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Connecting Beliefs and Practices: A Fine-Grained Analysis of a College Mathematics Teacher’s Collections of Beliefs and Their Relationship to His Instructional Practices

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Findings about mathematics teachers’ beliefs typically involve broad characterizations of those beliefs that are compared with general descriptions of practices. Teacher development research suggests that changes happen effectively from attention to specific practices. Few investigations of beliefs and practices are done at this level of detail. Thus, little is known about how beliefs shape practices at the very grain-size where development appears to happen most productively. This study focused on fine-grained details of beliefs, practices, and connections between them. Findings indicate that particular units of analysis (“collections of beliefs”) are useful for investigating connections between beliefs and specific practices. Certain collections were also found to be especially influential, including beliefs about evidence of student understanding and about how learning happens.

THE STATE OF RESEARCH ON TEACHERS’ BELIEFS

Over a decade ago, Thompson (1992) claimed that although prior research had established connections between teachers’ beliefs and practice, examinations had not been detailed enough to give insight into why it is so difficult for teachers to modify their beliefs and adopt new practices. As part of this enterprise, she said it was important to “study individual teachers in depth and to provide detailed analyses of their cognitive processes” (p. 140). In the years since that
publication, work in the area of beliefs has escalated, publications abound, and evidence has accumulated indicating that teachers’ beliefs about mathematics, teaching, learning, and students are related to their teaching practices (Borko & Putnam, 1996; Calderhead, 1996; Leder, Pehkonen, & Torner, 2002; McLeod & McLeod, 2002; Thompson, 1992). The actual connections between teachers’ beliefs and specific in-class decisions, however, are implicit in much research and rarely examined in the detailed manner for which Thompson advocated. In addition, much of the work relies on the same research designs and methods that were used in the work that prompted Thompson’s charge to the community, hindering efforts to shed light on the entailments that particular beliefs have for teachers’ in-class instructional decisions. Data collection and analysis methods typically found in the literature are coarse-grained and speak to broad categories of beliefs and general aspects of practice. Moreover, typical methods have not provided a close connection to actual in-class teaching practices. In short, findings lack explanatory power (Schoenfeld, 2000) for the role of beliefs in shaping practices, a characteristic that requires more than just describing what people can or will do and instead explains how and why things work in particular ways.

The situation becomes even more critical when attention turns to findings from research on teacher change and professional development. As with the more general research on teachers and teaching practices, findings support claims that beliefs interact with teachers’ learning from preparation and professional development (Borko & Putnam, 1996; Richardson, 1996). As a result, some programs make attention to beliefs an explicit focus. Of these professional development programs, among the ones reporting substantial success in helping teachers develop reform-oriented practices, that success appears to stem from focusing on small, but meaningful, aspects of practice (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema et al., 1996; Franke, Carpenter, Levi, & Fennema, 2001; Franke & Kazemi, 2001; Schifter, 1993). While these programs attend to teachers’ beliefs, detailed aspects of practice are not where the research community has amassed much understanding of the influence of beliefs. As indicated earlier, most research links broad characterizations of beliefs with similarly broad descriptions of teaching practices. Hence, very little is known about the influence of beliefs on teaching practices at the very level of detail where it appears development most productively occurs.

Findings from some reform-oriented professional development programs are encouraging. For example, some programs have demonstrated that it is possible for teachers’ beliefs to change in ways that are more consonant with the aims of reform and for them to develop reform-oriented practices (Fennema et al., 1996). What remains unexamined, however, are the connections between particular beliefs and specific moment-to-moment instructional practices that teachers are encouraged to adopt, as well as the mechanisms that underlie such connections. If
such connections and mechanisms were uncovered, researchers and designers of professional development would be better able to examine the particular practices they wish to promote and to devise programs aimed at shaping teachers’ beliefs and practices in the desired ways.

The research reported here focused on the following research questions:

1. Is there a unit of analysis that captures teachers’ beliefs in sufficient detail that entailments of those beliefs are evident in specific instructional practices?
2. If such a unit of analysis is found, are there particular sets of beliefs that appear especially influential on (a particular set of) teachers’ instructional practices?

This research made use of data collection and analysis methods designed expressly to illuminate connections between teachers’ moment-to-moment practices and beliefs with data grounded in specific teaching practices and of a grain size that permits examination of how particular beliefs influenced the teacher’s in-class decisions. Specifically, collections of beliefs is proposed as a construct to characterize beliefs at a level of detail appropriate to illuminate the influence of beliefs on teachers’ moment-to-moment instructional decisions.

It is not the case that beliefs in general, much less a particular set of beliefs, can fully explain teachers’ decisions or actions. What emerged from the data analysis, however, was consistency in the belief-based reasons teachers stated for their decisions while interacting with students. Teachers also demonstrated consistencies in how they responded to students in different contexts. These consistencies are the focus of the analysis in this article. No claim is made that the particular collections of beliefs discussed are the sole determining factor in the decisions made by the teacher. It appears, however, that those beliefs describe much of the rationale the teachers gave for their decisions, and that fact suggests that examining these beliefs might be productive place to start in the larger quest to understand the broader set of factors influencing teachers’ decisions and actions.

The following section contains an overview of literature related to the study of beliefs and teaching practices as well as discussion of data collection and analysis methods typically used in this area of research. Following that is a description of the particular methods used to pursue the research questions. With those sections as background, attention then turns to data, analysis, and discussion of connections between the teacher’s beliefs and the moment-to-moment decisions made while interacting with students who were working on challenging mathematical tasks in collaborative groups. The final section contains conclusions and implications of the study for research and practice.
Teachers’ Beliefs in Mathematics Education Research

Uncovering the role of beliefs

Lines of inquiry into beliefs and practices have grown out of researchers’ desires to understand teaching practices and factors that influence teachers to teach in particular ways. Findings in teacher cognition and related sub-fields of educational research indicate that many factors influence teaching practices (Borko & Putnam, 1996; Clark & Peterson, 1986). Teachers’ knowledge is one factor that researchers have found to shape teachers’ practices, along with the curriculum in use, teachers’ goals, and various social and contextual factors. Among all these factors, however, researchers have found beliefs to be a significant influence on teachers’ use of these cognitive (and other) resources (Calderhead, 1996; Pajares, 1992; Richardson, 1996; Thompson, 1992). Understanding teachers’ practices, therefore, requires understanding not only what resources they possess, but also how they decide what resources to use and when to use them. Such decisions are influenced by what teachers believe is important and plausible (Pajares, 1992).

In addition to influencing teaching, beliefs play a significant role in how teaching practices develop and evolve (Ball, 1988; Borko & Putnam, 1996; Fennema & Scott Nelson, 1997; Richardson, 1996; Schifter & Simon, 1992). Existing beliefs and knowledge shape how new information and experiences are understood and how new knowledge develops. In short, “experienced teachers’ attempts to learn to teach in new ways also are highly influenced by what they already know and believe about teaching, learning, and learners” (Borko & Putnam, 1996, p. 684).

Researchers have also documented the influence of beliefs on teachers’ efforts to adopt reform-oriented practices, for example, as characterized by the National Council of Teachers of Mathematics (NCTM, 1989, 1991, 2000). Researchers have described, among other things, situations in which teachers modified ideas of reform to fit into their existing beliefs about teaching (Cuban, 1982; Sykes, 1990). Other researchers have documented additional influences of existing beliefs on the implementation of reform (Darling-Hammond, 1990; Franke, Fennema, & Carpenter, 1997; Peterson, 1990; Scott Nelson, 1997; Simon, 1997; Sykes, 1990; Wiemers, 1990; Wilson, 1990).

Strengths and limitations of findings

It is clear that the educational research community’s understanding of teachers and teaching has been enhanced by investigations of beliefs that teachers possess and the ways in which beliefs appear to shape teacher practice and change. This line of inquiry has helped explain observed teaching practices and the resulting student learning opportunities in ways that cannot be accounted for with, for example, standard measures of teachers’ knowledge (Ball, Lubienski, & Mewborn,
2001; Begle, 1979). Yet few convincing and detailed explanations for particular findings have emerged from these studies—it seems apparent that beliefs play a role in shaping practices and changes to those practices, but it is not at all clear how and why beliefs are connected to practices. Conclusions and implications sections of many research reports include discussions of how the phenomena in question turned out to be more complex than was initially thought and how future studies are needed to disentangle and illuminate the relationship between beliefs and practices.

Recall Thompson’s 1992 charge to the community: examinations of teachers’ beliefs have not been detailed enough to give insight into why it is so difficult for teachers to modify their beliefs and adopt new ideas and practices. Taking this charge seriously requires that research questions focus on the details and intricacies of teachers’ cognition and that data collection and analysis methods be utilized that give researchers access to data at levels of detail appropriate for examining and answering such questions. As described next, this is not what most findings and research methods in studies of beliefs and practices provide.

Nature of Findings and Methods Used to Obtain Beliefs

Categorizations of beliefs

Findings most often take the form of categorizations of teachers’ beliefs. Researchers have proposed many different categories of beliefs (cf., Ernest, 1985, 1988, 1989; Kuhs & Ball, 1986; Lerman, 1990; Prawat, 1992). Some researchers seek relatively comprehensive classifications of as many of a teacher’s beliefs as they find feasible. Others have chosen to center their investigations on a single category of belief or a small set of categories. One noteworthy feature of work in this area is the plethora of categorization schemes found in research reports. Beyond top-level categories (e.g., beliefs about teaching, beliefs about mathematics), there are nearly as many sets of categories as there are researchers.

