Spherical Topological Relations

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Abstract. Analysis of global geographic phenomena requires non-planar models. In the past, models for topological relations have focused either on a two-dimensional or a three-dimensional space. When applied to the surface of a sphere, however, neither of the two models suffices. For the two-dimensional planar case, the eight binary topological relations between spatial regions are well known from the 9-intersection model. This paper systematically develops the binary topological relations that can be realized on the surface of a sphere. Between two regions on the sphere there are three binary relations that cannot be realized in the plane. These relations complete the conceptual neighborhood graph of the eight planar topological relations in a regular fashion, providing evidence for a regularity of the underlying mathematical model. The analysis of the algebraic compositions of spherical topological relations indicates that spherical topological reasoning often provides fewer ambiguities than planar topological reasoning. Finally, a comparison with the relations that can be realized for one-dimensional, ordered cycles draws parallels to the spherical topological relations.

1 Introduction

GIS applications that deal with phenomena that spread across the entire globe need semantic models of spatial relations that consider the particular properties of the sphere (Usery 2002). For example, an atmospheric scientist studying global warming needs a spherical geometric representation of the Earth to model accurately the dynamic processes of long-term climate change. Likewise spatio-temporal analyses of the worldwide diffusion of diseases benefit from models based on the sphere. The sphere is a two-dimensional space that is embedded in a three-dimensional setting such that it separates the embedding universe (typically $\mathbb{R}^3$) into two disconnected parts—the interior and the exterior of a globe. Models for qualitative spatial relations, particularly topological relations, have received much attention in the GIS and spatial-database literature over the last decade (Egenhofer and Franzosa 1991; Hadzilacos and Tryfona 1992; Smith and Park 1992; Clementini *et al.* 1993; Cui *et al.* 1993; Clementini *et al.* 1994; Egenhofer *et al.* 1994; Clementini *et al.* 1995; Egenhofer and

Franzosa 1995; Papadias *et al.* 1995; Winter 1995; Cohn and Gotts 1996; Papadimitriou *et al.* 1996; Clementini and di Felice 1997; Billen *et al.* 2002). Implementations in commercial GISs (e.g., Intergraph's MGA and ESRI's SDE) and spatial database systems (e.g., Oracle10*g* Spatial) exist and several standards and drafts of standards have incorporated various versions (e.g., SAIF, ISO TC/211, OGC's Simple Feature Specification). Most of the focus has been on relations in two-dimensional, occasionally three-dimensional space (Pigot 1991; Hazelton *et al.* 1992), but little attention has been paid to investigating models of such qualitative spatial relations on the surface of a sphere.

This paper derives the set of binary topological relations that can be found between two regions on the sphere $\mathbb{P}^2$, with $\mathbb{P} \subset \mathbb{R}$ such that $\mathbb{P}$ is connected and $\min(\mathbb{P}) = \max(\mathbb{P})$. For this purpose, this paper employs the 9-intersection (Egenhofer and Herring 1991) as a model for binary topological relations. It further analyzes the qualitative reasoning power of this set of relations in terms of its conceptual neighborhoods—a measure for the similarity of relations—and its compositions—a foundation for symbolic reasoning in terms of a relation algebra. Two comparisons are made throughout the paper. The first benchmark is the set of topological relations that can be realized in the two-dimensional plane $\mathbb{R}^2$. The second benchmark is the transition from a one-dimensional space $\mathbb{R}^1$, as used for temporal reasoning, to a cyclic one-dimensional space $\mathbb{P}^1$. Within these settings, we are particularly interested in answering the following four questions:

- Does the mapping from $\mathbb{R}^2$ onto $\mathbb{P}^2$ reduce the number of relations found in $\mathbb{R}^2$ but not in $\mathbb{P}^2$?
- Do additional binary topological relations exist in $\mathbb{P}^2$ that cannot be realized in $\mathbb{R}^2$?
- Are the conceptual neighborhoods of all relations in $\mathbb{P}^2$ a consistent theoretical framework for organizing binary spherical topological relations according to their similarity?
- Are inferences based upon the composition of topological relations in $\mathbb{P}^2$ less crisp than in $\mathbb{R}^2$?

The significance of the findings from this investigation is twofold. First, it is of immediate interest for a spatial inference engine to know what types of global spatial relations may be realized on a sphere but cannot be found in a plane. Such knowledge will provide the basis for future spatial query processors that apply to three-dimensional spatial data models or augment early versions, such as the Geodetic DataBlade (IBM 2002), which offers a three-dimensional data model that features only three binary topological relations—*inside*, *intersect*, and *outside*. Second, finding parallels between relations in one-dimensional and two-dimensional spaces—as well as parallels in the transition from linear to cyclic spaces—may give us new insights about the scalability of certain spatial properties. The latter is part of investigations into spatial theories and forms a fundamental aspect of any such formalization in geographic information science.

The remainder of this paper is structured as follows: Section 2 compares similarities and differences between a cyclic one-dimensional and spherical two-dimensional space, followed in Section 3 by a summary of the model for binary topological relations in $\mathbb{R}^2$. Section 4 develops the set of binary topological relations that can be realized on a sphere and compares the results with the relations realized in a cyclic one-dimensional space. Section 5 proves the completeness of this set of spherical topological relations. Section 6 derives systematically the compositions of spherical topological relations and compares the inference power with that of the topological relations in $\mathbb{R}^2$. Conclusions in Section 7 summarize the major findings.

2 Similarities between Cyclic One-Dimensional and Spherical Two-Dimensional Relations

Until recently the embedding space for one-dimensional (temporal) relations has been primarily the linear timeline that corresponds to the real numbers $\mathbb{R}^1$, while the setting that gives rise to cyclic temporal relations (i.e., relations that are embedded in a cyclic, one-dimensional space, denoted by $\mathbb{P}^1$) has been largely ignored. Cyclic one-dimensional relations expose the following properties (Hornsby *et al.* 1999; Balbiani and Osmani 2000):

- one pair of relations that can be realized in $\mathbb{R}^1$ collapses to a single relation in $\mathbb{P}^1$;
- in $\mathbb{P}^1$ additional binary relations exist that cannot be realized in $\mathbb{R}^1$; and
- the conceptual neighborhoods of the relations in $\mathbb{P}^1$ form a framework for a systematic analysis of the completeness of the relations.

We want to verify that similar conclusions can be drawn when the embedding two-dimensional space $\mathbb{R}^2$ gets warped into the surface of a sphere $\mathbb{P}^2$, much like forming a one-dimensional cycle $\mathbb{P}^1$ from a linear, one-dimensional space $\mathbb{R}^1$. Investigations of these comparisons are enabled by the existence of two similar frameworks for organizing such spatial relations in $\mathbb{R}^1$ and $\mathbb{R}^2$. In both cases, the basic sets of relations in $\mathbb{R}^1$ (Allen 1983) and $\mathbb{R}^2$ (Egenhofer and Franzosa 1991) have been widely popular and provide foundations for studies of relations in $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. The analogy between cyclic one-dimensional relations and spherical two-dimensional relations stems from common properties found in both embedding spaces.

One property is that both types of relations are located in a space that is embedded in a higher-dimensional space—at least a two-dimensional plane for cycles and at least a three-dimensional space for spheres. We call such an embedding space the *co-space*. If the *co-dimension*—the difference between the dimension of the co-space and the dimension of the reference object's space—is equal to 1, then the reference space acts as a Jordan curve (or its higher-dimensional equivalents), separating the co-space into two disconnected parts, an inner and an outer co-space. This property holds for the cyclic one-dimensional space as well as for the spherical two-dimensional space. Cyclic one-dimensional relations and spherical two-dimensional relations both attempt to capture qualitative information!(Hernández 1994). Such information typically relies on properties that are invariant under certain types of transformations.

