996)
42. Varzi, A.: Spatial Reasoning and Ontology: Parts, Wholes, and Locations. In: Aiello, M., Pratt-Hartmann, I., van Benthem, J. (Eds.) Handbook of Spatial Logics, pp. 945-1038 (2007)

43. Vasardani, M., Egenhofer, M.: Comparing Relations with a Multi-Holed Region. In: Hornsby, K., Claramunt, C., Denis, M., Ligozat, G. (Eds.) Conference on Spatial Information Theory (COSIT'09), Lecture Notes in Computer Science Vol. 5756 pp. 159-176 (2009)
44. Worboys, M., Bofakos, P.: A Canonical Model for a Class of Areal Spatial Objects. In: Abel, D., Ooi, B. (Eds.) Third International Symposium on Advances in Spatial Databases, Lecture Notes in Computer Science Vol. 692, pp. 36-52 (1993)