How do Mathematicians Learn to Teach?
Implications from Research on Teachers and Teaching for Graduate Student Professional Development

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**Scenario 1** At a pre-semester orientation session for mathematics graduate students, the following question and student work were presented:

If \( y = (3 - x^2)^3 \), what is \( \frac{dy}{dx} \)?

Student work: \( \frac{dy}{dx} = 3(3 - x^2)^2 \cdot (-2x) \cdot (-2) \).

The graduate students were asked to describe what a student might have been thinking when producing such an answer. After a few moments, the question was repeated, but none of the graduate students offered a potential explanation.

Then, a professor who was sitting in the room said, “Well, it’s not a bad answer.” He then explained how the student’s answer showed a pretty solid understanding of the chain rule, but that the student had applied the rule repeatedly instead of just the one time required.

Why could the professor explain what the student had done but the graduate students were unable to do so? Does the professor know more about the chain rule than the graduate students? Will the graduate students know more about the chain rule when they finish their degrees and then be able to make sense of students’ answers in the way the professor was able to? Or does the professor know other things from years of experience?

**Scenario 2** A graduate student was grading calculus exams. In order to calculate the area between two curves, one problem required that students solve \( 3y + 4 = 6x - 8 \) for \( y \). Once the equation was transformed into \( y = \) a function of \( x \), the problem could be solved by taking the appropriate difference and computing the integral. The graduate student showed the course professor one student’s work:

\[
3y + 4 = 6x - 8 = 6x - 12 = \frac{6x - 12}{3} = 2x - 4
\]

The graduate student asked how many points to deduct since the student, “had not taken the time to write out each step on a separate line and the resulting solution was sloppy.”

The professor examined the work and said, “The student wasn’t sloppy, she’s just confused about the equals sign. She probably learned one thing in elementary school and didn’t notice that things changed when she got to algebra.”

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1 These scenarios are taken from the experiences of the first author.
Is the professor’s ability to make sense of the student’s work a function of the amount of mathematics known? Is there a course that the graduate student could take that would teach the things about the equals sign that the professor knows?

The goal of this chapter is to present findings from research on teachers and teacher professional development to help people think about how to prepare graduate students to teach. In particular, we examine the role that teachers’ knowledge plays in shaping instructional practices and student outcomes. The claim that “what students learn is a function of what their teachers know” turns out to be difficult to establish empirically. From research that has been accumulated in this area, it appears that a subset of “knowledge” has more explanatory power. We review findings from research on teachers’ knowledge as well as from successful teacher professional development programs and we discuss potential implications of these findings for professional development of mathematics graduate students.

As the other chapters of this volume demonstrate, there is a substantial body of literature about student learning of college mathematics. Less research exists on the teaching of college mathematics. In the history of educational research, studies of teachers and teaching have typically lagged behind in number compared with those of student learning. Research at the K–12 levels has a long history and has resulted in significant findings about teachers and teaching. At this stage of development in research on undergraduate mathematics teaching (where research is relatively scarce), it may be productive to look to areas of K–12 research on teaching to inform peoples’ thinking about how best to assist graduate students as they learn to teach.

There is another reason why research on K–12 teachers and teaching is relevant at the undergraduate level: a significant part of the mathematics taught in colleges (e.g., college algebra, pre-calculus, calculus) is also taught in high school. While there are certainly factors unique to teaching this content at the college level, it is reasonable to assume that research on teaching this content has relevance to undergraduate teaching. We seek to draw from research at K–12 levels to inform work at the undergraduate level—both in terms of practices (preparation for future faculty) and directions for future research. Of course, for graduate students to benefit from what is known about factors that shape teachers’ practices (and their students’ learning) there need to be opportunities for them to participate in professional development programs and activities. The hope is that the remainder of this chapter provides both an argument for the importance of graduate student professional development as well as suggestions for those who have the responsibility for providing that professional development.

**What Makes Someone A “Good” Teacher?**

For a long time, educational researchers have asked, “What makes someone a good teacher?” While much is known about what teachers do and why, connecting teachers’ traits to students’ learning is a complex task and no simple answers have emerged (Darling-Hammond, 1999; Mewborn, 2003). Traits examined in the history of educational research include teachers’ general academic ability, subject matter knowledge, and teaching skills. Research on professors and graduate students is sparse in comparison, but similar issues are starting to be examined (Speer, Gutmann, & Murphy, 2005).