Despite these commonalities, there are some significant differences between a one-dimensional and a two-dimensional embedding, which make it impossible to generalize all findings from the one-dimensional space and apply them to a two-dimensional space. At the outset, the two approaches differ in the way they make use of the order of the space. Whereas the set of one-dimensional relations that disregards the order (Pullar and Egenhofer 1988) typically finds its applications in higher-dimensional spaces (e.g., cartographic applications featuring line relations with co-dimension 1), Allen's interval relations are tailored to representations of time and, therefore, exploit the order of $\mathbb{R}^1$, which is based on an order relation ($\leq$) with the usual algebraic properties of reflexivity, antisymmetry, and transitivity. With the transition from a linearly ordered one-dimensional space to a cyclically ordered one-dimensional space, the orientation is reduced to a less powerful relation that lacks transitivity. This difference in properties has implications on what relations can be distinguished. While A *before* B and A *after* B are two distinct relations in $\mathbb{R}^1$, they blend in $\mathbb{P}^1$ into a single relation, *disjoint* (Hornsby *et al.* 1999). On the other hand, the difference between A *meets* B and A *metBy* B, which is also due to the underlying order relation, is retained in $\mathbb{P}^1$ due to the orientation of the cycle. In $\mathbb{R}^2$ and $\mathbb{P}^2$, however, the setting is different. The orientation of a two-dimensional space has no observable influence on the choice of topological relations—although an enhancement of topological relations with cardinal directions provides an extension that offers additional expressive power (Sharma 1999). Therefore, one could expect that the transition from $\mathbb{R}^2$ and $\mathbb{P}^2$ does not offer the same contraction in a pair of relations as the transition from $\mathbb{R}^1$ to $\mathbb{P}^1$ does.

Another important difference relates to a property of the boundaries of a one-dimensional and a two-dimensional object. In a one-dimensional space the basic object of interest is an interval, which is a non-empty, closed, connected, and proper subset of a one-dimensional space. The boundary of such an interval forms a separation, that is, in order to connect all parts of the boundary it is necessary to traverse the interval's interior or exterior. On the other hand, in a two-dimensional space $\mathbb{R}^2$ the basic object is a *region*—defined as a non-empty proper subset of a connected topological space such that the region's interior is connected and the region is identical to the closure of the region's interior (Egenhofer and Franzosa 1992). It is closed, bounded, homogeneously two-dimensional, and homeomorphic to a 2-disk. Unlike the interval's boundary, a region's boundary is connected, that is, any parts of its boundary can be connected by a line without a need to traverse the region's interior or exterior. This difference between one-dimensional and two-dimensional elements in their corresponding spaces already led to different properties of one pair of topological relations. In 1-D the *overlap* relation has an empty boundary-boundary intersection, while in 2-D the corresponding relation requires the two boundaries to intersect (Egenhofer *et al.* 1993).

These differences indicate that the transition from $\mathbb{R}^1$ to $\mathbb{P}^1$ is not fully parallel to the transition from $\mathbb{R}^2$ to $\mathbb{P}^2$. Still a significant similarity exists between the two scenarios, and we study them subsequently.

3 Topological Relations in $\mathbb{R}^2$

The 9-intersection defines binary topological relations between two regions, A and B, in terms of A's interior (A°), boundary (∂A), and exterior (A^-) with B's interior (B°), boundary (∂B), and exterior (B^-) (Egenhofer and Herring 1991). The nine intersections between these six object parts describe a topological relation and can be concisely represented by a 3×3-matrix, called the *9-intersection* (Equation 1).

$$I_9 = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix} \tag{1}$$

Topological invariants of these nine intersections (i.e., properties that are preserved under topological transformations) are used to categorize topological relations. Examples of topological invariants, applicable to the 9-intersection, are the content (i.e., emptiness or non-emptiness) of a set, the dimension, and the number of separations (Egenhofer and Franzosa 1995). The content invariant is the most general criterion, because other invariants can be considered refinements of non-empty intersections. By considering the values empty ($\varnothing$) and non-empty ($\neg\varnothing$) for each of the nine intersections, one can distinguish $2^9 = 512$ binary topological relations. Eight of these 512 relations can be realized between two regions embedded in $\mathbb{R}^2$. They are subsequently referred to as the $\mathbb{R}^2$-relations. Although the subset of the four intersections of the regions' interiors and boundaries—called the 4-intersection—is sufficient to distinguish the eight $\mathbb{R}^2$-relations, the 9-intersection captures critical information for making inferences about combinations of topological relations (Egenhofer 1994a).

4 Topological Relations on a Sphere

We develop the spherical topological relations in two steps. First, we build on the eight $\mathbb{R}^2$-relations and examine whether they can be realized in $\mathbb{P}^2$ (Section 4.1), before we investigate what relations are particular to $\mathbb{P}^2$ and, therefore, beyond the set of eight $\mathbb{R}^2$-relations (Section 4.2).

The definition of a region in $\mathbb{P}^2$ is identical to that of a region used for the study of topological relations in the plane (Egenhofer and Franzosa 1992) and, therefore, allows direct comparisons. A region has a non-empty interior, a non-empty boundary, and a non-empty exterior, and interior and exterior are simply connected. This definition eliminates some borderline cases of objects that may occur on spheres but are not subject of the present study, such as the entire sphere (because the boundary and the exterior would be empty), a sphere with a crack (because the exterior would be empty), and subsets of $\mathbb{P}^2$ with disconnected exteriors (e.g., regions with holes) and disconnected interiors (e.g., regions with separations).

While the union of two regions in the plane cannot cover the entire embedding space $\mathbb{R}^2$ (Egenhofer and Franzosa 1992), it is possible on the sphere that the union of two regions is identical to $\mathbb{P}^2$. A study of the properties of regions on the sphere (Gotts 1996)—not relations between regions—used the region-connected calculus (Randell *et al.* 1992), a formalism that yields results comparable to those of the 9-intersection.

4.1 Realizability of $\mathbb{R}^2$-Relations in $\mathbb{P}^2$

The first question addresses whether all of the eight $\mathbb{R}^2$-relations can be found in $\mathbb{P}^2$ and if so, whether they can be distinguished uniquely in $\mathbb{P}^2$ as well. A straightforward task is to warp a two-dimensional plane, with two regions on it, so that it forms a sphere. On that sphere we find that all eight region-region relations from $\mathbb{R}^2$ have 1:1 corresponding topological relations (Figure 1). Since the same underlying assumptions of the 9-intersection apply to $\mathbb{R}^2$ and $\mathbb{P}^2$—a connected boundary separates a simply connected exterior from a simply connected interior—the 9-intersection serves as a valid model to distinguish these eight relations in $\mathbb{P}^2$ as well. Therefore, we have found the answer to the initial question about the scalability of cyclic relations:

- While Allen's temporal interval relations, which rely on an order relation, do not scale up immediately from $\mathbb{R}^1$ to $\mathbb{P}^1$—in this process one pair of $\mathbb{R}^1$-relations gets merged into a single $\mathbb{P}^1$-relation (Hornsby *et al.* 1999)—the transition from $\mathbb{R}^2$ to $\mathbb{P}^2$ does not have a similar impact on the topological relations, as it retains all $\mathbb{R}^2$-relations in $\mathbb{P}^2$.

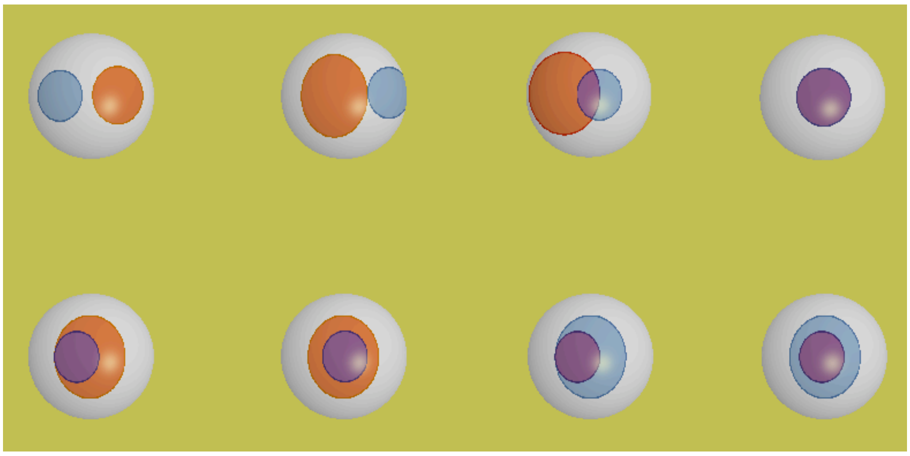


Fig. 1. Examples of the eight topological relations that can be realized in $\mathbb{R}^2$ and in $\mathbb{P}^2$.