To describe beliefs about mathematics, researchers sometimes use categorizes from philosophy of mathematics (e.g., Ernest’s (1989) descriptions of problem solving as Platonist or instrumentalist). Other categorization schemes come from different views of the nature of mathematical knowledge (e.g., Lerman’s (1990) “absolutist” and “fallibilist” views). Beliefs about teaching and learning have also been the focus in some studies and researchers have devised various categories based on characteristics of instruction they were interested in (cf. Kuhs & Ball, 1986) where categories such as “learner-focused” and “classroom-focused” are used. Many other researchers have proposed additional categorization schemes (for reviews of such work, cf. Leder et al., 2002; Thompson, 1992). Additional categories that are sometimes found in this area of research include beliefs about self and beliefs about self as a teacher (Calderhead & Robson, 1991), and
beliefs about purposes of school and the processes of learning to teach (Bullough, Knowles, & Crow, 1991).

Typically, investigations of beliefs and practices center on comparing categorizations of beliefs with categorizations of teachers’ practices. Findings suggest that these relationships are complex. Rarely are clear-cut or consistent correspondences reported. In some cases, beliefs appeared consistent with observations of classroom practice (e.g., Thompson, 1985) and in others inconsistencies are reported (e.g., Cohen, 1990). These studies link general characterizations of teachers’ beliefs with general characterizations of researchers’ observations of their practices.

Although some studies have identified correlations between teachers’ beliefs and their teaching practices, few are designed to examine specific beliefs and then trace them to specific practices. Some of these shortcomings may be linked to data collection and analysis methods (described later) that are typical of this area of research. For a more extensive discussion of these methods-related issues, see Speer (2005).

Often data on beliefs are obtained from teachers’ self-reports (e.g., questionnaires, interviews). Categorization schemes such as the ones described above are sometimes derived empirically from these data and other times predefined categorization schemes are used. In some instances, beliefs teachers state are augmented with ones inferred from observations, as advocated by some researchers (e.g., Thompson, 1992).

Categorization schemes such as the ones described earlier provide useful tools for capturing beliefs at a broad level of detail. Although such descriptions may be very helpful in conveying general trends of teachers’ views, such classifications are not meant to be descriptive of very particular beliefs. For example, knowing that a teacher holds “learner-centered” beliefs about teaching and learning does not provide enough information to know what questioning practices the teacher believes are most appropriate to use to create a “learner-centered” classroom—that level of detail has not been the goal of most studies of teachers’ beliefs, yet it is specifics such as questioning practices that can influence students’ learning opportunities in important ways.

Data on teachers’ practices typically come from two sources: observations and teacher self-reports (Calderhead, 1996; Thompson, 1992). Classroom observations are sometimes recorded and field notes or structured systems for recording observations are sometimes used. In some cases, data about instructional practices come not from observations but instead from teacher self-reports about their teaching. Using data collected in this manner, some researchers examine the relationship between teachers’ beliefs and their in-class teaching practices. To do this, data from questionnaires and interviews is categorized and data on observed or reported practices is categorized. The two categorizations are then examined for consistencies or other kinds of patterns.
Discussion about findings and methods

Despite much research into teachers’ beliefs, many issues remain underexamined. Findings have yet to make substantial contributions to the development and/or refinement of theories that illuminate the underlying processes of teaching. Coarse grain sized characterizations of beliefs and general descriptions of teaching practices appear unlikely to do justice to the complex, contextually dependent acts of teaching. Categories of beliefs are portrayed as essentially static, well-defined entities, presumed to function in cognition in consistent, contextually independent ways yet several researchers have argued that viewing the phenomena in this manner does not do justice to the subtle complexities (diSessa, Elby, & Hammer, 2002; Skott, 2001; Thompson, 1992). The design of the research, however, often fails to provide access to the phenomena at a level of detail that might provide explanatory power, enriching the existing understandings of how and why things happen in particular ways.

Aside from the importance of advancing theories of teacher cognition through finer-grained studies, there are also potential benefits to practice and teacher education. As described next, findings from professional development suggest that supporting teachers to adopt reform-oriented instruction can be effective when the focus is on specific practices. If, as prior research indicates, beliefs play a role in such processes, it follows that understanding connections between beliefs and specific practice will contribute to work in educational reform.

Professional development programs

Assisting teachers to teach in ways that may differ substantially from how they were taught presents complex and substantial challenges. Implementing reform in ways that do justice to the ideals of the designers of the reform is also extremely challenging (Cuban, 1990; Fennema & Scott Nelson, 1997; Sykes, 1990). Although the relationships appear to be complex, change in teaching practices seems linked to change in beliefs. No simple cause and effect pattern appears to exist (Grossman, 1992; Guskey, 1986; Kagan, 1992), but there is some evidence that change in beliefs and change in teaching practices happen in an interconnected, cyclic fashion (Kagan, 1992; Thompson, 1992). It has also been shown that beliefs play important roles in how teachers engage with and learn from specific professional development (PD) programs and activities (Darling-Hammond, 1990; Franke et al., 1997; Peterson, 1990; Scott Nelson, 1997; Simon, 1997; Sykes, 1990; Wiemers, 1990; Wilson, 1990).

Professional development programs that report success in enabling teachers to develop reform-oriented practices often share several characteristics. These characteristics include (1) an extended focus on small, but meaningful, aspects of practice and (2) a recognition that people make sense of new information in light
of their existing knowledge, beliefs, and practices. One particularly successful and well-researched example is Cognitively Guided Instruction (CGI). CGI (Carpenter, Fennema, & Franke, 1996; Carpenter et al., 1989; Fennema et al., 1996; Fennema & Scott Nelson, 1997; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998; Franke et al., 1997) focused on very specific aspects of practice—what teachers did to gain access to their students’ strategies and difficulties and how they responded to students in light of what they knew about students’ thinking. CGI achieved a level of success that is not typically reported in the professional development literature, showing changes in teachers’ practices and connections between those practices and student learning. Teachers developed practices, including particular ways of asking questions, that were more “cognitively guided” (i.e., based on knowledge of students’ ideas and aimed at learning more about how students were thinking) and those changes created richer learning opportunities for their students. The Purdue Problem-Centered Mathematics Project (Cobb, Wood, & Yackel, 1990) and SummerMath for Teachers (Schifter, 1993) provide other, similar examples of programs that help teachers develop specific kinds of practices.

These reform-oriented professional development programs have reported successes; however, the research foci have not been on the role of beliefs in shaping specific practices that are part of the programs. The amassed body of research does not shed light on the role of beliefs in shaping practices at the very level of detail where change and development have been shown to occur.

METHODS

This case is drawn from a larger study that involved additional participants and included comparisons and contrasts among participants’ beliefs and practices (Speer, 2001). Because the purpose of this article is to present findings that illustrate the power of particular theoretical constructs and methods for illuminating the role the beliefs play in shaping teachers’ instructional decisions, findings from only one teacher are discussed. In this section I describe the methods used to obtain and analyze data on the teacher’s beliefs, practices, and connections between particular collections of beliefs and specific instructional practices.

Participant and Setting

At the time of the study, the participant, Zachary, was a doctoral student in the mathematics department of a large university in the western United States. Zachary was active in the department and involved in many of the graduate student organizations. He came to graduate school directly after completing his
undergraduate mathematics degree at a medium-sized university and planned to pursue a career as a research mathematician.

Zachary was a teaching assistant (TA) for a first-semester calculus course for physical science and engineering majors and he was responsible for discussion sections that accompanied a large lecture given by a mathematics faculty member. These discussion sections met three times per week, for a total of about three hours.

The department stipulates how time in discussion sections is to be spent. Instead of the “traditional” use of discussion sections where TAs review material from lecture, present sample problems and solutions, and answer homework questions, in these discussion sections TAs assist students as they work on problems in small groups. The problems are designed to be quite challenging (some similar in difficulty to ones students might see on an exam) so that working collaboratively in groups is advantageous to the students. Students do their work at the blackboards and the TA circulates in the room and assists students. The goal is for the TA to act as a facilitator and a resource to the groups, asking guiding questions, and encouraging students to explain and justify their solutions. This format for running discussion sections was modeled in part on other programs that make extensive use of collaborative group work in calculus classes (Fullilove & Treisman, 1990; Treisman, 1985; Treisman, 1992). These changes were made only to the discussion sections—the department did not ask faculty to alter the content or format of their lectures or their exams.

Although Zachary had taught calculus before, during the time of the study he was teaching this kind of discussion section for the first time. He and other TAs new to this approach to teaching participated in a half-day orientation session prior to the start of the semester. The orientation session included presentations about this kind of teaching as well as opportunities for TAs to experience what it is like to be a student in these kinds of classes, to discuss strategies for working with student groups, and to look at and discuss some of the curriculum materials. TAs who were teaching in the department for the first time were required to take a semester-long course in teaching concurrent with their first teaching assignment. Zachary had taken that course several years earlier, which was before the department had changed the discussion section format to include group work.