In this chapter, we chose to focus on knowledge for three reasons. First, it seems obvious that what teachers know should determine, at least in part, what their students learn. Second, teachers’ knowledge has been the subject of much investigation and there have recently been significant findings in this area. Lastly, some of the findings from K–12 professional development programs that focus on aspects of knowledge seem especially well-suited to adaptation for the professional development of mathematics graduate students.

**Does What You Know Determine What Your Students Learn?**

While it is natural to assume that what students learn is influenced by what their mathematics teachers know, finding empirical support for this claim has been extremely difficult (Ball, Lubienski, & Mewborn, 2001). Such research has examined the amount of mathematics that teachers know (as measured by number of courses taken, number of credits earned, etc.) as a measure of teachers’ mathematical knowledge. If mathematical knowledge is measured in these ways, there is no definitive relationship between teachers’ knowledge and their students’ learning of mathematics.

One of the most-cited studies in this area is from Begle (1979). He reported findings from a meta-analysis of studies conducted from 1960 through the mid-1970s that examined effects of teacher traits on student performance.
His meta-analysis found no significant positive relationship between schoolteachers’ highest academic degree, post-bachelor’s course credits, or majoring in mathematics, and student achievement. If only courses beyond calculus are considered, only 10% of the time did teachers’ taking such courses produce greater student performance. Even more stunning was his finding that about 8% of the time, having more courses post-calculus led to lower levels of student achievement.

More recently, however, Monk (1994) found “positive relationships between the number of undergraduate mathematics courses in a teacher’s background and improvement in students’ performance in mathematics” (p. 130). The courses under consideration in this study were those at the sophomore and junior levels. This encouraging picture is tempered a bit by specifics of the findings: taking an additional mathematics course translates into, at most, a 1.2% increase in student performance (0.2% for sophomores). It is also important to note that increases of this size are only apparent for the first five undergraduate mathematics courses teachers take—taking additional courses beyond five is associated only with a 0.2% gain in student performance.

These counter-intuitive findings prompted the educational research community to seek other measures of teacher knowledge that correlate with student learning. After all, finding such correlations could inform the design of teacher preparation and professional development programs with some confidence that such programs would increase student achievement. This has resulted in two lines of research, both of which are refinements to more traditional definitions of mathematics content knowledge. One line of research expanded the scope of what is taken as knowledge, while the other proposes a particular kind of content knowledge that is closely connected to teaching. Both of these areas of research are discussed below.

Different kinds of knowledge
Several alternative ways of characterizing knowledge appear to explain more about student learning opportunities than the measures used in the research summarized in the previous section. Two of these refinements to examinations of knowledge are: pedagogical content knowledge (PCK) and mathematical knowledge tied specifically to teaching. For more extensive reviews of research on teachers’ knowledge see, (Ball, Lubienski, & Mewborn, 2001; Calderhead, 1996; Mewborn, 2003). In this section, these two kinds of knowledge are described and research about the roles of this knowledge in teaching and student learning are discussed.

Pedagogical content knowledge overview
For much of the history of mathematics education research, two areas of knowledge were examined: knowledge of subject matter and knowledge of pedagogy. As noted above, however, standard measures of knowledge do not have much predictive power for how effective teachers will be. Neither do the myriad measures of teachers’ general pedagogical knowledge and skills (classroom management, organization, etc.).

In the mid-1980s, researchers identified another kind of knowledge possessed by teachers. This includes much of what teachers draw on while teaching, planning for teaching, and making sense of student thinking. In particular, it includes information about typical student difficulties, typical ways in which students approach particular tasks (both unsuccessfully and successfully), examples that are especially illuminating of the ideas, etc.

This kind of knowledge is referred to as pedagogical content knowledge (Grossman, Wilson, & Shulman, 1989; Shulman, 1986) and combines subject-matter knowledge and knowledge about teaching that subject matter. In mathematics, for example, there are things that teachers know that are specific to particular topics (e.g., typical errors students make when they first learn the quotient rule in calculus; common misunderstandings of the definition of limit) that are not part of what they were taught in mathematics courses. Pedagogical content knowledge is in large part what enables teachers to understand why what students write (or say) makes sense to them, even if it is not correct.

Pedagogical content knowledge in undergraduate teaching and learning
In the first vignette of this chapter, the professor was able to make sense of the student’s work on the derivative problem because he knew that students sometimes “over generalize” and carry out the chain rule process on expressions more times than is appropriate. To the student, applying the chain rule in this repeated fashion may make sense—it may just be that the student has not yet developed the ability to distinguish between situations where it applies and where it does not. In such situations, students are making an effort to apply what they know and the work they produce
provides clues to what they do (and do not) understand. You will not find this “over generalizing” difficulty described in any calculus textbook. Knowing this about student thinking is not knowledge of the chain rule (although having knowledge of the chain rule may play a role in how the professor responds to the student’s error).