4.2 Exclusively Spherical Relations

What binary topological relations does a sphere reveal that $\mathbb{R}^2$ would not permit? To answer this question, we start with the topological relation that occurs when two half-spheres are attached to each other so that their union forms a complete surface of a sphere (Figure 2a). In this case, the two boundaries coincide, while each object's interior coincides with the other object's exterior. The same relation holds for any configuration homeomoprhic to this setting with two half-spheres. We call this relation *attach*. To distinguish *attach* from *meet*, we need to use the 9-intersection, because the difference between the two relations is captured by the way the boundaries lay with respect to the exteriors, which is a property that cannot be captured by the

4-intersection (Equation 2). The *attach* relation cannot be realized between two regions in $\mathbb{R}^2$, because for two regions in the plane the coincidence of the two boundaries would imply a coincidence of the two interiors, which represents the relation *equal*.

$$\begin{pmatrix} \varnothing & \varnothing & \neg\varnothing \\ \varnothing & \neg\varnothing & \varnothing \\ \neg\varnothing & \varnothing & \varnothing \end{pmatrix} \tag{2}$$

Another spherical topological relation occurs if the *attach* relation is deformed such that parts, but not all, of the boundary of each region runs through the interior of the other region (Figure 2b). This relation is called *entwined*. Again the 9-intersection is needed to describe this relation, because the 4-intersection alone cannot distinguish it from *overlaps* (Equation 3). *Entwined* cannot be realized between two regions in $\mathbb{R}^2$, because for two regions in the plane the inclusion of one region's boundary in the other region's closure (such that it intersects with the interior and boundary) would imply the relation *covers*.

$$\begin{pmatrix} \neg\varnothing & \neg\varnothing & \neg\varnothing \\ \neg\varnothing & \neg\varnothing & \varnothing \\ \neg\varnothing & \varnothing & \varnothing \end{pmatrix} \tag{3}$$

The third exclusively spherical relation is one in which each region's boundary is located completely in the interior of the other region's interior, while each region's exterior is located completely in the other region's interior (Figure 2c). This relation is called *embrace*. It is the only $\mathbb{P}^2$-relation that can be distinguished with the 4-intersection from the eight $\mathbb{R}^2$-relations (Equation 4). It is impossible, however, to realize it between two regions in $\mathbb{R}^2$, because for two regions in the plane the complete inclusion of one region's boundary in the other region's interior implies the relation *contains*.

$$\begin{pmatrix} \neg\varnothing & \neg\varnothing & \neg\varnothing \\ \neg\varnothing & \varnothing & \varnothing \\ \neg\varnothing & \varnothing & \varnothing \end{pmatrix} \tag{4}$$

All exclusively spherical relations are such that the union of the two regions forms the entire sphere. This property does not hold for any of the eight topological relations that were projected from $\mathbb{R}^2$ into $\mathbb{P}^2$, nor did it hold for any of the eight region-region relations in $\mathbb{R}^2$. Furthermore, all exclusively spherical relations are symmetric, because their 9-intersection matrices are symmetric with respect to the main diagonal.

With the identification of these three exclusively spherical relations, we have found the answer to the second question about the scalability of cyclic relations:

- Similar to the mapping from $\mathbb{R}^1$ to $\mathbb{P}^1$, the mapping from $\mathbb{R}^2$ to $\mathbb{P}^2$ gives rise to new binary topological relations between two regions that cannot be found between two regions in $\mathbb{R}^2$.

(a) (b) (c)

Fig. 2. Examples of the three topological relations that can be realized only on the sphere $\mathbb{P}^2$: (a) *attach*, (b) *entwined*, and (c) *embrace.*

It is an interesting observation that the three exclusively spherical relations are such that the union of the two regions coincides with $\mathbb{P}^2$. One might argue that these new relations could have been obtained in $\mathbb{R}^2$ as well if one allowed two regions to be such that their union forms $\mathbb{R}^2$. Such an approach, however, would require a modification of the basic definition of a spatial region (Egenhofer and Franzoa 1992) to include non disk-like configurations such as a half plane. In order to stay within the scope of the established setting for geographic applications, such modifications are not desired.

4.3 Completeness of the Set of Topological Relations in $\mathbb{P}^2$

Analogous to the discovery of the topological relations in $\mathbb{R}^2$ (Egenhofer and Herring 1991), we prove the completeness of this set of spherical topological relations by examining what 9-intersection combinations cannot be realized between two regions on the sphere. We capture impossible relations as constraints among the elements of the 9-intersection matrix. Some of these constraints are common to regions in $\mathbb{R}^2$, while others that applied to $\mathbb{R}^2$ do not hold true in $\mathbb{P}^2$.

Ten constraints apply to the interactions between interiors, boundaries, and exteriors for two regions on the sphere (Equations 5a-j). They also apply in the reverse direction, from B to A, by exchanging systematically A and B in Equations 5a-k.

Constraint 1: The two interiors A° and B° cannot be disjoint at the same time as A° is disjoint from the exterior of B (Equation 5a).

$$\nexists\ A,B: \quad A^\circ \cap B^\circ = \varnothing \ \wedge \ A^\circ \cap B^- = \varnothing \tag{5a}$$

Proof: A° and B° must be non-empty (Section 4). Since at least one part of B must be non-empty, it follows that if $A^\circ \cap B^\circ$ is empty, A° must have a non-empty intersection with ∂B or B^-. Assume that $A^\circ \cap B^-$ is empty, then A° would have to be totally included in ∂B, which is impossible. On the other hand, if $A^\circ \cap \partial B$ is empty, then A° would have to be totally included in B^-, that is, $A^\circ \cap B^- = \neg\varnothing$, which contradicts $A^\circ \cap B^- = \varnothing$. ∴

Constraint 2: The two interiors A° and B° cannot be disjoint at the same time as A° intersects with B's boundary (Equation 5b)

$$\nexists\ A,B:\quad A^\circ \cap B^\circ = \varnothing \ \wedge\ A^\circ \cap \partial B = \neg\varnothing \tag{5b}$$

Proof: Detailed proof was included in Egenhofer and Franzosa (1991). ∴

Constraint 3: A's interior A° cannot intersect with B's boundary at the same time as A° is disjoint from B's exterior (Equation 5c).

$$\nexists\ A,B:\quad A^\circ \cap \partial B = \neg\varnothing \ \wedge\ A^\circ \cap B^- = \varnothing \tag{5c}$$

Proof: Follows from proof of Constraint 2. ∴

Constraint 4: A's interior cannot be disjoint from B's exterior B^- at the same time as A's boundary intersects with B^- (Equation 5d).

$$\nexists\ A,B:\quad A^\circ \cap B^- = \varnothing \ \wedge\ \partial A \cap B^- = \neg\varnothing \tag{5d}$$

Proof: Follows from proof of Constraint 2. ∴

Constraint 5: A's interior A° cannot intersect with B's interior at the same time as A° is disjoint from B's boundary and A° intersects with B's exterior (Equation!5e).

$$\nexists\ A,B:\quad A^\circ \cap B^\circ = \neg\varnothing \ \wedge\ A^\circ \cap \partial B = \varnothing \ \wedge\ A^\circ \cap B^- = \neg\varnothing \tag{5e}$$

Proof: The three parts of B—B°, ∂B, and B^-—form a complete partition of space. They are also arranged such that B° is adjacent to ∂B and ∂B is adjacent to B^-. Since ∂B forms a Jordan curve, separating B° from B^-, there is no connection from B° to B^- without going through ∂B. Therefore, if A° has non-empty intersections with B° and with B^-, it must have a non-empty intersection with ∂B as well, which contradicts $A^\circ \cap \partial B = \varnothing$. ∴

Constraint 6: A's boundary cannot intersect with B's exterior B^- at the same time as A's exterior is disjoint from B^- (Equation 5f).

$$\nexists\ A,B:\quad A^- \cap B^- = \varnothing \ \wedge\ \partial A \cap B^- = \neg\varnothing \tag{5f}$$

Proof: Analog to proof of constraint 2, replacing A^- and B^- with A° and B°, respectively. ∴

Constraint 7: A's boundary ∂A cannot be disjoint from B's interior at the same time as ∂A is disjoint from B's boundary and ∂A is disjoint from B's exterior (Equation!5g).

$$\nexists\ A,B:\quad \partial A \cap B^\circ = \varnothing \ \wedge\ \partial A \cap \partial B = \varnothing \ \wedge\ \partial A \cap B^- = \varnothing \tag{5g}$$

Proof: The three parts of B—B°, ∂B, and B^-—form a complete partition of space. Also, ∂A must be non-empty. Therefore, ∂A must have a non-empty intersection with at least one part of B. ∴

Constraint 8: A's boundary ∂A cannot intersect with B's interior at the same time as ∂A is disjoint from B's boundary and ∂A intersects with B's exterior (Equation 5h).