During the semester, TAs met periodically with the faculty member who was giving the lectures for the course. These meetings were devoted to discussion of schedules for up-coming exams, grading of exams, and other administrative issues. Faculty also occasionally sent email messages to their TAs asking them to discuss specific topics or to have students work on particular problems during class. Faculty members were required to observe each TA’s class once during the semester and to complete a short written report on the adequacy of the TA’s performance.
Several characteristics of Zachary’s classroom setting were advantageous for obtaining data on beliefs and the decisions that shaped the teacher’s practices. First, Zachary spent a substantial portion of class time talking with students as they worked on problems in groups, creating many more student–teacher interaction opportunities than would occur during a more traditionally formatted class where lecture or teacher presentation of solutions might dominate the time. Because students used the blackboards that lined the walls of the classroom, their work was visually accessible without having to intrude on their interactions. This made it possible to document the discussions Zachary had with his students in ways that would not have been possible if they were working on a common piece of paper (e.g., students’ written work and their gestures to particular parts of the solution were clearly visible).

The fact that this was Zachary’s first time teaching this kind of class was also advantageous. Once teachers are experienced with particular content, they typically develop routines and scripts for particular purposes when interacting with students (Clark & Peterson, 1986). Because these routines and scripts are specific to particular kinds of teaching and content, and take time to develop, Zachary was unlikely to have ones that were well rehearsed for this setting. This made it more likely that he would be able to articulate the reasoning behind his instructional decisions. In short, he was likely to be more aware of his decision-making process because the setting and decisions he faced were new to him. Zachary’s inexperience could create a situation where his beliefs and practices were likely to be relatively unstable and inconsistent. Findings revealed, however, that his practices (and the belief-related reasons he gave for his decisions) were very consistent during the time of the study.

Data Collection

As noted earlier, connections between beliefs and practices as characterized in the literature are somewhat tenuous. To generate results that are closely tied to practices and that can enrich the research community’s understanding of how and why beliefs and practices interact, data on classroom practices was collected and excerpts were used in “video clip” interviews about the teacher’s decisions and beliefs.

This approach to interviewing was based on other researchers’ use of “video clubs” as a professional development activity and as a source of data on teacher cognition (Frederiksen, Sipusic, Sherin, & Wolfe, 1998; Nathan, Knuth, & Elliot, 1998; Sherin, 1996, 2002). Because the data on beliefs are linked to specific examples of teachers’ practices, a more detailed type of data are generated than what is possible in traditional, de-contextualized interviews or in a combination of interviews and observations. For a more extensive discussion of these methods

**Interview preparation procedures**

To generate video clips for the interviews and to gather data on his practices, the teacher’s classes were videotaped. Eight of Zachary’s 50-minute classes were videotaped and video clips for the interviews were selected from those classes. Those interviews (a total of nine, lasting 60–120 minutes each) were audio taped and transcribed. The classes occurred during the second half of the semester and were chosen to include days when students were beginning their work on a new topic as well as times when they were in the middle of studying a topic. Classes devoted to review or tests were not included since the primary focus of the research was on student–teacher discussions.

Initial analysis indicated that there were major “modes” of teacher–student interaction during class. Specifically, all substantive interactions (i.e., conversations about things other than administrative details or classroom procedures) could be grouped into the following four categories:

- discussions with groups when students had completed a problem correctly
- situations when the teacher detected an error in students’ work before initiating a discussion
- situations when the teacher detected an error in students’ work during a discussion
- instances when students were struggling with a problem

Episodes of these four modes of interaction were chosen for analysis because, as noted earlier, most of the group work time fell into one of these categories. In addition, in the corpus of classroom video data, teachers were very consistent in how they handled discussions within each of these modes of interaction. The number of episodes discussed per interview varied, as did the length of the episodes. Some episodes were very short (20 seconds) and others were longer (3 minutes) or came from a set of episodes (for example, struggling episodes might be made up of an initial conversation Zachary had with the group of students and then the longer, follow up discussion he had when the issues were addressed in detail). Some episodes prompted more discussion than others: during some interviews only three episodes were discussed, while during others twice as many were discussed. In total, 50 episodes were used in the analysis: 16 involved correctly completed student work; 7 contained discussions of pre-existing errors; 5 contained an example of an emergent error; and 22 had instances of students struggling with a problem.
For each category of interaction, several kinds of episodes were used in the interviews. Selecting episodes (as well as some of the questions asked during the interviews) was based on the techniques of “convergent interviewing” (Dick, 2000). This section included episodes that were representative of the way the teacher typically interacted with students. For example, if a teacher always started a discussion by asking students to explain their solution, episodes of this practice were chosen. As data collection and analysis progressed, when the videotapes were viewed, it became possible to correctly hypothesize what reasons teachers would provide (during the subsequent interviews) for their decisions. When episodes were encountered for which it was challenging to make such hypotheses, they were also selected (outlier episodes) because this indicated that there were aspects of the teacher’s practice and beliefs that were not yet fully explained by the analysis. Episodes were also selected of times when students were being introduced to a new topic as well as when they were engaged in more complex problem solving. These content variation episodes were used generate data on how the teacher’s beliefs shaped decisions about different types of teaching practices. Situations when different groups of students were working on the same problem were captured in comparison and contrast episodes. These sets of episodes included times when the teacher interacted with groups of students in substantially different or in very similar ways. Sets of episodes when the teacher interacted with the same group of students on multiple occasions as they worked on one problem (trajectory episodes) created opportunities to ask the teacher whether his decisions in the different episodes were based on the same or different factors.

**Interview procedures**

Interviews were semi-structured, but instead of following an ordered list of questions, the discussion centered on issues the teacher raised for each video clip. There were, however, three questions that were always posed at some point during the discussion of each episode:

- **What did you want students to get out of doing the problem?** This question provided data on the objectives for the problem as Zachary saw them and helped contextualize the rest of the discussion of the video clip.
- **What happened in this episode?** This gave teachers an opportunity to “narrate” what had happened during the clip after it was played. These narratives provided the teacher’s perspective on what had occurred in the episode and helped ensure that subsequent analysis was not based only on the assumption that the researcher “saw” the same things in the episode as the teacher.
- **What were you trying to accomplish and why?** As participants narrated the video clips, additional questions were asked to probe and clarify what
they were trying to accomplish and why. They were questioned about their instructional decisions. This generated data on belief-based decisions that were closely tied to specific moments in the videotape.

Although video clip interview data were the focus of the analysis, additional data used included: classroom videotapes; transcripts of selected video clip; and field notes from all class that were videotaped.

Data Analysis

One goal of the analysis was to determine if there were consistencies in Zachary’s instructional practices either within or across the different kinds of interactions described earlier (i.e., correctly completed, pre-existing errors, emergent errors, and students struggling). Another goal was to analyze the beliefs Zachary stated when giving justifications for his instructional decisions so consistencies within and across the various types of interactions could be examined. The third and final goal of the analysis was to determine if any connections were apparent between Zachary’s consistent instructional practices and his beliefs.

Although in what follows they are described separately, these three aspects of the analysis were conducted in a highly iterative and cyclic manner, based in part on Grounded Theory (Glaser, 1992; Glaser & Strauss, 1967; Strauss & Corbin, 1990). To provide insight into the teacher’s moment-to-moment practices, methods were used to parse video clip transcript data and identify factors shaping the teacher’s decisions (Aguirre & Speer, 1999; Schoenfeld, 1998, 1999, 2000; Schoenfeld, Minstrell, & van Zee, 1996) Analysis of the interview data was used to understand the practices and the practices were examined in conjunction with the interview data. This cyclic process was used to develop an understanding of the explanations Zachary gave for his decisions as captured in the video data.

The three components of the data analysis are described in what follows. Each section is meant as an overview of the process. Additional details and illustrations are provided in conjunction the findings that are presented in later sections.

Analysis of beliefs

To provide context for the detailed analysis of classroom episodes, a profile of Zachary’s beliefs was constructed from analysis of the interview data. This profile captured major aspects of Zachary’s beliefs, based on analysis that used iterative open coding and axial coding (Strauss & Corbin, 1990), with triangulation from the episode analysis. Efforts were made to ensure representativeness and importance in the following ways. Beliefs included in the profile were ones that appeared in varied contexts (because beliefs seen across many contexts were presumed to
be most prominent or significant in the teacher’s overall set of beliefs). Second, the list of beliefs was examined in the light of the completed episode analyses. The belief profiles informed the more detailed analysis of the episodes and in turn, the episode analysis served as a check on the selection of beliefs from the earlier interview transcript analysis.

In addition to providing a narrative description of Zachary’s beliefs, diagrams are included that depict different parts of his belief system to help readers follow the narrative. Organization of beliefs in these representations was based on the work of Green (1971). After selecting beliefs that appeared most important (based on frequency and range of episodes in which they were stated), they were arranged in a hierarchy. The most general belief statement in a particular category was selected and then beliefs that were either logical consequences of the general belief or were examples or instantiations of the general belief were added. These representations are for rhetorical purposes only. The organization and the use of top-level categories (e.g., mathematics, teaching, learning, students) is not meant to imply the existence of any particular cognitive structure.

In the profile of Zachary’s beliefs, quotes from interviews are used to illustrate the beliefs. Because of the way beliefs were selected for inclusion in the profile (i.e., they surfaced in multiple, varied interview discussions), there were several interview quotes that could illustrate each belief. The decision of which quotes to include was based on how easy they would be for the reader to understand. In particular, quotes were selected where Zachary did not refer back to other parts of the conversation or to very specific parts of the video clips. Although these rich discussions certainly contributed to the analysis of Zachary’s beliefs, for the purposes of presentation, quotes were chosen that illustrated the beliefs such that it was not necessary to explain an inordinate number of references or details to the reader. These beliefs are summarized in the Data Case section below and the more detailed profile can be found at http://www.msu.edu/~nmspeer/Appendix/Connecting_beliefs_and_practices.pdf.