In the second vignette we saw a graduate student and a professor confronting a student’s algebraic reasoning while solving part of a calculus problem. The student produced the following string of computations:

\[ 3y + 4 = 6x - 8 - 4 = 6x - 12 = \frac{6x - 12}{3} = 2x - 4 \]

when the task required solving for \( y \) in \( 3y + 4 = 6x - 8 \). Instead of beginning with the original equation and writing the result of the transformation of each line separately (thereby maintaining the equality), the student had created a trace of only the transformations of the right-hand side of the equation. This gives the correct expression for \( y \) in the final term, but creates a string of expressions that are not equal to one another.

Why do students produce work like this? Are they just failing to be careful and systematic with their work? Do such errors arise just because students are trying to take a short cut by not writing down each step? This kind of error is actually fairly common and the result of how students interpret the use of the equals sign from their study of arithmetic and/or their weak understanding of how the equals sign is used in algebra. For most of their elementary schooling, students encountered problems such as \( 5 + 7 = \ldots \). In this example, and most others involving basic arithmetic, students are asked to perform a calculation and put the answer to the right of the equals sign. Repeated exposure to these kinds of tasks instills in students the belief that the equals sign is a “do something” operator (Behr, Erlwanger, & Nichols, 1980; Kieran, 1981; Saenz-Ludlow & Walgamuth, 1998).

When students encounter expressions as \( 3 + x \) in algebra they often have difficulty leaving them as is and feel compelled to “do something,” resulting in expressions such as \( 3 + x = 3x \). If students do not develop a different view of the equal sign, when they are faced with solving equations for a particular variable, they may manipulate expressions in ways that violate the equality. In the example from the vignette, the student understands the processes for solving the equation for \( y \), but is carrying out those processes only on the right-hand side of the equation. At each step, the student is “doing something” to the expression, without regard to how what they are doing alters the equality of the expressions.

Knowing that the equals sign is treated in a particular way in elementary arithmetic and that some students carry that understanding into their study of algebra and beyond is an example of pedagogical content knowledge. This is knowledge that is both about mathematics (in this case, about equality and algebraic transformations) and about student understanding of that mathematics (including errors that are symptoms of students’ under-developed knowledge of the equals sign).

There are other examples of pedagogical content knowledge (PCK) possessed by professors. Similar to the chain rule example, “over generalizations” happen when students apply L’Hopital’s Rule multiple times as they evaluate limit problems. Although the rule states that it is only applicable in situations that meet certain criteria (e.g., the expression being evaluated must be a quotient of some indeterminant form), students sometimes keep applying the rule even after the expression they have obtained is no longer an indeterminant form. This kind of thinking results in chains of expressions such as:

\[ \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{3x^2} = \lim_{x \to 0} \frac{\cos x}{6x} = \lim_{x \to 0} \frac{-\sin x}{6} = 0 . \]

The source of these kinds of errors is different than the source of other errors (e.g., “simplifying” \( \frac{\sin x}{x} \) and getting 1, evaluating \( \frac{0}{0} \) as 0, etc.) and being able to distinguish among these types of errors draws on a professor’s pedagogical content knowledge.

Other examples of pedagogical content knowledge include: knowing that when students are first learning to construct delta-epsilon proofs for limits, they are likely to have difficulty understanding and producing expressions that involve absolute values (e.g., \( |x - 3| < \epsilon \)); knowing that students will have difficulty understanding the relationship between a function being continuous and being differentiable (mixing up “continuous functions are differentiable” with “differentiable functions are continuous”); knowing that errors that students make when using partial fractions to evaluate integrals may come more from their skills with fractions than from their understanding of integration; etc.
Research on pedagogical content knowledge

In the time since the specialized knowledge described above was named “pedagogical content knowledge,” researchers have sought to document the extent to which teachers possess such knowledge and how that knowledge relates to their teaching practices and their students’ learning. A particular generative line of investigations has come from researchers associated with a project called “Cognitively Guided Instruction.” Research has included investigations of elementary school teachers’ knowledge of how students think about particular content and examinations of the role such knowledge plays in teachers’ instructional practices (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Researchers have developed instruments to evaluate the extent of teachers’ knowledge of student difficulties and student strategies. It has also been possible to assess the extent to which teachers use their knowledge of student thinking as they teach.