$$\nexists\ A,B:\ \ \partial A \cap B^{\circ} = \neg\varnothing\ \wedge\ \partial A \cap \partial B = \varnothing\ \wedge\ \partial A \cap B^{-} = \neg\varnothing \tag{5h}$$

Proof: Follows from proof of constraint 5. ∴

Constraint 9: A's exterior A^{-} cannot intersect with B's interior at the same time as A^{-} is disjoint from B's boundary and A^{-} is disjoint from B's exterior (Equation!5i).

$$\nexists\ A,B:\ \ A^{-} \cap B^{\circ} = \neg\varnothing\ \wedge\ A^{-} \cap \partial B = \varnothing\ \wedge\ A^{-} \cap B^{-} = \neg\varnothing \tag{5i}$$

Proof: Follows from proof of constraint 5. ∴

Constraint 10: A's exterior A^{-} cannot be disjoint from B's interior at the same time as A^{-} is disjoint from B's boundary and A^{-} is disjoint from B's exterior (Equation!5j).

$$\nexists\ A,B:\ \ A^{-} \cap B^{\circ} = \varnothing\ \wedge\ A^{-} \cap \partial B = \varnothing\ \wedge\ A^{-} \cap B^{-} = \varnothing \tag{5j}$$

Proof: Follows from proof of constraint 7. ∴

With the help of a Prolog program we determined the set of 9-intersections that do not violate any of these constraints. The resulting set consists of the 9-intersections of the eleven spherical topological relations determined in Sections 4.2 and 4.3. The ten constraints are not redundant, as could be demonstrated by an attempt to remove each constraint from the set of ten and recalculate the set of possible relations. For all possible combinations of selecting only nine out of ten constraints, the resulting set of 9-intersection combinations was larger than the set obtained by using all ten constraints. A different—possibly smaller, but equivalent—set of constraints might be found in the future, but it would not change the purpose or the confirmation of this set's completeness.

5 Similarity Among Topological Relations in $\mathbb{P}^2$

Conceptual neighborhoods have been used successfully in the analysis of sets of relations for similarity (Egenhofer and Al-Taha 1992; Freksa 1992; Egenhofer and Mark 1995). The conceptual neighborhood graph captures for each relation those relations that are conceptually closest to it. Two relations are neighbors if a continuous transformation can be performed between the two relations without the need to go through a third relation. Since relations to be related typically lack a total order, their conceptual neighborhoods are used as the primary tool to provide insights about the closeness or similarity of the relations (Bruns and Egenhofer 1996). They also provide a foundation for the selection of appropriate natural-language terminology when people communicate with information systems (Mark and Egenhofer 1994).

The conceptual neighborhood for the eight topological relations in $\mathbb{R}^2$, denoted by N_8, forms a connected graph in which pairs of relations that are connected directly by an edge correspond to transitions that can be obtained by applying topological transformations—translation, rotation, or scaling—to one or both objects. On the other hand, pairs of relations that are not directly connected cannot be obtained through such topological transformations. Further connections—from *inside* to *equal* and from *equal* to *contains*—could be established by considering a scaling that changes boundaries at all points simultaneiously. Likewise, for objects with the same size, shape, and orientation a direct transition from *overlap* to *equal* could be established. Such additional links, however, would not change the overall layout and properties of the conceptual neighborhood graph. N_8 has a vertical symmetry axis that coincides with all symmetric relations and the mirror images along this axis correspond to pairs of converse relations.

The conceptual neighborhood for the eleven topological relations in $\mathbb{P}^2$, denoted by N_{11}, can be derived with the same rationale as N_8 (Egenhofer and Al-Taha 1992). For each pair of spherical topological relations, r_a and r_b, the number of differences in the 9-intersection at corresponding intersections, denoted by $I_{r_a}[i,j]$ and $I_{r_b}[i,j]$, provides a metric for the topological difference of the relations (Equation 6), where the difference is 0 between two empty elements, 0 between two non-empty elements, and 1 between an empty and a non-empty element, as well as between a non-empty and an empty element. Therefore, the sum over all nine interior-, boundary-, and exterior-intersections, denoted by $\tau(r_a, r_b)$, is a cumulative, equal-weight difference value.

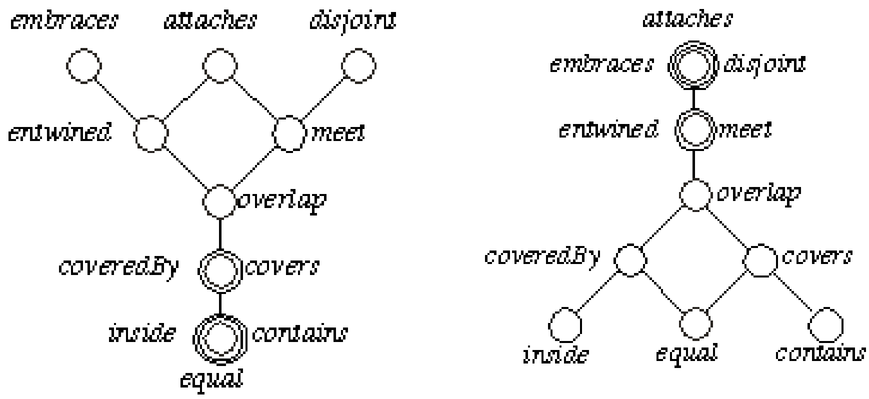
$$\tau(r_a, r_b) = \sum_{i=\circ}^{-} \sum_{j=\circ}^{-} (I_{r_a}[i,j] - I_{r_b}[i,j]) \tag{6}$$

The conceptual neighbors of a relation r_a comprise the set of those relations r_x with the smallest non-zero difference $\tau(r_a, r_x)$ (Table 1). This constraint is not necessarily symmetric, because a relation r_b may be found to be among the least different relations from r_a without the requirement that r_a is among the least different relations from r_b. Since the conceptual neighborhood graph is a non-directed graph, these differences are not captured in N_{11}.

The conceptual neighborhood graph for the eleven topological relations that can be realized on the sphere shows how the three spherical relations fan off from the relations *meet* and *overlap* in the upper half of the graph (Figure 3a). There is no connection to any of the three spherical relations in the lower half of the graph. The six relations located in the upper half of the graph, denoted by N_{11}^+, are symmetric. This property differs from the six relations in the lower half of the graph (Figure 3b), denoted by N_{11}^-, where the vertical axis forms a symmetry axis and corresponding relations form pairs of converse relations (A *inside* B $\Leftrightarrow$ B *contains* A and A *covers* B $\Leftrightarrow$ B *coveredBy* A). Elements that are located on the symmetry axis are symmetric. *Overlap* is part of N_{11}^+ and N_{11}^-; therefore, it fulfills the properties of both sets of relations. *Overlap* is also referred to as the *center element* of N_{11}. These properties would not change if one considers the additional connections that apply to identical region sizes or isotropic scalings that change boundaries at all points simultaneously. To account for such transitions, the neighborhood graph would add a vertical link from *equal* through *overlap* to *attaches* (for two identical half spheres) and two horizontal links—one from *inside* through *equal* to *contains*, and another one from *embraces* through *attaches* to *disjoint* (for isotropic scalings).