Analysis of practices and selection of “focal practices”

Recall that capturing teachers’ moment-to-moment instructional practices is central to the research design. Once captured on videotape, analysis methods were used that enable teachers’ practices and interactions with students to be represented in very fine-grained detail (Aguirre & Speer, 1999; Schoenfeld, 1998, 1999, 2000a; Schoenfeld et al., 1996). This approach uses empirical methods for parsing transcript records of classroom actions. The parsed transcript consists of a series of segments that cohere phenomenologically—they are either about the same thing or have the same activity structure (at some level of grain size). For each lesson segment, associated goals are identified and their development is
traced through the interaction. Using fine-grained analysis of videotape, a lesson is parsed into a sequence of episodes, each of which contains a coherent set of actions. Elements of the teacher’s decision-making processes are then inferred, including goals and “action plans” (which describe, at different levels of grain-size, what the teacher does to carry out his goals) that influence the moment-to-moment interactions. Each videotape was analyzed in this fashion, producing descriptions of the class activities at multiple levels of detail. These levels of analysis included the whole class session as well as the fine-grained detail of short exchanges between the teacher and students.

Analysis of the classroom video data made it possible to determine which teaching practices were used most consistently and frequently. In the case of Zachary, two such focal practices were selected. These are his questioning practices and his problem-solving support practices. These were the two instructional practices that he used consistently, in varied contexts and in the greatest number of episodes. These practices will be discussed in more detail later.

Connections between beliefs and practices

The link between Zachary’s two focal practices and his beliefs was established using the data from video clip interviews where his use of the particular practice was discussed. These data, along with the profile of Zachary’s beliefs, were used to create “arguments” for why he would be unlikely to use any of a number of theoretically plausible practices during the episode with the students. This aspect of the analysis relied on methods similar to “competitive argumentation” (cf. Schoenfeld, Smith, & Arcavi, 1993; VanLehn & Brown, 1982). Details of this aspect of the analysis (and the construct “collections of beliefs”) are presented in conjunction with the data in the next section.

DATA CASE

In this section, data are presented to illustrate findings related to the main research questions:

1. Is there a unit of analysis that captures teachers’ beliefs in sufficient detail that entailments of those beliefs are evident in specific instructional practices?
2. If such a unit of analysis is found, are there particular sets of beliefs that appear especially influential on (a particular set of) teachers’ instructional practices?
Findings indicate that there is indeed a unit of analysis where connections between beliefs and practices are evident. Specifically, there are particular collections of beliefs that are influential on teachers’ instructional decisions associated with specific practices. In this data case, two collections of beliefs are presented and their connections to two focal instructional practices are demonstrated with data from video clip interviews and from classroom video recordings.

The data presentation begins with a summary of Zachary’s beliefs. Next, Zachary’s two most prominent instructional practices (questioning and problem-solving support) are described. Subsequently, the construct of collections of beliefs is discussed. For the balance of the section, two of Zachary’s collections of beliefs are described and data analysis is presented to demonstrate the connections between his collections of beliefs and the focal practices.

Caveats

No specific evaluative claims about the merits of Zachary’s teaching practices are made in the analysis presented here. Characterizations of the nature of the interactions he had with students are presented. Determining if an approach is “good,” however, is a function of the particular goals and context of the class. Making such judgments was not part of the work in this study. The presentation does include analysis of the learning opportunities the teacher created within the context of group work where having students work on challenging tasks was a major objective. Many of Zachary’s teaching practices are consistent with goals for reform-oriented instruction and within this particular context, readers may make their own judgments about the merits of his teaching practices.

Summary of Zachary’s Beliefs

This is a general description of Zachary’s beliefs, intended to give the reader an introduction to his views and serve as background for the more detailed discussion of beliefs and practices that is the main focus of this section. His beliefs are summarized in Table 1. As mentioned earlier, a more detailed presentation of Zachary’s beliefs can be found at http://www.msu.edu/~nmspeer/Appendix/Connecting_beliefs_and_practices.pdf.

Zachary described learning as the process of “making something your own.” According to him, this occurs when one makes connections between ideas and comes to see those ideas in new ways. This happens as students work on and think about problems. This is a process that the students themselves must engage in—it is not possible for a teacher to provide the understanding to the students. Zachary also drew a distinction between not “knowing” and not “understanding.”
TABLE 1
Summary of Zachary’s Beliefs

<table>
<thead>
<tr>
<th>Students</th>
<th>Teaching</th>
<th>Learning</th>
<th>Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students should be independent, resourceful problem solvers</td>
<td>Teachers should be guides who ask students questions and scaffold problem solving</td>
<td>Learning is “making something your own,” which involves epiphanies and making connections</td>
<td>Calculus entails learning to think and acquiring skills but does not showcase the beauty of mathematics</td>
</tr>
</tbody>
</table>

He believed these are two different situations and require different instructional remedies.

Zachary believed that it was very important that his students learn to solve problems without substantial assistance. He felt this idea of “independence” was important because working in this manner was what would be expected of students in their futures. He also identified features of schooling (in general and of mathematics class in particular) that distinguish it from the “real world” that students would later encounter.

In support of “independence” and in quest of learning, Zachary believed that his primary mode of interaction with students should be asking questions. He saw his role as “guide” where his goal was to help students learn to use their resources (cognitive and others) well, to support their problem solving, and to ensure that their understanding of the ideas was strong. Zachary used questions to highlight the important ideas in the problems and to point students toward important mathematical ideas. He wanted to make sure his students really understood the ideas behind their answers. He believed it was quite possible for students to give correct answers even when they did not have a full or fully correct understanding of the underlying ideas.

Zachary believed that learning mathematics entails both the ability to solve problems and the acquisition of specific techniques to carry out that problem solving. This belief was reflected his view of the nature of mathematics. He felt that mathematics was more than just a set of skills or techniques. He believed he should teach students how to think and help them acquire calculus-specific skills. He also expressed some frustration because he felt unable to really demonstrate the beauty of mathematics through calculus.

Focal Practices

As described earlier, two of Zachary’s instructional practices were selected as the focus of the analysis. The choice of practices was based in part on how consistently he exhibited the practices in different contexts and in part on how frequently he
used them overall. Zachary’s most common and frequent way of interacting with students was to ask questions. He responded to nearly everything students said by asking them a question to help them think through the ideas. For the purposes of this article, questioning practices are defined to mean both when he chose to ask questions and the particular nature of the questions he asked. Zachary also had particular ways of providing support for students as they worked on problems. These problem-solving support practices included the kind of guidance he gave when students were trying to make sense of a problem or struggling as they were producing a solution. Examples of these practices and more detailed discussion of their features are provided later in conjunction with the analysis of various classroom episodes.

Collections of Beliefs Construct

A main claim of this article is that there exists a unit of analysis for beliefs that makes it possible to connect teachers’ beliefs to their moment-to-moment decisions and instructional practices. This unit of analysis is a collection of beliefs. Based in part on findings from other research (Aguirre & Speer, 1999), a collection is defined to be a small set of related beliefs that, in combination, describe a teacher’s perspective on a particular topic. For example, when Zachary explained decisions he made during class, there were several sets of beliefs that he referenced very frequently and consistently as the basis for many decisions he made.

Additional details about this construct will be discussed later, but for now, some general characteristics are provided. A collection may contain one or more beliefs. These beliefs may come from one or more categories typically used to describe beliefs (e.g., teaching, learning, students, and the nature of mathematics). Combining beliefs in this way makes it possible to describe a teacher’s perspective in a manner that reflects the interconnected, distributed nature of beliefs. For example, disentangling beliefs about teaching from those about learning presents methodological and analytical challenges. The resulting separate descriptions may in fact fail to fully capture the complexity of the teacher’s views. Collections of beliefs, in contrast, allow the researcher to describe several beliefs at once, preserving the integrity of the individual beliefs when possible while simultaneously creating a unit of analysis that acknowledges the related and interconnected nature of beliefs.

Zachary’s First Collection of Beliefs: Criteria for Evidence of Student Understanding

In this section, data and findings are presented related to one of Zachary’s collections of beliefs and its connections to one of his most prominent instructional practices (his questioning practices). This collection of beliefs relates to Zachary’s
criteria for evidence of student understanding and is defined by the following questions:

- What is taken by the teacher as sufficient evidence that students understand an idea or problem?
- What explanations does the teacher generate when students are unable to produce evidence of understanding (either when they do not respond or respond incorrectly)?

Zachary’s collection of beliefs about what counts as evidence of student understanding:

- Students understand if they are able to explain their answers and respond to follow-up questions.
- If they cannot answer, the students do not understand the ideas. If they answer incorrectly, they neither understand the correct ideas nor the ideas related to their incorrect answer.

These beliefs are illustrated later with interview excerpts and the examples presented are representative of the reasons Zachary gave for his decisions across the various modes of interaction.

Zachary believed that even when students were able to state an answer correctly, they may or may not understand the ideas. He believed students were capable of recalling words associated with the problem even when they really did not fully understand the concepts. When a student answered, Zachary believed further investigation was called for to determine if the student understood the meaning behind their answer.

If students were unable to answer, Zachary took that as evidence they did not understand the ideas. He also believed that being able to explain an answer was important and demonstrated understanding of the ideas. When students made incorrect statements, Zachary believed this indicated two things. First, it meant they had failed to understand the correct idea. Second, it indicated their incomplete understanding of ideas related to the incorrect answer. For example, suppose a student gives “points of inflection” as the answer to a question when the correct response is “critical points.” Zachary believes that means the student does not understand critical points and does not fully understand what points of inflection are because if they did, they would not have given that as an answer inappropriately.