In conjunction with such work, professional development that focuses on developing teachers’ knowledge of student thinking (described in more detail in a later section) appears to be a promising approach to improving teaching (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema et al., 1996; Fennema & Scott Nelson, 1997). In short, research findings demonstrate that it is possible to help teachers increase the depth and breadth of their knowledge of student thinking and that such changes in knowledge can be linked to positive changes in teaching practices. These changes create more opportunities for students to think about and understand mathematical ideas.

Researchers have also taken this line of work one step farther and examined the consequences of such changes in teachers’ practice on student achievement. It appears that the more use that teachers make of their knowledge of student thinking while teaching, the more mathematics their students will learn (Fennema et al., 1996).

Similar lines of work are just beginning to appear at the college level. While this research is relatively sparse, researchers have begun to examine mathematics graduate students’ knowledge of student thinking for some key concepts from calculus (e.g., limit, derivative). Findings indicate that graduate students do not necessarily have extensive knowledge of student strategies and difficulties for these topics (Speer, Strickland, & Johnson, 2005) but that it is possible for graduate students to develop rich and detailed understanding of student thinking (Kung, 2005).

In the next section, a second kind of knowledge is described and research is discussed that examines this kind of knowledge in teachers and its role in students’ learning.

Mathematical knowledge for teaching overview

In addition to pedagogical content knowledge, researchers have proposed other refinements to the basic categorizations of knowledge. In particular, researchers have found evidence that there are certain kinds of knowledge of mathematics that play important roles in how teachers teach.

This type of knowledge is distinct from pedagogical content knowledge discussed above: it includes knowledge of mathematics content and is not inherently related to student learning or understanding. This knowledge may come, in part, from the special kind of mathematical work that teachers engage in—a kind of work in which those who use mathematics outside of teaching are unlikely to engage. This kind of knowledge, as well as how knowledge is used in teaching, has received considerable attention in both the K–12 and undergraduate mathematics education community in recent years (Ball & Bass, 2000; Ferrini-Mundy, Burrill, Floden, & Sandow, 2003; Hill, Rowan, & Ball, 2004, 2005; Hill, Schilling, & Ball, 2004; Ma, 1999).

Ma (1999) provided a window into the rich and connected knowledge of mathematics possessed by some school teachers by investigating the extent to which teachers had knowledge of the complex relationships among mathematical ideas that arise in elementary mathematics teaching. By comparing how Chinese and U.S. teachers responded to mathematics tasks and teaching-related questions associated with those tasks, she highlighted the depth and breadth of mathematical knowledge that teachers can bring to their work.

Ma investigated teachers’ knowledge of mathematics in certain domains (subtraction, multiplication, fractions, area and perimeter) and teachers’ knowledge that is linked to the teaching of topics in those domains. For example, teachers in her research completed division of fractions tasks and then constructed word problem examples to reflect particular division of fractions computations. While most teachers were able to carry out the computations accurately, some struggled to generate a word problem that correctly represented the division of fractions process. Teachers who were successful in creating a word problem that modeled the mathematics displayed a type of knowledge of fractions and division that is distinct from what is typically gained through regular schooling (at least in the U.S.). Such teachers
also knew alternative strategies for solving division of fractions tasks that could be used to explain the ideas to students and could be used as a basis for the design of word problems.

These findings demonstrate that computational fluency is not necessarily an indicator of deep understanding of mathematical processes. While this finding is not unique, what Ma’s research also showed was that teachers make use of a particular kind of knowledge of mathematics when they engage in the work of teaching. Moreover, this kind of knowledge for teaching is not necessarily a natural and automatic by-product of knowledge of mathematical content. Ma concludes that having such knowledge enables teachers to make sense of student thinking and contributes to the learning opportunities that teachers can create for their students.

Mathematical knowledge for teaching in undergraduate teaching and learning

To date, analogous studies have not been conducted with people who teach undergraduate mathematics. It remains to be seen what kinds of specialized knowledge of mathematics graduate students and professors develop as a result of interpreting students’ ideas and engaging in other teaching-related mathematical activities. The next section describes research on these issues for K–12 teachers.