	$\tau(r_a, r_b)$	d	m	o	cb	cv	i	ct	e	a	en	em
d	$\begin{pmatrix} \varnothing & \varnothing & \neg\varnothing \\ \varnothing & \varnothing & \neg\varnothing \\ \neg\varnothing & \neg\varnothing & \neg\varnothing \end{pmatrix}$	0	**1**	4	5	5	6	6	6	4	7	6
m	$\begin{pmatrix} \varnothing & \varnothing & \neg\varnothing \\ \varnothing & \neg\varnothing & \neg\varnothing \\ \neg\varnothing & \neg\varnothing & \neg\varnothing \end{pmatrix}$	**1**	0	3	4	4	5	5	5	3	6	7
o	$\begin{pmatrix} \neg\varnothing & \neg\varnothing & \neg\varnothing \\ \neg\varnothing & \neg\varnothing & \neg\varnothing \\ \neg\varnothing & \neg\varnothing & \neg\varnothing \end{pmatrix}$	4	**3**	0	**3**	**3**	4	4	6	6	**3**	4
cb	$\begin{pmatrix} \neg\varnothing & \varnothing & \varnothing \\ \neg\varnothing & \neg\varnothing & \varnothing \\ \neg\varnothing & \neg\varnothing & \neg\varnothing \end{pmatrix}$	5	4	3	0	5	**1**	6	3	5	4	5
cv	$\begin{pmatrix} \neg\varnothing & \neg\varnothing & \neg\varnothing \\ \varnothing & \neg\varnothing & \neg\varnothing \\ \varnothing & \varnothing & \neg\varnothing \end{pmatrix}$	5	4	3	5	0	7	**1**	3	5	4	5
i	$\begin{pmatrix} \neg\varnothing & \varnothing & \varnothing \\ \neg\varnothing & \varnothing & \varnothing \\ \neg\varnothing & \neg\varnothing & \neg\varnothing \end{pmatrix}$	6	5	4	**1**	7	0	6	4	6	5	4
ct	$\begin{pmatrix} \neg\varnothing & \neg\varnothing & \neg\varnothing \\ \varnothing & \varnothing & \neg\varnothing \\ \varnothing & \varnothing & \neg\varnothing \end{pmatrix}$	6	5	4	6	**1**	6	0	4	6	5	4
e	$\begin{pmatrix} \neg\varnothing & \varnothing & \varnothing \\ \varnothing & \neg\varnothing & \varnothing \\ \varnothing & \varnothing & \neg\varnothing \end{pmatrix}$	6	5	6	**3**	**3**	4	4	0	4	5	6
a	$\begin{pmatrix} \varnothing & \varnothing & \neg\varnothing \\ \varnothing & \neg\varnothing & \varnothing \\ \neg\varnothing & \varnothing & \varnothing \end{pmatrix}$	4	**3**	6	5	5	6	6	4	0	**3**	4
en	$\begin{pmatrix} \neg\varnothing & \neg\varnothing & \neg\varnothing \\ \neg\varnothing & \neg\varnothing & \varnothing \\ \neg\varnothing & \varnothing & \varnothing \end{pmatrix}$	7	6	3	4	4	5	5	5	3	0	**1**
em	$\begin{pmatrix} \neg\varnothing & \neg\varnothing & \neg\varnothing \\ \neg\varnothing & \varnothing & \varnothing \\ \neg\varnothing & \varnothing & \varnothing \end{pmatrix}$	6	7	4	5	5	4	4	6	4	**1**	0

Table 1. The topological distance (Egenhofer and Al-Taha 1992) between the eleven topological relations in $\mathbb{P}^2$. Highlighted is the shortest distance of the paths from the target relation (vertical) to the reference relation (horizontal), defining conceptual neighbors.



(a) (b)

Fig. 3. Two orthogonal views of the conceptual neighborhood graph of the eleven spherical topological relations highlighting (a) the upper half and (b) the lower half.

The following interpretations in terms of gradual movements can be made about the conceptual neighborhoods of the three exclusively spherical relations:

- Starting with the topological relation *meet*, where the boundaries partially intersect ($\partial A \cap \partial B = \neg\varnothing$ and $\partial A \cap B^- = \neg\varnothing$ and $A^- \cap \partial B = \neg\varnothing$) such that the two objects do not share any interior ($A^\circ \cap B^\circ = \varnothing$). If the two objects are gradually transformed such that they share more and more of their boundaries, without having their interiors intersect, then the relation *meet* will change into *attach* the moment the two boundaries coincide ($\partial A \cap \partial B = \neg\varnothing$ and $\partial A \cap B^- = \varnothing$ and $A^- \cap \partial B = \varnothing$); therefore, *meet* and *attach* are conceptual neighbors.
- Starting with the topological relation *attach* ($\partial A \cap \partial B = \neg\varnothing$ and $\partial A \cap B^\circ = \varnothing$ and $A^\circ \cap \partial B = \varnothing$). By pushing part of the boundary of one object into the interior of the other ($\partial A \cap \partial B = \neg\varnothing$ and $\partial A \cap B^\circ = \neg\varnothing$ and $A^\circ \cap \partial B = \neg\varnothing$), the two objects are *entwined*. A similar transition is possible from *overlap* to *entwined* by moving the entire part of the boundary that is located in the other object's exterior ($\partial A \cap B^- = \neg\varnothing$, $A^- \cap \partial B = \neg\varnothing$, and $\partial A \cap \partial B = \neg\varnothing$) from the exterior into the boundary ($\partial A \cap B^- = \varnothing$, $A^- \cap \partial B = \varnothing$, and $\partial A \cap \partial B = \neg\varnothing$) while maintaining a non-empty interior-interior intersection ($A^\circ \cap B^\circ = \neg\varnothing$). Since both of these transformations can be performed without the need of going through a third relation, *entwined* is a neighbor of both *attach* and *overlap*.
- Starting with the topological relation *entwined*, where part of the boundary is located in the other object's interior ($\partial A \cap B^\circ = \neg\varnothing$ and $A^\circ \cap \partial B = \neg\varnothing$), the remainder intersects with the other object's boundary ($\partial A \cap \partial B = \neg\varnothing$, $\partial A \cap B^- = \varnothing$, and $A^- \cap \partial B = \varnothing$). If the part of the boundary that intersects with the other object's boundary is moved completely into the other object's interior such that all of A's boundary is located in B's interior ($\partial A \cap B^\circ = \neg\varnothing$, $A^\circ \cap \partial B = \neg\varnothing$, and $\partial A \cap \partial B = \varnothing$), then A *embraces* B.

For display reasons, we employ a flattened graph, in which N_{11}^{+} has been rotated by 90° such that all eleven relations fall into the same plane (Figure 4). This diagram also highlights the role of *overlap* as the center element of N_{11}. Relations are *at the same level* if they are located in the same part of the neighborhood graph (i.e., the upper half or the lower half) and if they have the same shortest path length from *overlap*. For instance, *inside*, *equal*, and *contains* are at the same level, because they are all in the lower half and the length of their shortest paths from *overlap* is 2.

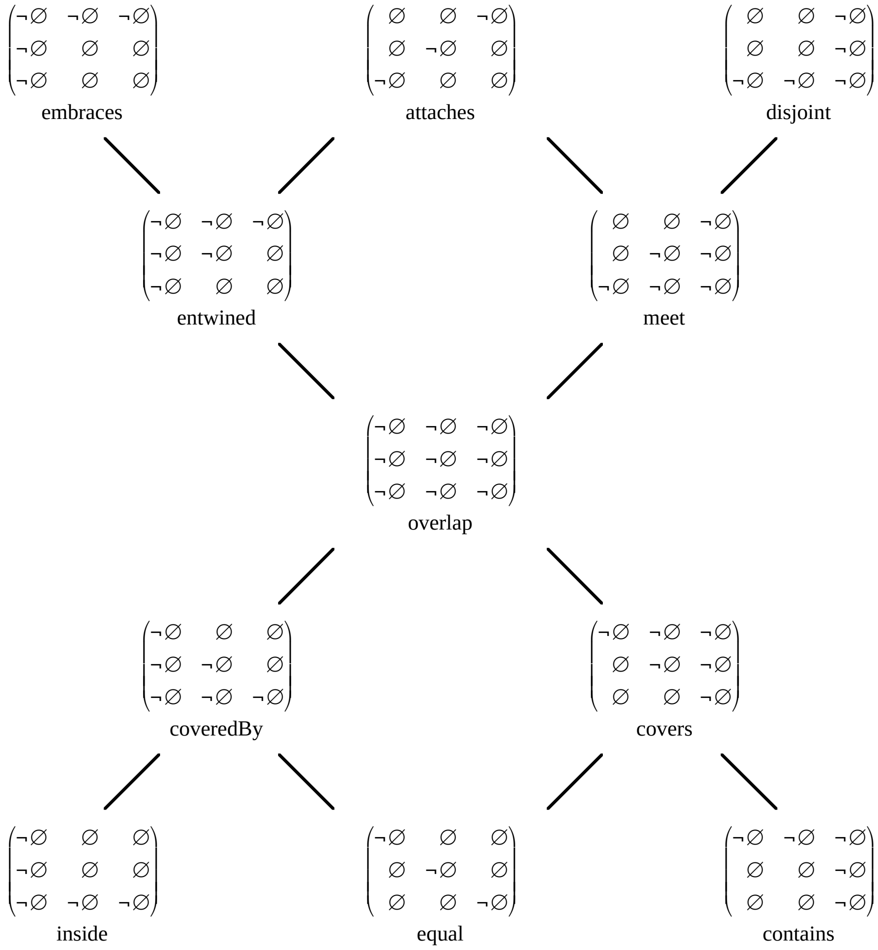


Fig. 4. The flattened conceptual neighborhood graph of the spherical topological relations.