Zachary made a clear distinction between circumstances when students did not understand and when they did not know something. If students did not know a definition, theorem or other fact, he would either tell them to look it up in the textbook or he would provide the information for them. On the other hand, when they did not understand a concept, he asked questions to help them come to understand without telling them the answer explicitly.
Zachary’s beliefs about students’ understanding surfaced in many of the interviews. For example, when asked about his practice of asking questions he made an analogy to training monkeys:

Zach . . . because, you know, students, students can think and they’re also like monkeys. I mean, you can train them to ring the bell to get the treat.

Int. Um huh.

Zach So you can train them that when they see certain words that they do certain things and they may have no idea what it really means.

Int. Un huh.

Zach So I don’t want to train them. [2.7.2–7]

Instead of training them to respond in a particular way, Zachary wants to ensure that his students understand what they are doing and why.

When students answered, Zachary did not take that as sufficient evidence they understood and he always asked them to explain or provide an elaboration of their answer. Here, again, he said students could give answers correctly without actually understanding and described how he used his questioning technique to guard against this situation:

Int. . . . it seems like often your mode is they say something like “It’s about critical, well it’s about mins, local extrema” or whatever he said, and then you turn around and you say, “Well, what is a local extrema?”

Zach Um huh. Often I do that.

Int. And you’re doing that because you want to, I mean they could say anything . . . why are you choosing to do that, why are you choosing to turn it around and ask them what it is that they’ve just said?

Zach To see if they had an understanding or if they were just parroting something,

Int. OK, OK, and what are you then doing?

Zach I mean somehow we’re building a house together and they’re providing like, I don’t know, the two-by-fours. I’ve got to knock on them and see if they’re solid. Because I don’t want our house to fall down. [2.11.13–22]

Zachary also believed that if students were unable to answer his questions, or answered incorrectly, then they were lacking some understanding of the ideas. When asked why it was that students were unable to answer a particular question,
his responses were almost always the same: the student had yet to build the appropriate connections among the ideas for themselves. Sometimes he attributed an error to faulty recall of a fact, but he appeared to believe the primary explanation for an incorrect or non-answer was a lack of understanding. Here he described how he inferred from the students’ incorrect answer that they had not understood all of the concepts in the problem:

These guys drew these graphs and they were kind of right, but they aren’t, weren’t thinking like, “Ah first derivative test, this is about testing critical points.” And so they didn’t actually realize that in each case there should be a critical point. So what I was trying to do was, first of all, get them to realize that the first derivative test is for testing, determine, sort of categorize critical points. And that, therefore, in these four cases there has to be a critical point everywhere. And I could tell that they hadn’t made this clear because, their plus/plus and minus/minus graphs didn’t have a critical point. [2.2.17–22]

Because the students’ graphs were incorrect, he concluded that they had not internalized the ideas related to the first derivative test.

Episode Analysis for First Collection of Beliefs

Some preliminaries

To illustrate the role beliefs played in shaping teacher–student interactions, four different categories of teaching episodes were analyzed: completed work, pre-existing error, emergent error, and stuck or struggling (refer to the left hand side of Table 2). These four modes of interaction were representative of the teacher’s practices, captured the range of interactions he had with students, and are of the sort that are apt to shape students’ opportunities to learn. The episodes chosen for analysis are not meant to be an exhaustive catalog of the teacher’s practices. For example, teachers have routines for announcing the problems for the day and for handling students who arrived late for class. There is probably consistency in the decisions teachers made in these situations, but the content of the interaction did not seem mathematically meaningful or likely to shape the students’ learning opportunities in significant ways. Examples for each mode of interaction are presented, two for each of Zachary’s collections of beliefs that are discussed.

Ideal knower

In conjunction with the description of the task the students were working on in the episode, there is a discussion of what one ideally might want students to
<table>
<thead>
<tr>
<th>Mode of Interaction</th>
<th>Generic Plausible Teacher Responses</th>
</tr>
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</table>
| **Correctly completed work:**  
The work students have produced is complete (they are done with a problem or part of a problem) and is correct. | a) Tell students their answer is correct and do not discuss the answer/solution.  
b) Tell students their answer is correct and review their answer/solution.  
c) Tell students their answer is correct and ask them to explain their answer/solution.  
d) Ask students to explain their answer/solution without evaluating its accuracy directly.  |
| **Emergent error:**  
The work students have produced contains an error that is noticed by the teacher in the midst of the discussion or an error is introduced during the discussion. | a) Tell students they have made an error and tell them how to correct it.  
b) Tell students they have made an error and ask questions that direct them to correct it.  
c) Ask students about their work and ask questions to guide them to seeing and correcting their error.  |
| **Pre-existing error:**  
The work students have produced contains an error that is noticed by the teacher prior to initiating a discussion about their solution. | a) Tell students they have made an error and tell them how to correct it.  
b) Tell students they have made an error and ask questions that direct them to correct it.  
c) Ask students about their work and ask questions to guide them to seeing and correcting their error.  |
| **Struggling or stuck:**  
Students may or may not have completed some work. They are unable to make progress or are having difficulty responding to the teacher’s questions during a discussion. | a) Tell students what they are doing wrong and/or tell them what they need to do next to solve the problem.  
b) Tell students what they are doing wrong and ask questions that guide them to seeing what they need to do next to solve the problem.  
c) Ask students about their work and ask questions that guide them to seeing what they need to do next to solve the problem.  |

Think about and learn from doing the task. These discussions are not meant to represent an exhaustive list of all things related to the problem. Instead, they are meant to characterize the set of ideas that one could expect students to be capable of thinking about and understanding given where they are in the course and their study of the particular topic.

**Decision points**

During an episode, a teacher is faced with making many decisions, including deciding when and how to begin a discussion with a group of students. Once the
interaction is underway, the teacher must make a decision at each turn of talk about what, if anything, to say in response to what the students have done or said. At some point, the teacher must also make the decision to leave the group. In the data and analyses presented next, certain decision points have been selected for examination. Each turn of talk is not discussed, but instead the presentation focuses on decisions that either lead to a shift in the ensuing discussion or were in reaction to a significant response from students, and appeared to have the potential to shape the nature of the students’ learning opportunities.

**Plausible alternatives**

To illustrate the potential influence of particular collections of beliefs on Zachary’s decisions, a set of decisions a teacher might have made in the same circumstances is described. Table 2 contains a description of each mode of interaction and general descriptions of some high-level plausible teacher responses.

For each mode of interaction, the order of the plausible alternatives corresponds (roughly) to decreasing specificity. For example, in pre-existing error interactions, a teacher might tell students they have made an error and tell them how to correct it. A less specific (or directive) response would be to ask the students about their work without evaluating their answer and then guide them to see the error on their own. This decreasing specificity corresponds to increasing opportunities to access students’ ideas and understandings. Specific plausible alternatives to what Zachary did at various decision points are presented and potential learning opportunities generated by the different responses are discussed for each episode.

**Some comments about the mathematics**

The data come from a first-semester calculus course for physical science and engineering majors. While having some familiarity with calculus concepts may make reading the transcripts easier, it is not necessary to have such knowledge to follow the analysis. The transcript data are used to illustrate claims about Zachary’s questioning and problem-solving support practices. Although it is certainly the case that these practices are shaped by the content of the course and Zachary’s knowledge of mathematics, the claims made about Zachary’s beliefs and practices are not particular to the specific topics in the tasks.

**Correctly completed work episodes**

In this kind of episode, students have correctly completed a problem or a sub-section of a problem before the interaction with Zachary begins. Such a discussion can provide opportunities for rich examination of mathematical ideas,
there are, however, many plausible ways a teacher might handle these situations. For example, a teacher might choose to discuss students’ solution in some circumstances, but not in others. In some cases, the teacher may read over the solution and let the students know it is correct. Alternatively, the teacher may ask the students to present the solution before providing the evaluation. In addition, the teacher may or may not ask follow-up or extension questions. These discussions are one way students have access to what is important about the problem or the concepts contained in the problem and they provide opportunities for students to solidify their problem-solving techniques. During the discussion, the teacher can, in theory, note especially important techniques, highlight especially significant concepts, or draw students’ attention to potential stumbling blocks.

Next we see the type of discussion Zachary had with students about correctly completed work.

**Task and ideal knower description**

<table>
<thead>
<tr>
<th>a. State the two limit definitions of the derivative.</th>
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<tbody>
<tr>
<td>b. Draw diagrams to represent each of these definitions and label them appropriately.</td>
</tr>
</tbody>
</table>

In part (a) of this question, students are asked to state the two limit definitions of the derivative. These two definitions are:

\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]  
(1)  
\[ f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \]  
(2)  

In addition to being able to write the two definitions accurately, one would like students to understand their similarities and differences. One would like them to understand that two different algebraic statements can represent the same limit. One would also like them to understand the specifics of how the definitions are different. In the first one, the variable \( x \) is approaching \( a \), where \( a \) is a value on the \( x \)-axis and \( x - a \) represents distance. In the second definition, the variable \( h \) is used to represent the distance and \( h \) is made to approach zero. Despite these differences, the two definitions are just different ways of representing the same quantity: the slope of secant lines as two points on the graph get closer and closer together. One would also like students to understand the difference between what is represented “inside” the limit (i.e., the slope of a secant line) and what is represented by the entire limit statement (i.e., the slope of the tangent line at a point).
In part (b), the students are asked to illustrate the two definitions with appropriately labeled diagrams. Both representations show the slope of a secant line as it approaches the slope of the tangent line. The way the limiting value is represented, however, is different in the two diagrams (see Figure 1).