Research on mathematical knowledge for teaching

Complementary to the line of work (described above) that identified particular examples of knowledge connected to teaching, others have extended this research by identifying other examples of knowledge for teaching, creating assessments of that knowledge, and examining connections between having such knowledge and student achievement (Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004). It appears that it is possible to create assessment items that tap into knowledge that is particular to teaching—knowledge that people with extensive non-teaching backgrounds in mathematics are unlikely to possess. In this work, a distinction is made between content knowledge and knowledge particular to teaching. This distinction has been described by Hill, Schilling, & Ball (2004) as follows:

One way to illustrate this distinction is by theorizing about how someone who has not taught children but who is otherwise knowledgeable in mathematics might interpret and respond to these items. This test population would not find the items which tap ordinary subject matter knowledge difficult. By contrast, however, these mathematics experts might be surprised, slowed, or even halted by the mathematics-as-used-in-teaching-items; they would not have had access to or experience with opportunities to see, learn about, unpack and understand mathematics as it is used at the elementary level. (p. 16)

The following sample (from Hill, Rowan, & Ball, 2005) illustrates the kind of assessment item generated in this research program:

| Imagine that you are working with your class on multiplying large numbers. Among your students’ papers, you notice that some have displayed their work in the following ways: |
|---|---|---|
| Student A | Student B | Student C |
| 35 | 35 | 35 |
| \( \times 25 \) | \( \times 25 \) | \( \times 25 \) |
| 125 | 175 | 25 |
| 875 | 875 | 100 |
| + 750 | + 700 | 150 |
| + 600 | | |
| + 875 | | |

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

<table>
<thead>
<tr>
<th></th>
<th>Method would work for all whole numbers</th>
<th>Method would NOT work for all whole numbers</th>
<th>I’m not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Method A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) Method B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) Method C</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The researchers contend that, “To respond in such situations, teachers must draw on mathematical knowledge: inspecting the steps shown in each example to determine what was done, then gauging whether or not this constitutes a method,” and, if so, determining whether it makes sense and whether it works in general” (p. 388). They go on to say that doing so is not common work for adults who do not teach, however, this work “is entirely mathematical, not pedagogical; to make sound pedagogical decisions teachers must be able to size up and evaluate the mathematics of these alternatives” (p. 388).

Similar work is also being conducted specifically in the domain of algebra. Researchers are developing assessment items and frameworks for analyzing the nature of the knowledge that is utilized in the teaching of algebra (Ferrini-Mundy, Burrill, Floden, & Sandow, 2003).

In conjunction with some of the projects described above, researchers have looked for connections between teachers possessing such knowledge and the learning of their students. Findings from this work indicate that, “teachers’ content knowledge in mathematics, as measured by items designed to be close to the content and its uses that teachers deploy, is positively related to student achievement” (Hill, Rowan, & Ball, 2004, p. 35). Findings from the same study also, “offer evidence that such effects are due to more than general intelligence, or mere course-taking” (p. 35).

How do Teachers Acquire Knowledge for Teaching?

The research programs involving K–12 teachers are just beginning. Among the long-term goals are: figuring out how teachers develop this kind of knowledge, and figuring out how to best support teachers in the acquisition of this knowledge. Very little is known about precisely how this knowledge is acquired, but we can speculate about what is not the source: formal course work in mathematics. This type of content is not part of the curriculum and so it is likely that it is acquired through on-the-job learning.

This section contains descriptions of research on professional development for teachers and explores the question of how teachers develop knowledge for teaching. With that as background, the remainder of the chapter includes a discussion of the issue of what these research findings might mean for the professional development of people who teach college-level mathematics.

For elementary and high school teachers, preparation for teaching consists of a mixture of mathematics content courses, teaching methods courses, and supervised experiences teaching. Each of these elements of teacher preparation contributes to the learning opportunities that teachers are able to create for their students. Content knowledge is obtained from mathematics courses, teaching methods courses expose teachers to some pedagogical content knowledge, and additional knowledge develops during experiences teachers have in conjunction with their preparation programs. When teachers emerge from preparation programs, however, there is still a great deal to be learned.

Much of that learning occurs as teachers plan and carry out lessons, interact with students, and examine students’ homework and exams. Teachers develop extensive mental catalogs of the difficulties that students have, of errors that occur during particular chapters, and of examples that are especially illuminating for certain topics. This learning in the context of teaching is a major way in which teachers develop pedagogical content knowledge and mathematical knowledge for teaching.

For mathematics professors, the process of becoming a teacher is different. Professors do not participate in preparation programs and do not typically take courses in education as part of their graduate schooling. Some professors, however, have supervised teaching experiences while in graduate school, often as a teaching assistant. It is during these experiences (teaching discussion sections, possibly teaching full courses, talking with students during office hours, grading exams, etc.) that graduate students begin to develop pedagogical content knowledge and pedagogically useful mathematical knowledge as we saw in the examples above.