Considering the 9-intersection matrices within the organization of the conceptual neighborhood graph, common properties of a partially ordered set (Birkhoff 1967) are found:

- The least upper bound of any two topological relations at the same level is the intersection of the two relations' 9-intersection matrices.
- The greatest lower bound of any two topological relations at the same level is the union of the two relations' 9-intersection matrices.

The neighborhood graph also shows other regularities about the distribution of the elements in the 9-intersection matrices *I* (Equation 7).

$$I = \begin{pmatrix} i_{00} & i_{10} & i_{20} \\ i_{01} & i_{11} & i_{21} \\ i_{02} & i_{12} & i_{22} \end{pmatrix} \tag{7}$$

- Under the transposition along the horizontal axis through N_{11}'s center element, corresponding 9-intersection matrices, denoted by I^{T-}, are *horizontal* mirror images of each other (Equation 8).

$$\forall I \in N_{11}: \quad I^{T-} = \begin{pmatrix} i_{02} & i_{12} & i_{22} \\ i_{01} & i_{11} & i_{21} \\ i_{00} & i_{10} & i_{20} \end{pmatrix} \tag{8}$$

- Under the transposition along the vertical axis through N_{11}'s center element, corresponding 9-intersection matrices, denoted by $I^{T|}$, are mirror images along the *minor diagonal* (from top right to bottom left) for all intersections in N_{11}^{+} (Equation 9a). The same property applies to all mirror images along the *main diagonal* (from top left to bottom right) for all intersections in N_{11}^{-} (Equation 9b).

$$\forall I \in N_{11}^{+}: \quad I^{T|} = \begin{pmatrix} i_{22} & i_{21} & i_{20} \\ i_{12} & i_{11} & i_{10} \\ i_{02} & i_{01} & i_{00} \end{pmatrix} \tag{9a}$$

$$\forall I \in N_{11}^{-}: \quad I^{T|} = \begin{pmatrix} i_{00} & i_{01} & i_{02} \\ i_{10} & i_{11} & i_{12} \\ i_{20} & i_{21} & i_{22} \end{pmatrix} \tag{9b}$$

- Under the transposition along N_{11}'s main diagonal, corresponding 9-intersection matrices, denoted by $I^{T/}$, are *vertical* mirror images of each other (Equation 10).

$$\forall I \in N_{11}: \quad I^{T/} = \begin{pmatrix} i_{20} & i_{10} & i_{00} \\ i_{21} & i_{11} & i_{21} \\ i_{22} & i_{12} & i_{02} \end{pmatrix} \tag{10}$$

- Finally, under the transposition along N_{11}'s minor diagonal, corresponding 9-intersection matrices, denoted by $I^{T\backslash}$, are also *vertical* mirror images of each other (Equation 11).

$$\forall I \in N_{11}: \quad I^{T\backslash} = \begin{pmatrix} i_{20} & i_{10} & i_{00} \\ i_{21} & i_{11} & i_{21} \\ i_{22} & i_{12} & i_{02} \end{pmatrix} \tag{11}$$

With these insights about the conceptual neighborhood graph of the topological relations in $\mathbb{P}^2$ we can answer the third question.

- The conceptual neighborhoods of all relations in $\mathbb{P}^2$ provide a consistent and regular framework for organizing the binary topological relations according to their similarity.

6 Inferences about Topological Relations in $\mathbb{P}^2$

The relations derived in the previous sections allow us to process topological queries on the sphere in a consistent fashion, but these relations *per se* do not allow us to perform any higher-level inferences about combinations of the relations. Such combinations are of interest if a query response cannot be derived directly from the stored base relations (Egenhofer and Sharma 1993). They are also relevant to assess whether a more complex query of conjunctions of such relations can produce a result at all or whether it is internally inconsistent (Egenhofer 1994b). The latter is also useful for assessing formally whether two or more independently collected sets of spatial descriptions conform or whether they contradict each other.

6.1 Single-Relation Inferences in $\mathbb{P}^2$

Some basic inferences over single relations can be made simply based on the properties of the conceptual neighborhood graph N_{11} and the relations' 9-intersection matrices. Among the eleven spherical relations we find two pairs of converse relations (Equations 12a-b), while each of the remaining seven relations is symmetric (Equation 12c-i).

$$\textit{inside}\,(\mathrm{A},\mathrm{B}) \Leftrightarrow \textit{contains}\,(\mathrm{B},\mathrm{A}) \tag{12a}$$

$$\textit{covers}\,(\mathrm{A},\mathrm{B}) \Leftrightarrow \textit{coveredBy}\,(\mathrm{B},\mathrm{A}) \tag{12b}$$

$$\textit{disjoint}\,(\mathrm{A},\mathrm{B}) \Leftrightarrow \textit{disjoint}\,(\mathrm{B},\mathrm{A}) \tag{12c}$$

$$\textit{meet}\,(\mathrm{A},\mathrm{B}) \Leftrightarrow \textit{meet}\,(\mathrm{B},\mathrm{A}) \tag{12d}$$

$$\textit{overlap}\,(\mathrm{A},\mathrm{B}) \Leftrightarrow \textit{overlap}\,(\mathrm{B},\mathrm{A}) \tag{12e}$$

$$\textit{equal}\,(\mathrm{A},\mathrm{B}) \Leftrightarrow \textit{equal}\,(\mathrm{B},\mathrm{A}) \tag{12f}$$

$$\textit{attaches}\,(\mathrm{A},\mathrm{B}) \Leftrightarrow \textit{attaches}\,(\mathrm{B},\mathrm{A}) \tag{12g}$$

$$\textit{entwined}\,(\mathrm{A},\mathrm{B}) \Leftrightarrow \textit{entwined}\,(\mathrm{B},\mathrm{A}) \tag{12h}$$

$$\textit{embraces}\,(\mathrm{A},\mathrm{B}) \Leftrightarrow \textit{embraces}\,(\mathrm{B},\mathrm{A}) \tag{12i}$$

6.2 Composition Table in $\mathbb{P}^2$

The basis for inferences over multiple relations is the composition (Tarski 1941). For a pair of spatial relations $A\ r_i\ B$ and $B\ r_j\ C$, it determines the relation (or set of relations) that may hold between A and C. Typically composition of two relations is written as $r_i\ ;\ r_i$, omitting the references to the objects involved. For a set of *n* relations, the *composition table* captures all n^2 compositions. Subsequently we derive the composition table for the eleven topological relations in $\mathbb{P}^2$ and compare their inference power with that of the eight topological relations in $\mathbb{R}^2$.

To display the result of compositions in a compact format, we employ an iconic representation, in which each icon is based on the conceptual neighborhood graph (Figure!4). If a relation is part of the composition, the icon highlights it in the graph (Figure 5a). An icon with more than one highlighted relation implies that the composition results in multiple alternatives (Figure 5b). If all relations are highlighted, the composition of those particular relations yields the universal relation, which does not provide any inference information (Figure 5c).

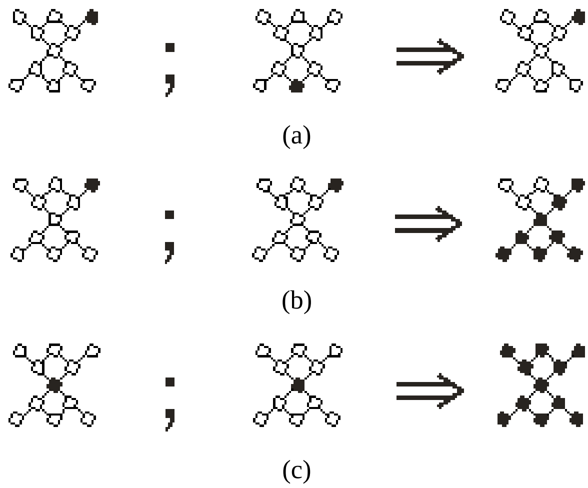


Fig. 5. Iconic presentations of compositions: (a) with a unique result, (b) with alternatives, and (c) with the universal relation as the result.

We developed systematically the compositions of the spherical topological relations using the same method as for the composition of the topological relations in $\mathbb{R}^2$ (Egenhofer 1994a).

- A non-empty intersection between two parts A and B implies a non-empty intersection between the parts A and C if B is a subset of C (Equation 13a).
- An empty intersection between the parts A and B implies an empty intersection between the parts A and C if C is a subset of B (Equation 13b).