As with part (a), one would like students to understand both how these representations are similar and how they are different. One would like them to understand how all the components of the symbolic definitions from part (a) can be represented graphically.

**Episode summary and existing student work**

In this episode, the students had completed both parts of the task correctly and the limit expressions and diagrams were written on the board. The episode began with Zachary asking one of the students to explain the first part of the problem. After the student explained, Zachary asked several follow-up questions and other students joined the discussion. Zachary continued to ask questions as they discussed the second part of the problem.

**Transcript**

01 Zach OK, so somebody tell me about, Jenny you want to tell me about part a?
02 Jenny Part a? Um, these are the limit definitions.
03 Zach Ok, so what’s the difference between them?
04 Jenny Um, this one, the movable points are just labeled differently.
05 Zach OK. What is, so in the first one you’re thinking about the point x going towards a?
06 Jenny Yeah.
07 Zach And what are we thinking about moving in the second one?
08 Jenny The point x
09 Alexa h toward zero.
10 Zach Yeah, what does h represent?
11 Jenny Distance between \( x \) gestures with fingers
12 James Distance between the two points [gestures with fingers]
13 Zach Between the two points, right. So it’s exactly the same definition they’ve just got
14 different names. Ok how about part (b). What to tell us about part (b) Keri?
15 Keri Yeah. It’s graphically showing
16 Zach Ok.
17 Keri this is y, delta y, this is delta x [pointed to corresponding parts of the diagram]
18 Zach And
19 Keri this is your slope.
20 Zach Ok, and right, what is the ratio inside the limit, what does that represent on the graph?
21 Keri Slope
22 James Slope
23 Zach Slope, OK, slope of what?
24 James Tangent line.
25 Zach Is it
26 James Tangent to the graph at that point [gestured to indicate tangent to the curve]
27 Zach Ok, so is the slope, is the tangent, is the slope of the tangent before you take the limit or after?
28 Keri After
29 James After
30 Zach OK, so what is it before you take the limit?
31 Keri It’s the secant.
32 Alexa Secant.
33 Zach Secant lines, right. So, you guys are great. OK, that looks really good.

**Discussion**

In this episode there were many places where Zachary had to decide how to react to the students. The main decision points occurred at lines 1, 3, and 13. In line 1, Zachary asked Jenny to explain the first part of the question. Given that Zachary could see the students’ correct work on the board, there were several

\[2\text{Underlined transcript from two different people indicates that they were speaking simultaneously.}\]
alternative ways he could have handled the situation. He could have looked at the
work and told them it was correct. He could have said it was correct and provided a
summary or elaboration. He also could have told them they were correct and then
asked them to explain their answers, with or without asking follow-up questions.
Zachary, however, asked them to explain the problem without telling them whether
or not the answer was correct.

Zachary’s questioning practices in this situation are closely connected to his
beliefs about what counts as evidence of student understanding. Recall from the
earlier discussion of Zachary’s collection of beliefs that when students’ work was
correct, Zachary did not take that as conclusive evidence that they understood the
problems and related concepts. Recall also that “questioning practices” involve
both when a teacher decides to ask questions as well as the nature of those
questions. Given his beliefs, at this decision point it is highly unlikely that Zachary
would have looked at the students’ answers without also questioning them about
what their work meant. Here we see how having information about his collection
of beliefs enables us to eliminate many of the “plausible” alternatives and to
explain his decision to question the students at this point. Note that having general
information about his beliefs (as was presented earlier and summarized in Table 1)
does not make it possible to explain which of the alternatives Zachary is most likely
to pursue. Most of the proposed alternatives could be considered compatible with
the general characterization of his beliefs. At that level of detail, it is not possible
to make accurate predictions about what he would do at these decision points,
but with the finer-grained description in the collections of beliefs, the connections
between his decisions and his beliefs are more evident.

As the discussion continued, Zachary proceeded in a similar manner and asked
follow-up questions to nearly all of the students’ responses. After Jenny responded
to his initial question by saying they had written the limit definitions, Zachary asked
what the difference was between the two definitions (line 3). As with his initial
question to the group, Zachary had several other plausible ways of responding
to Jenny’s answer at this decision point. He could have told them the definitions
were correct and moved on to the second part of the question. He could have told
them they were correct and then provided some elaboration. Because we know
from his collection of beliefs that Zachary did not take the students’ use of the
correct words as evidence of understanding, in this context he did not assume that
the students understood the difference between the two definitions merely because
they were able to produce them. Consequentially, it is unlikely that he would have
decided to end the discussion at this point.

A very similar pattern of questions occurs throughout the rest of the episode.
To probe their understanding, Zachary asked several detailed questions about the
variables they used. In line 5, Zachary offered a bit of information about the
labeling of the points and then asked related questions in lines 7 and 10. He asked
them to explain what other variables represented and how they appeared in the
diagrams. He asked them to explain their answers or to respond to a follow-up
question during almost all of the turns of talk. After he provided a short summary (line 13) of part (a), he asked a student to describe part (b) of the question. He worked through part (b) in a similar manner, asking follow-up questions in lines 20, 23, 27, and 30.

Although the students had a complete and correct solution written on the board, Zachary discussed many aspects of the problem with them. A teacher with a different collection of beliefs about student understanding is likely to respond in a different manner. For example, if a teacher takes students’ abilities to write or say answers as evidence of understanding, there would be no need to question students about their answers to determine if they understood the ideas. In the course of the discussion, Zachary’s students not only learned that their solution was correct, but also had opportunities to enrich their understanding of a variety of concepts related to the limit definitions of the derivative.

In the next section we see an additional example of how Zachary’s collection of beliefs about student understanding was evident in his decisions.

Emergent error episodes

In these episodes, an error came up as Zachary and the students discussed a problem. As with other kinds of episodes, how a teacher addresses an emergent error can shape the opportunities students have to learn during the interaction. There are many potentially plausible ways teachers might handle such situations. Some teachers, in some situations, might choose not to address the error at all. Alternatively, if a teacher does notes the error, he or she might tell the students they have made a mistake or engage them in a discussion about their reasoning. If a teacher brings the error to the students’ attention and discusses the ideas, this could give students an opportunity to see the source of their error as well as to deepen their understanding of the related concepts. If a teacher only points out the error, it is possible that students are less likely to have the opportunity to internalize the metacognitive skills that will help them sort out similar difficulties in the future. Engaging students in a discussion about their errors may also give them an opportunity to deepen their understanding of ideas, even if they are not the concepts directly related to the problem at hand.

Task and ideal knower descriptions

When can’t you apply the second derivative test? Give examples of $f$ and $c$ when the second derivative test doesn’t work, where

(a) $f(c)$ is a local maximum and $f'(c) = 0$.
(b) $f(c)$ is a local minimum and $f'(c)$ does not exist.
(c) $f''(c) = 0$ but $f(c)$ is not a local extremum.
This task is designed to help students with two concepts. First, they should develop a more robust understanding of the second derivative test by examining some conditions under which it cannot be applied. This problem also gives students the opportunity to construct functions to meet specific criteria. In doing this problem, one would like students to develop a better understanding of how to construct functions from parameters as well as a better sense of what functions look like in situations where the second derivative test cannot be applied.

**Episode summary and existing student work**

In this episode, we see how Zachary responded when a student made an incorrect statement in the midst of a discussion. Recall that Zachary believed different responses were called for when students did not understand and when they did not know something. In this instance, the students were mistaken about the definition of the second derivative test. They thought the test had to do with points of inflection when in fact it involved classification of critical points. After the students answered Zachary’s questions about points of inflection correctly, he told them that they were mistaken about the second derivative test.

**Transcript**

01 Zach  So, so, what’s this problem about?
02 Dave  Second derivative
03 Liz    Second derivative
04 Zach  OK and what is that for?
05 Liz    For points of inflection.
06 Zach  OK, what’s a point of inflection
07 Liz    Where it changes
08 Nicole Where it changes from positive to negative.
09 Zach  OK, and so how do you find points of inflection?
10 Nicole The fact that the first, second derivative equals zero.
11 Zach  OK, and what do you need to know about the second derivative in order to be sure that that will?
12 Nicole It has to be continuous.
13 Zach  Yeah. If the second derivative is continuous you can find out where it changes sign by setting it equal to zero.
14 Zach  So everything you just said is true, but it’s just not what’s called the second derivative test.
15 Liz    Oh, OK.
16 Zach So what is the second derivative test?
17 Liz Well, that was our answer.
18 Zach What, what, what’s the first derivative test?
19 Liz Ummm.
20 Nicole [inaudible] where there’s a maximum
21 Zach Uh huh, this is classifying critical points to see if they are maxs or mins. The second derivative test is another way to do that using the second derivative. Does that ring any bells?
22 Nicole Yeah.
23 Zach Did, did any of you come to lecture yesterday?
24 Dave When you have a local maximum or local minimum.
25 Zach Yeah, if you have, if you have a critical point you can try to use the second derivative to try to tell if it’s a max or a local min.
26 Dave So
27 Liz At the critical point
28 Dave OK.
29 Zach So what do you think happens to your critical point and the first, second derivative is positive?
30 Dave It’s concave up.
31 Zach OK. So what kind of critical point is it?
32 Dave and a local minimum.
33 Zach So it’s a local min. And if the second derivative is negative?
34 Nicole Then it’s concave down.
35 Zach And what kind of critical point do you have?
36 Nicole Local max.
37 Zach OK, great, so this problem is about when that test fails, like the second derivative might be zero and you can’t tell what kind of critical point that is. So they’re asking you to come up with examples of it failing in these three ways.