Some people have written about the preparation of university faculty to teach, but reports of research in this area are scarce. In most articles or volumes about teaching and learning at the university level, there is much information about how particular kinds of teaching take place and what the outcomes are for teachers and students (e.g., Holton, 2001, “The Teaching and Learning of Mathematics at University Level: An ICMI Study.”) Such pieces about teaching, however, speak mostly about what teachers do or might do and how these practices relate to particular learning goals for students. Implicit in such work is the assertion that professors need to acquire the abilities and knowledge to carry out the particular kind of instruction described in the reports. It is possible to read articles and chapters.
about university teaching with an eye toward the kind of knowledge that might underlie the teaching practices being described or proposed, but such information is not typically an explicit part of such reports. For a particular interesting venue in which to attempt this exercise of inferring the knowledge need to teach, see Mason (2001). In this chapter (“Mathematical teaching practices at tertiary level: Working group report,”) working group members describe approaches to teaching that they value. Nearly all place particular emphasis on anticipating and finding out what students already know about a topic and organizing class activities in ways that provide students opportunities to learn mathematics and also provide the professor with a window into how students are thinking about the mathematical ideas in question.

Given the substantial demands on teachers’ knowledge that teaching well entails, many programs exist that provide K–12 teachers with opportunities to acquire more knowledge after they begin their teaching careers. Similar opportunities for mathematics professors are scarce (NSF, 1992). In the next section, a particularly effective form of professional development for elementary schoolteachers is described and in a later section, we discuss how elements of this program might be incorporated into graduate student professional development.

Providing teachers with opportunities to develop knowledge

If factors other than traditional content knowledge and basic pedagogical skills influence what students learn, what kinds of efforts have been made to provide teachers with opportunities to learn these things? Educational research on pedagogical content knowledge has a longer history than analogous work on mathematical knowledge for teaching. As a result, programs for teachers and associated research are more extensively developed for pedagogical content knowledge. Among various aspects of PCK, knowledge of student understanding (including knowledge of how students typically understand particular concepts, how ideas from prior courses interact with their learning of new content, typical student strategies for solving problems, and common student difficulties/errors) has been the focus of some programs with effective results.

One of the best-documented and most extensively researched programs is Cognitively Guided Instruction (CGI). Teachers participating in this program acquire knowledge about student thinking and learning of particular mathematics topics. Teachers participate in workshops where mathematics educators present research on how students think about specific topics (e.g., addition, subtraction, etc.). Teachers also read reports of research about student strategies and common difficulties for particular kinds of problems. During the workshop, teachers also watch videos of students working on problems and being interviewed about how they are thinking about the problems. In addition, teachers investigate how students think about particular topics by conducting interviews and investigate how findings from educational research relate to what students did and said during the interviews. These activities give teachers opportunities to see and understand the (sometimes surprising) variety of ideas that students have about particular problems.

Researchers who study participants in these workshops have found that teachers develop richer knowledge of student thinking and they also change their teaching practices to include more requests for students to explain or justify their answers (Carpenter, Fennema, & Franke, 1996; Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loe, 1989). In addition, research on CGI has documented significant gains in student achievement as a result of this kind of professional development (Carpenter, Fennema, & Franke, 1996; Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loe, 1989; Fennema et al., 1996). Student problem solving and concept knowledge increased, and in many cases, overall student achievement improved by as much as one standard deviation. Unlike the research on teachers’ content knowledge, these findings indicate a strong relationship between teachers’ pedagogical content knowledge (in this case, knowledge of student thinking and learning) and their students’ learning.

Themes in Research and Implications for Graduate Student Professional Development

In this section we discuss three themes evident in the research reviewed earlier in this chapter and describe how professional development for graduate students might be designed in light of these themes. We describe each theme, discuss what makes the theme suited to implementation with mathematics graduate students, and provide some specific suggestions for implementing professional development consistent with the theme.
This section is inherently speculative — research into the professional development of mathematics graduate students is scarce (Speer, Gutmann, & Murphy, 2005). What we do know, however, is that most graduate students receive preparation for their first job as a teacher. This may take the form of university-wide orientation sessions for graduate students or it may be a program designed specifically for mathematics. While these programs help graduate students adapt to their new role as a teacher, the emphasis is typically on administrative responsibilities and mechanical aspects of teaching (clear writing on the board, collecting homework efficiently, etc.). While these are certainly important aspects of learning to teach, there is a lot for graduate students to learn about how students think about mathematical ideas and most of that kind of learning happens once they set foot in the classroom and begin interacting with students and with the mathematics that their students produce. The suggested activities described below are modeled after elements of the professional development programs connected with the research on teachers’ knowledge that was summarized earlier in this chapter. Using activities designed for K–12 teachers, however, is not always appropriate or feasible in the undergraduate teaching context. The suggestions below come from modifying such activities in ways to make them well-suited for graduate students and to take advantage of the rich learning environment that graduate students are in as they teach.