- A non-empty intersection between the parts A and B implies a non-empty intersection with the union of the two parts C_0 and C_1 if B is a subset of the union of C_0 and C_1 such that B intersects with both C_0 and C_1 (Equation 13c).
- An empty intersection between A and the union of B_0 and B_1 implies an empty intersection between A and C if C is a subset of the union of B_0 and B_1 (Equation 13d).

$$A \cap B = \neg\varnothing \quad \wedge \quad B \subseteq C \quad \Rightarrow \quad A \cap C = \neg\varnothing \tag{13a}$$

$$A \cap B = \varnothing \quad \wedge \quad B \supseteq C \quad \Rightarrow \quad A \cap C = \varnothing \tag{13b}$$

$$\begin{aligned} A \cap B = \neg\varnothing \quad \wedge \quad B \subseteq (C_0 \cup C_1) \quad \wedge \quad B \not\subseteq C_0 \quad \wedge \quad B \not\subseteq C_1 \\ \Rightarrow \quad A \cap C_0 = \neg\varnothing \quad \wedge \quad A \cap C_1 = \neg\varnothing \end{aligned} \tag{13c}$$

$$\begin{aligned} A \cap (B_0 \cup B_1) = \varnothing \quad \wedge \quad (B_0 \cup B_1) \supseteq C \quad \wedge \quad B_0 \not\supseteq C \quad \wedge \quad B_1 \not\supseteq C \\ \Rightarrow \quad A \cap C = \varnothing \end{aligned} \tag{13d}$$

The composition of all 121 pairs of spherical topological relations was determined computationally with a Prolog program with a total of 44 lines of code (11 ground axioms for the 11 base relations and 33 predicates to determine the inferred compositions). It ran 6.5 seconds on a 266 MHz Macintosh PowerBook G3. Figure 6 displays the 11×11 composition table for the topological relations that can be realized on the sphere between two regions.

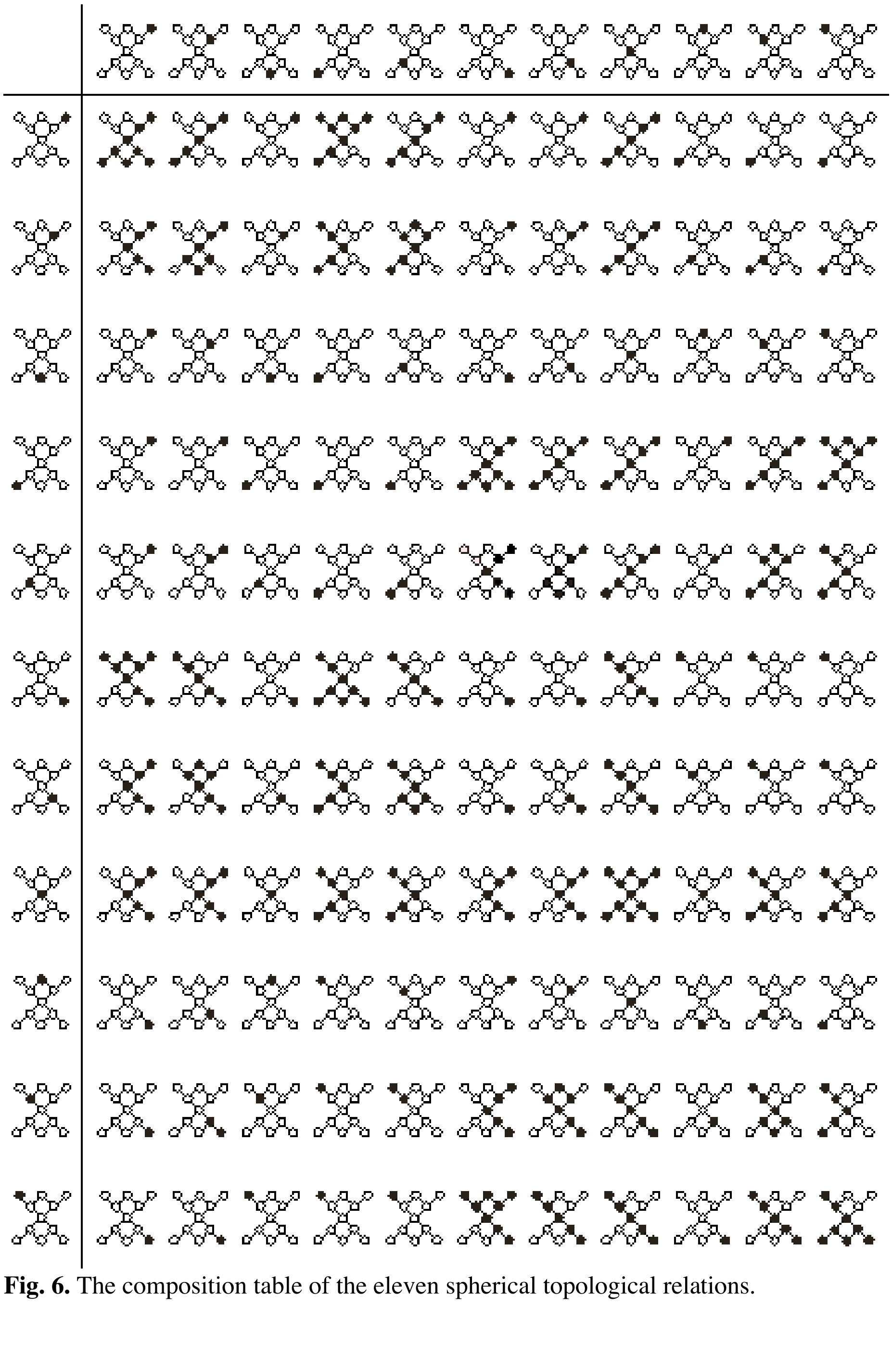


Fig. 6. The composition table of the eleven spherical topological relations.

A comparison of the counts of relations in each composition reveals interesting similarities among the eight planar relations and the three exclusively spherical relations:

- All compositions with *equal* have the same cardinality (i.e., number of relations) as the compositions with *attach*. One interpretation is that the coincidence of the boundaries, which is common to both relations, is a strong factor for making inferences.
- All compositions with *coveredBy* have the same cardinality as the compositions with *entwined*. This analogy has the same roots as the matching between *equal* and *attach*.
- All compositions with *inside* have the same cardinality as the compositions with *embrace*. In both cases, one region's boundary is completely contained in the interior of the other region's boundary.

The composition table is the foundation for assessing whether or not the spherical topological relations form a relation algebra (Tarski 1941). Using the set-theoretic operations union ($\cup$), intersection ($\cap$), and complement (–), and considering *equal* as the identity relation and $\bar{r}$ as the converse relation of r (Equation 12a-i), we found that all seven properties of a relation algebra are fulfilled by the set of eleven spherical topological relations:

- Each composition with the identity relation is idempotent, because $\forall r: \ r\, ; equal = r$.
- The composition with a set of relations is equal to the union of the compositions with each of the elements of the set, because $\forall (r_i, r_k)\, \exists r_j: \ (r_i \cup r_j)\, ; r_k = (r_i\, ; r_k) \cup (r_j\, ; r_k)$.
- The converse of a converse relation is equal to the original relation, because $\forall r: \ \bar{\bar{r}} = r$.
- The converse of a set of relations is equal to the union of the converse relations of each of the elements of that set, because $\forall (r_i, r_j): \ \overline{(r_i \cup r_j)} = \bar{r}_i \cup \bar{r}_j$.
- The converse relation of a composition is equal to the composition of the converses of the two relations, taken in reverse order, because $\forall (r_i, r_j): \ \overline{(r_i\, ; r_j)} = \bar{r}_j\, ; \bar{r}_i$.
- A variation of De Morgan's Theorem K holds, because $\forall (r_i, r_j): \ \bar{r}_i ; -(r_i\, ; r_j) \cup -r_j = -r_j$.
- The composition is associative, because $\forall (r_i, r_k)\, \exists r_j: \ (r_i\, ; r_j)\, ; r_k = r_i\, ; (r_j\, ; r_k)$.

6.3 Comparing the Inference Power of Topological Relations in $\mathbb{R}^2$ and $\mathbb{P}^2$

It was expected that spherical topological reasoning would be more complex than topological reasoning in $\mathbb{R}^2$. If this assumption was true then the composition tables for $\mathbb{R}^2$ and $\mathbb{P}^2$ should reveal that the addition of the three spherical relations makes the inferencing less crisp.