Discussion

In this episode, the students’ incorrect understanding of the second derivative test became apparent when Liz said that the test was for points of inflection (line 5). There were a variety of things that Zachary could have done at this decision point (and at the other main decision points that occurred in lines 9 and 16). He could have told them that they were wrong, he could have ignored what they said, he could have reflected their answer back as a question (“Points of inflection?”), or he could have made some other move to indicate that their response was not what he was looking for. From the general characterization of his beliefs, it is not clear what choice Zachary would make because all of these alternatives could be considered compatible with his beliefs in promoting independence and consistent with his more general belief in the importance of questions.
Zachary chose to pursue the students’ answer and asked, “OK, what’s a point of inflection?” As with similar situations, Zachary believed that their answer might indicate both that they did not know what the test was and that they did not understand what points of inflection were. Consistent with this belief, Zachary pursued students’ responses even when they were not correct.

During the next part of conversation, Zachary asked several questions about points of inflection. After Nicole and Liz said a point of inflection was where the sign of the second derivative changed from positive to negative, Zachary asked them how to find points of inflection (line 9). Nicole said that one needs to find where the second derivative equals zero. Zachary responded by asking about properties of the second derivative that would allow you to find points of inflection in this manner. Nicole replied correctly that the second derivative needed to be continuous. In line 13, Zachary summarized the discussion (“If the second derivative is continuous you can find out where it changes sign by setting it equal to zero”).

It is plausible that Zachary asked these questions to determine whether the source of the student’s error was a misunderstanding about what points of inflection or if they were just mistaken about the proper definition of the second derivative test. Once he had determined that they did in fact understand what points of inflection were and how to find them, he needed to address the fact that they did not know what the second derivative test was.

After telling them that their answers were correct, but not related to the second derivative test (line 14), he asked again what the second derivative test was. When an answer to his question was not forthcoming, he asked about the first derivative test. At this decision point, he could have told them what the second derivative test was, how to apply it, and what role it played in the problem. After Nicole offered information about the first derivative test, Zachary reiterated her answer and told them that the second derivative test was another way of classifying critical points as minima or maxima. During the rest of the interaction, Zachary asked them more questions until the main features of the second derivative test came to light.

This episode is representative of how Zachary handled all varieties of student errors. Consistently, his initial response was to ask students to explain their answer. As we see in this episode and as we saw in the correctly completed work episode as well, Zachary devoted considerable time to discussion of the answers and related ideas. Only when the students had realized their error or when concepts had been thoroughly discussed did Zachary move on to other aspects of the problem.

The analysis of these episodes demonstrates how Zachary’s collection of beliefs about evidence of student understanding shaped the decisions he made about when to ask questions of the students and the nature of those questions. We see that Zachary consistently chose to ask questions of the students and chose to ask questions that would reveal how they were thinking about some aspect of the
problem, whether their work was correct or incorrect. A teacher with a different collection of beliefs about what counts as evidence of understanding might not have invoked questions in this manner. In addition, armed only with a general description of Zachary’s beliefs, a rationale for his decisions would not be as evident.

In the next section we see how a second collection of beliefs connected to decisions Zachary made as he scaffolded students’ problem solving.

Zachary’s Second Collection of Beliefs: Learning and How it Happens

There were many aspects of Zachary’s beliefs about learning. From the interview data, it appeared, however, that two components were especially influential in shaping how he provided problem-solving support to his students. This is not meant to imply that other beliefs are not relevant in these contexts. In fact, readers may notice times when beliefs about student understanding also appear connected to his decisions.

For the purposes of the current discussion, the collection of beliefs about student learning is defined as:

- What the teacher thinks learning is.
- How the teacher thinks learning happens.

Zachary’s collection of beliefs about learning and how it happens:

- Learning means, “making something your own,” by forming connections between what you already know and new ideas.
- Learning happens when students form connections between ideas and have “epiphanies” about ideas and learning happens when students work on and make sense of problems for themselves.

Zachary believed that “to learn something is to encounter it and make it yours” [2.7.17]. He felt people were able to make an idea “their own” by forming connections between the new idea and what they already understood. He also believed learning entailed “epiphanies,” which he defined as the experience of coming to understand something in a new way.

Although “epiphany” seemed to define moments of significant insight, making connections seems to be the main means by which students made something their own (i.e., learned). To Zachary, just knowing disconnected pieces of information was not the same as learning:

Because what the learning is, . . . I said this before, learning something and making it your own. Learning is also making connections. I mean you’ve got
something in your head and something else in the head, you’ve go the ability, . . . you may know it, you have the ability to make the connection, but you haven’t made the connection yet. [2.25.16–20]

Zachary believed that learning (connections and epiphanies) could occur while students did problems and struggled with ideas, especially when they were able to figure things out for themselves. He also believed it was especially important that students make the connections between ideas for themselves. He believed he could not make the connections for them. Therefore, his role was limited to creating opportunities for them to form the connections on their own. In the continuation of the previous quote, he described why he asked questions and how that helped students make ideas “their own:”

So the question helps, the question is to help with that, it kind of helps catalyze that. And that’s why I [ask questions], so they make the connections themselves in their head. [2.25.16–20]

While discussing his use of questions, he reinforced this idea about students needing to make the connections for themselves:

I’m just trying to prod them a little bit. I can’t really make the connections themselves and I feel like asking them the questions, given they’ve already been exposed to the statements they’re supposed to, I mean there are these statements, you give them this lecture and you tell them stuff it’s supposed somehow directly engender the connections in their heads, but that doesn’t always work, doesn’t usually work. The questions, given that that’s already been exposed to them, that somehow that building pieces, the connections are floating around somewhere, maybe at least a little, the questions are then the most effective tool for assimilating that connection. [2.25.15–30]

When describing this process of “making something their own,” he also said “And they’ve had a chance to encounter it and maybe it’s closer to being theirs. And if I’m standing in the way to tell them . . . like I get in the way of their making it theirs” [2.7.19–20].

**Episode analysis for second collection of beliefs**

In this section, we see how Zachary’s collection of beliefs about student learning was connected to his decisions. Zachary’s decisions were consistent with his belief that learning is about making connections. His decisions also reflect the premium he placed on having students make those connections for themselves.
The questions he asked and the ways he orchestrated the discussions provided a roadmap to the big conceptual ideas in the problems and guided students toward the ideas they needed to connect.

As in the previous section, for each category of episode, the task is presented, followed by a summary of the episode and the students’ existing work, and then the relationship between Zachary’s beliefs and the decisions he made during the interaction are discussed.

**Pre-existing error episodes**

Making errors is a natural and expected part of the problem-solving process. There are, however, many possible ways that teachers may choose to manage these situations. Whether or not a teacher discusses errors and assists students in detecting and correcting them for themselves may have consequences for students’ learning and their ability to successfully handle such situations in the future. For example, if a teacher tells the students about the error, that may create a one kind of learning opportunity. Alternatively, if the teacher scaffolds the discussion and helps the students see the error for themselves, the learning opportunities might be different. If a teacher provides the kind of problem-solving support that enables students to correct their own errors, students may have the opportunity to learn some meta-cognitive skills in addition to the specific content of the problem and they may not have this opportunity if the teacher handles the situation in different, although plausible, manner.

**Task and ideal knower descriptions**

| Sketch four graphs illustrating the four cases covered by the first derivative test. (Hint: These cases can be described by the notation $++$, $+/-$, $-/+$, and $-/-$.) On each graph label the local maximum or local minimum or say that the graph has neither at the point in question. |

In this problem, students are to produce graphs that show the different cases addressed by the first derivative test. This test is used to classify the behavior of the function at its critical points. The hint refers to the sign of the slope or first derivative of the function on either side of the critical point. One of the important features of a correct solution is clear indication of the slope of the derivative being zero at the critical points.

**Episode summary and existing student work**

When Zachary approached the group, two of their four graphs on the board contained errors (see Figure 2). The graphs for the $+/-$ and $-/+$. cases clearly
showed slopes of zero at the critical number $x = 0$. The $+/+$ and $-/-$ cases were not as accurate and did not clearly show a slope of zero at the critical number $x = 0$.

In addition to this pre-existing error, during the course of the discussion a student made an incorrect statement about points of inflection. Zachary’s beliefs about what learning is and how it happens provide insights into the decisions he made. He structured the discussion to support their problem-solving efforts. In particular, the questions Zachary asked highlighted some important issues in the problem and gave students opportunities to build or strengthen connections between their existing ideas and the concepts in the problem.