**Focusing on students’ thinking improves teachers’ teaching practices**

The first theme in the research discussed in this chapter is that knowing mathematics is a necessary but not sufficient condition for teachers to create good learning opportunities for their students. Findings indicate that students learn more when teachers have extensive knowledge of student strategies, student difficulties, and the mathematics tied to the teaching of particular ideas. Teachers develop this knowledge in various ways, but doing so requires focusing on issues of student thinking (in addition to other aspects of teaching).

As noted above, most pre-semester programs for mathematics graduate students do not focus on issues of student learning. This is, in some ways, very natural—graduate students are there to learn to teach and so time is spent discussing issues of teaching. This approach, however, may not help graduate students recognize the role that knowledge of student thinking plays in teaching and may not give them opportunities to begin to develop that knowledge.

**Reasons why this is well-suited to the undergraduate mathematics context.** There are several reasons why focusing on issues of student thinking might be a productive approach to professional development for graduate students. When designing professional development for K–12 teachers, one has to take into consideration the depth and breadth of the teachers’ mathematical knowledge. Some K–12 teachers do not have as strong and deep knowledge of mathematical content as we might wish. As a result, preparation and professional development programs for K–12 teachers are often designed to address issues of mathematical content in addition to other issues (knowledge of student thinking, etc.). Professional development for graduate students, on the other hand, can take advantage of their extensive background and interests in mathematics. For example, some knowledge of student thinking can come from “unpacking” the mathematics contained in problems or topics. This unpacking involves thinking about all of the mathematical ideas that are connected to the particular problem. Then, one can envision a variety of ways someone might approach the problem if for some reason they chose to pursue a solution path that was not identical to the one first thought of by the graduate student.

**Ways this can be accomplished.** There are several ways that issues of student thinking could be added to or emphasized in professional development for graduate students.

- Talk about examples of student work. As part of graduate student instructor meetings, the professor who supervises those graduate students could orchestrate a discussion of examples of student work. These examples could relate to topics that everyone is going to be teaching soon. The professor (or experienced graduate students) could bring samples of work students produced when answering test or quiz problems on the topic. The discussion could be focused on trying to figure out why what the student did made sense to him or her and what the student might have been thinking as they worked on the problem. Faculty and experienced graduate students are apt to be able to provide quite a bit of insight into students’ thinking. In addition to sharing their knowledge with less experienced graduate students, such discussions also may help new graduate students appreciate the role that knowledge of student thinking plays in teaching well. Professors and graduate students could build up libraries of these examples of student work for each course that could then be used in the future
for discussions with new graduate students. Some examples can be found in the Boston College Case Studies materials (S. Friedberg et al., 2001a; S. Friedberg et al., 2001b).

- Use resources such as this volume. Graduate students and professors could read the chapter relevant to topics that are coming up in a course. Above, the suggestion was made that graduate students could examine examples of student work before teaching the related topics—in addition, graduate students can collect and examine student work from their students, read a summary of related research, and use the ideas from the research to analyze what their students did. As is done in the Cognitively Guided Instruction model, graduate students could also be given the task of interviewing a couple of students about how they solved a particular problem from class or a test. If there are several graduate students teaching the same course, each could select a different kind of problem and then report back to the group about what they learned about the students’ difficulties and strategies. While this activity might be most meaningful if it used problems from the courses the graduate students are teaching, research articles on student thinking could also be used as a source for problems. In addition to this volume, the Research in Collegiate Mathematics Education (RCME) series provides collections of articles (Dubinsky, Schoenfeld, & Kaput, 1994; Dubinsky, Schoenfeld, & Kaput, 2000; Kaput, Schoenfeld, & Dubinsky, 1996; Schoenfeld, Kaput, & Dubinsky, 1998; Selden, Dubinsky, Harel, & Hitt, 2003).

Experienced Teachers Have Rich Knowledge of Student Thinking

Among the best resources for learning about student thinking are professors and advanced graduate students who have already acquired some of this knowledge. Providing TAs with access to the authentic practices of professors can give less experienced graduate student instructors ideas about what professors do when planning for teaching and examining student work.