To assess the crispness of compositions, we use four different measures. First we count the number of relations in a composition (Equation 14). The more relations in a composition, the less crisp the inference and, therefore, the more possible cases a person or a machine needs to consider as the outcome of an inference.

$$C = \#(r_i ; r_j) \tag{14}$$

Since the largest number differs for relations in $\mathbb{R}^2$ and $\mathbb{P}^2$ (it is 8 vs. 11), we use a second crispness measure $\overline{C}$, a normalized count of relations (Equation!15). It has a different base for $\mathbb{R}^2$ and $\mathbb{P}^2$ (i.e., 8 and 11, respectively). This normalized crispness measure has the highest value if the composition is unique, while it is 0 for a composition that results in the universal relation.

$$\overline{C}_8 = 1 - \frac{\#(r_i ; r_j)}{8} \tag{15a}$$

$$\overline{C}_{11} = 1 - \frac{\#(r_i ; r_j)}{11} \tag{15b}$$

The last two measures are based on the number of undetermined compositions (i.e., compositions that result in the universal relation, which does not yield any inferences at all) and the number of determined compositions (i.e., compositions that result in a single relation, which allows for the most crisp inference). They indicate how often nothing can be derived, or how often the inference is unique.

Hypothesis 1: The composition of $\mathbb{P}^2$-topological relations is less crisp than the composition of $\mathbb{R}^2$-topological relations, because it has more undetermined compositions.

Dismissed. The $\mathbb{R}^2$-composition table has three universal relations—the results of (1)!*disjoint* ; *disjoint*, (2) *overlap* ; *overlap*, and (3) *inside* ; *contains*—while the $\mathbb{P}^2$-composition table has a single universal relation (the result of *overlap* ; *overlap*). Normalized over the total number of compositions, this means a decrease in undetermined compositions from 4.7% to 0.8%. None of the compositions with any of the three exclusively spherical relations is undetermined, and the least crisp compositions involving the exclusively spherical relations is 8 out of 11 (i.e., $\overline{C}_{11}$ = 0.273), which occurs for three compositions—(1) embrace ; contains, (2) inside ; embrace, and (3) embrace!; embrace. ∴

Hypothesis 2: The composition of $\mathbb{P}^2$-topological relations is less crisp than the composition of $\mathbb{R}^2$-topological relations, because it has fewer determined compositions.

Dismissed. The $\mathbb{R}^2$-composition table has 27 compositions of cardinality 1, while the $\mathbb{P}^2$-composition table has 64 of such crisp compositions. Normalized over the total number of compositions, this means an increase in determined compositions from 42.1% to 52.9%. The relations with the highest numbers of determined compositions in $\mathbb{P}^2$ are *equal* and *attach* (all compositions with *equal* and *attach* are determined), while the lowest number of determined compositions involving a particular relation is with *overlap*. ∴

Hypothesis 3: The composition of all eleven spherical topological relations is less crisp than the composition of the eight topological relations in $\mathbb{R}^2$.

Dismissed for relative counts, but confirmed for absolute counts. The average crispness of all 64 $\mathbb{R}^2$-compositions is $\overline{C}_8 = 0.623$, while $\overline{C}_{11} = 0.727$ for all 121 $\mathbb{P}^2$-compositions. The crispness of the 57 compositions that involve at least one exclusively spherical relation is also higher ($\overline{C}_{11} = 0.781$) than the average of all $\mathbb{R}^2$-compositions ($\overline{C}_8 = 0.623$). In absolute numbers, all compositions in $\mathbb{R}^2$ include 193 relations, while there are 363 compositions in $\mathbb{P}^2$. When compared with respect to the total number of compositions existing in $\mathbb{R}^2$ and $\mathbb{P}^2$, the two ratios are almost identical: on average there are 3.01 relations per composition in $\mathbb{R}^2$ and 3.00 in $\mathbb{P}^2$. ∴

Hypothesis 4: The addition of the three exclusively spherical relations reduces the crispness of the majority of the 64 compositions.

Dismissed. When projecting the eight $\mathbb{R}^2$-relations onto $\mathbb{P}^2$, 13 of their 64 compositions become less crisp (for these 13 relations, the crispness $\overline{C}_8$ is by an average of 7.5% greater than $\overline{C}_{11}$). Fifty of the 64 compositions become crisper (for these 50 relations, the crispness $\overline{C}_8$ is by an average of 9.1% smaller than $\overline{C}_{11}$). One composition—*overlap*!;!*overlap*—is equally crisp in $\mathbb{R}^2$ and in $\mathbb{P}^2$ (i.e., it has the same inference power in $\mathbb{R}^2$ as in $\mathbb{P}^2$). For all 64 compositions, this means an average increase in crispness by 5.6%.

While in absolute numbers the total count goes up from 193 in $\mathbb{R}^2$ to 226 in $\mathbb{P}^2$ (i.e., a 17% increase), the loss of crispness comes from 15 out of the 64 compositions (23.4%), while the remainder (76.6%) stays unchanged. Among the 15 compositions that loose inference crispness by mapping the relations from $\mathbb{R}^2$ onto $\mathbb{P}^2$, eight decrease in their crispness by 67%, two by 60%, three by 50%, one by 38%, and another one by 33%. The greatest decrease in crispness occurs for compositions that involve *overlap*, while—as expected—all compositions with *equal* remain perfectly crisp. ∴

Hypothesis 5: Compositions of topological relations on the sphere are more often undetermined than compositions of topological relations in the plane.

Dismissed. In $\mathbb{P}^2$ there is only one undetermined composition (i.e., 0.8% of all $\mathbb{P}^2$-compositions), whereas in $\mathbb{R}^2$ there are three undetermined compositions (which corresponds to 4.7% of all possible compositions between $\mathbb{R}^2$-relations). ∴

Hypothesis 6: Compositions of topological relations on the sphere are less often uniquely determined than compositions of topological relations in the plane.

Dismissed. In $\mathbb{P}^2$ there are 64 unique compositions (which is 52.9% of all 121 $\mathbb{P}^2$-compositions), while in $\mathbb{R}^2$ there are 27 unique compositions (i.e., 42.1% of the 64 compositions between $\mathbb{R}^2$-relations). ∴

7 Conclusions

Models of geographic space as they are used in current geographic information systems are typically oversimplified. They reduce, for instance, the 3-dimensional nature of geographic phenomena to a planar view, and they flatten the surface of the Earth from a sphere into the plane. Such simplifications are sufficiently good approximations for capturing locally limited geographic areas, but impose serious limitations for modeling global geographic phenomena, as required in such a setting as Digital Earth.

This paper investigated fundamental spatial properties that are preserved in the transition from a flat, two-dimensional embedding space to a two-dimensional surface embedded in a three-dimensional space. Such a setting corresponds to modeling and analyzing spatial phenomena on the surface of a sphere. With a focus on topological relations, we gained new insights about qualitative topological reasoning, comparing the planar with the spherical setting. While the sphere offers additional topological relations that cannot be realized in the plane—the set of possible relations grows by 37.5% from 8 to 11—the inferences that can be made with the composition of topological relations remain primarily crisp, which is a measure of the inference power of the algebraic system formed by these topological relations. Only 23.4% compositions are diluted if the setting for their analysis is on the sphere rather than in the plane. On the other hand, compositions that involve at least one relation that can only be realized on the sphere are on average crisper than compositions of relations that can be realized in the plane. Based on these analyses we conclude that the transition from planar to spherical topological reasoning is a small step that should require few additional logical reasoning abilities.

The second insight relates to the parallel between one-dimensional (e.g., temporal) reasoning and two-dimensional (e.g., spatial) reasoning. We found that the transition from the plane to the sphere (for the two-dimensional case) corresponds to the transition from a linear model to a cyclic model (for the one-dimensional case). This finding is based on the observation that both transitions give rise to additional qualitative relations. These additional relations extend the conceptual neighborhood graphs in parallel ways (even though the linear relations used in $\mathbb{R}^1$ are based on an orientation of the plane and, therefore, typically create pairs of converse relations where there is only one orientation-neutral relation in the two-dimensional setting). Such analogies are critical to increase our understanding about the relationship between spatial and temporal reasoning, in particular providing answers to why certain types of spatial and temporal concepts appear to be compatible.

A final observation relates to the stunning regularity of the numbers of unique compositions in $\mathbb{R}^2$ ($27=3^3$) and $\mathbb{P}^2$ ($64=4^3$).

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