Reasons why this is well-suited to the undergraduate mathematics context. For many graduate students, their teaching experiences include times when they are responsible for one section of a course or a recitation section associated with a lecture of a course. In many of these cases, there is a professor who supervises the graduate students who are all teaching some part of the same course. These professors have the responsibility for ensuring that various aspects of the course are coordinated among the graduate students (topics for the coming week, giving quizzes, grading exams, handling homework, etc.). To accomplish these things, often professors hold periodic meetings with the graduate students who are instructors for a course.

This kind of joint planning time is typically not found in K–12 teaching contexts, but having such time to discuss students and their thinking is often considered key to helping teachers improve their practice. In such meetings among graduate students and a professor, the focus is often on what is coming up next in the course—but such discussions could be focused on student thinking in addition to mechanics/administrative issues. Specific suggestions for how time during these meetings might be used are given below.

Ways this can be accomplished

- Have graduate students and professors discuss the examples that are used in particular sections of the textbook and what it is about those examples that makes them well-suited as an illustration of the particular ideas in the chapter.
- In staff meetings for large courses, have professors talk about planning for a particular lecture and how examples were selected. Discuss what specific ideas the examples are illustrating and/or how certain “classic” examples illustrate difficult ideas in particularly useful ways.
- Have graduate students write quizzes or problems and predict what students are likely to do with the problems and what difficulties they will have. Then talk with an experienced professor or more advanced graduate student to get feedback about other strategies or difficulties students might have to ensure that the problems are actually going to assess understanding of the ideas that the graduate student has in mind.

Some Teaching Practices Create Many Opportunities to Acquire Knowledge of Student Thinking

While graduate students and other teachers gain knowledge about student understanding in many ways, one of the most productive is from interactions with students when they are explaining their thinking. These interactions might
take place in office hours, during a class discussion, or on paper when a question asks students to explain their reasoning. These practices are useful for student learning and they provide graduate students with authentic access to student thinking and learning.

**Ways this can be accomplished.** Particular approaches or models of instruction focus extensively on having students explain and justify their reasoning. At the undergraduate level, one of the most successful has been the Emerging Scholars Program (known by other names at some institutions). The Emerging Scholars Program (ESP) was designed based on research conducted on how successful students learn calculus (Fullilove & Treisman, 1990; Treisman, 1985). In ESP classes, students work in collaborative groups on challenging problems and the graduate student instructor assists students and moves from group to group asking them to describe how they arrived at their answers. This provides graduate students with especially extensive and rich access to student thinking. There is some evidence that such teaching experiences enable people to acquire considerable knowledge of student thinking in the course of their graduate school teaching careers (Kung, in press). For more information about ESP classes and their influence on students’ learning of mathematics, see (Hsu, Murphy, & Treisman, this volume).

There are also small changes one can make to one’s teaching to get more information about student thinking. The simplest is to ask questions that require students to provide an explanation for their answers. After a student responds to a question, if a teacher asks, “How did you get that answer?” not only will other students have access to the thinking behind an answer, but there is the possibility that the teacher will learn something new about how students think about the ideas or about the kinds of difficulties they have when solving such problems.

**Concluding Thoughts**

Given the relatively rapid increase in research on undergraduate mathematics education over the past few decades, it is likely that the future will generate a rich research base about how graduate students and professors learn to teach as well as insights into how best to support the development of their teaching practices. Currently most research activity is focused on students and how they think about and learn mathematics. The undergraduate mathematics education research community, however, is in the fortunate position of having the option to utilize research on student thinking in the professional development of teachers. Researchers of K–12 mathematics education have also amassed a base of research on student thinking and more recently have discovered ways to use that base of research in improving teaching practices. As described above, researchers have shown that knowing how students are apt to approach particular problems, what their difficulties are likely to be, and why specific ways of thinking make sense to students, are the types of knowledge that are likely to influence how students learn mathematics. Professional development programs that create opportunities for teachers’ to enrich and expand their knowledge of student thinking have been successful in inducing changes in teaching practices that lead to increases in student learning.

As the other chapters in this volume demonstrate, much is known about how students think about and learn mathematics at the undergraduate level. Since this body of research exists (and is expanding), the undergraduate mathematics education community may very well already possess an important key to improving undergraduate education. The findings from K–12 research on teachers can play important roles in creating effective professional development for graduate students and professors. Questions of how best to adapt aspects of K–12 professional development for the undergraduate setting are among the issues to consider as more and more researchers take on the challenge of understanding how best to help people learn to be effective teachers of undergraduate mathematics.

**References**


23. How do Mathematicians Learn to Teach?


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