Transcript

01 Zach So I have a question. What is this first derivative test about?
02 Liz Increasing, decreasing.
03 Zach Increasing, decreasing.
04 Liz The $f$ graph, so if $f(x)$, or if $f'(x)$ is positive then $f(x)$ is increasing at that point.
05 Zach OK. And what does this plus minus, minus plus, minus minus have to do with?
06 Nicole It shows like that the slope here is negative and then it goes to the positive.
07 Zach Um hum.
08 David And, at like the point where it changes or not, like a point of inflection.
09 Liz That’s the second derivative test.
10 Zach OK, so, so what is the, what is it useful, these sign patterns they’re useful for doing what?
11 Liz Um, determining what the graph looks like.
12 Zach Un huh.
13 Nicole The slope of the graph.
14 Zach OK
15 Nicole If it’s positive or negative.
16 Zach And how ‘bout
17 David And critical points?
18 Zach Critical points. OK. And so what does it help you to do with critical points?
19 David Um.
20 Nicole It tells you if it’s a critical point or a point of inflection.
21 Zach So what’s a point of inflection?
22 Liz That’s
23 David That’s the second derivative.
24 Liz where it changes from sign to sign. If it changes from positive to negative or negative to positive, that would be a point of inflection.
25 Zach So, the first deriv, this is if the first derivative changes?
26 Liz Um, the second, the second derivative.
27 Zachy The second derivative changes. So actually points of inflection don’t
28 Liz They don’t come in
29 Zac They don’t come into this picture. So, this has to do with critical points
30 and the sign patterns help us figure out? Something about the critical points?
31 David How many there are? No.
32 Liz Oh, oh, oh if it goes from negative to positive, then you have a local minimum and if it goes from positive to negative you have a local maximum. Oh, wait, wait, is that right?
33 Zach Right, that sounds good
34 Nicole And if you have positive to positive it’s just [inaudible].
35 Zach So implicitly in this problem there’s a critical point in each case. Is there a
critical, do you have a critical point in the first graph?
36 Liz No. It doesn’t look flat.
38 Zach So, implicitly what this sign pattern means is that the first derivative is positive, then zero, and then positive again. So your graph should have a critical point.
39 Liz So that graph
So what do you think, does it have one? It almost does. But maybe you can fix the graph so that all of them have critical points. [Zachary walked away.]

Discussion

During an interview with Zachary about this episode, he said he had noticed the errors in the students’ graphs before he began talking with them. He initiated the discussion by asking them about the first derivative test. There were several other things he could have done at this point other than asking this relatively open-ended question. He could have told them their graph was incorrect, he could have told them how to correct it, or he could have told them what the first derivative test as for and what the sign patterns were indicating. He also could have focused on the technicalities of the graphs and not asked about the larger conceptual issue of the first derivative test. These options, however, would not have given students as extensive an opportunity to form connections between the first derivative test and the graphs they had produced. In addition, these plausible alternatives to what Zachary did would not have provided as much of an opportunity for students to make the important connections for themselves. Thus, the other alternatives would have stood in opposition to his collection of beliefs about how learning happens. Because the general description of Zachary’s beliefs does mention connections, such a characterization does not make it possible to explain the particular decisions Zachary made about the guidance he provided to the students.

After Liz responded to his inquiry about the first derivative test (line 2, “Increasing, decreasing”), Zachary repeated her answer. Here he could have evaluated her response more directly or said how the first derivative test can be used to determine where functions are increasing and decreasing. He could have also told them how the slope of the graph is related to the notation and the graphs they had produced. As with the start of the interaction, pursuing these other alternatives would have been in contradiction with Zachary’s desire to help students build connections between ideas for themselves. By only repeating Liz’s response, he is not introducing any new information but is instead helping to structure the discussion in a way that can help students to provide additional information and ideas themselves.

In line 4, Liz continued with her answer, explaining that if the first derivative is positive then the function is increasing at that point. Zachary replied, inquiring about the +/- notation. Alternatively, Zachary could have described how to solve the problem, he could have told them more about the +/- notation, or he could have continued to ask about the first derivative test in a very general way. The choice he made, however, was very consistent with his beliefs about student learning. He wanted to help them make connections so he provided a road map to the ideas in the problem with the hope that they would see the connections.
There is an important relationship between the first derivative test and the notation in the problem statement. Zachary pointed them toward this relationship by asking a question about the notation. If he had told them how the notation related to the ideas, he would have deprived the students of the opportunity to make the connections for themselves. If he had instead just repeated his general question about the first derivative test, that also might not have helped to move them in the appropriate direction or facilitated the connection-making process.

During the next few turns of talk (lines 6–9), the students offered up various pieces of information about the meaning behind the notation. When Zachary responded, he asked questions that were even more specific about the sign patterns. In line 11, Liz said the sign patterns were useful for determining what the graph looked like. The other students contributed various pieces of information until, in line 17, David mentioned critical points. Zachary confirmed his answer and then asked about the relationship between the test and critical points. Now that the idea of critical points had been aired, Zachary could have said something directly about their role in the graphs. That choice, however, might have denied the students the opportunity to make the appropriate connections. He could have repeated his general question about what the first derivative test is for, but that too might not have put the students in a better position to make the connections between the ideas themselves.

Nicole replied to Zachary’s question, stating incorrectly that it helped you tell if it was a critical point or a point of inflection. In response to Nicole’s incorrect comment, Zachary asked what points of inflection were. He could have told them that they were wrong and that points of inflection had nothing to do with the problem. He could have ignored what Nicole said and just continued with the discussion. Had Zachary pursued either of these options, they would not have had the chance to connect what they were currently thinking (incorrectly) with the ideas in the problem. If he had not taken up the issue at all, they would not have had the opportunity to try to make sense of the situation for themselves and use their own resources to get themselves out of the confusion.

In lines 22–24, David and Liz sorted out some of the confusion and, after an additional question from Zachary to which Liz responded, Zachary repeated Liz’s correct answer. He summarized her answer, saying it meant points of inflection were not relevant in the situation. He also reiterated the main point, stating that the problem had to do with critical points (lines 27–29).

Because the students still had not articulated the final connection between the classification of critical points and their graphs, Zachary asked if the sign patterns helped them figure out something about the critical points (line 30). After the students provided an appropriate answer, Zachary summarized the discussion, saying that “implicitly in this problem there’s a critical point in each case.”

Next, Zachary asked if they had a critical point in their graph. Liz said they did not because it was not flat. Zachary provided more of a summary, highlighting
the central connection and then suggested that they fix their graphs so all would represent critical points. He could have chosen to not provide this kind of summary or connection to the problem at hand. He could have just left things as they were, not having the students correct their graphs. Because he thinks connections are what learning is, it is unlikely that he would have left the discussion without tying together what was discussed in the problem. He left them to carry out the last step and they were now in a position where they were likely to make progress while understanding what they were doing.

In this episode, we see how Zachary directed the students toward the central issues captured by the problem and highlighted the ideas they should consider as they worked. He provided summaries of important ideas at various points. He left the group when they appeared to be in a position to make the necessary corrections to their graphs and to understand what the corrections meant in terms of the big ideas from the problem. In the next section we see how Zachary’s collection of beliefs shaped the decisions he made about problem solving support when students were having trouble with a problem.

Students stuck and/or struggling episodes

When students are unable to make progress on a problem or are struggling in some way, different teachers may react differently. For example, if students are struggling, some teachers might leave them to sort out the difficulty on their own. Other teachers might decide to tell them how to solve the problem or to assist them in solving the problem. These alternatives provide students with opportunities to learn different things. In addition, if a teacher does provide students with problem-solving support, it can come in a variety of forms. Some forms may highlight the relevant problem-solving skills and others may not. For example, if a teacher consistently tells students how to resolve confusion or how to move beyond an impasse, students may not have the opportunity to develop their own skills for dealing with such situations in the future. Alternatively, if a teacher consistently leaves students to struggle for prolonged periods of time or does not provide sufficient guidance, the students may be unable to make progress, may not understand, and may become discouraged.

In this episode, we see how Zachary’s beliefs about learning were apparent in how he responded when students were stuck and struggling. In particular, we see how his belief that learning is about students making connections for themselves is connected to the kind of scaffolding and problem-solving support he provided.

Task and ideal knower descriptions

Note: This is the same task Zachary’s students were working on in the Emergent Error episode. See that earlier section for the ideal knower discussion.
When can’t you apply the second derivative test? Give examples of \( f \) and \( c \) when the second derivative test doesn’t work, where

(a) \( f(c) \) is a local maximum and \( f'(c) = 0 \).
(b) \( f(c) \) is a local minimum and \( f'(c) \) does not exist.
(c) \( f'(c) = 0 \) but \( f(c) \) is not a local extremum.

**Episode summary and existing student work**

The students had been working on part (a). They had been unsuccessful in finding a function and point where the value of the function at that point was a local maximum and the second derivative was zero. They had tried second- and third-degree equations and both had failed to meet the criteria from the problem statement. Zachary helped them work through these examples and then left them to investigate an additional case.

**Transcript**

01 Zach  So did you guys finish problem 2?
02 Liz    We can’t.
03 Zach  Problem 1 rather.
04 Liz    We can’t.
05 Zach  You can’t do problem 1?
06 Liz    We understand, we understand like the concepts behind it
07 Zach  OK
08 Liz    but we don’t know how to answer the problem, like what they’re asking.
09 Zach  Yeah, it’s a little tough, so, let’s do part (a).
10 Liz    Ok.
11 Zach  What’s a function with a local max?
12 David [Offered the chalk to Zachary]
13 Zach  No thanks, I’m trying to cut down.
14 Liz    OK. We have a local max.
15 Zach  What’s the simplest function you know with a local max?
16 Liz    \(-x^2\).
17 Zach  OK. Great.
18 Nicole [Drew \(-x^2\)]
19 Zach  Now does it satisfy part (a)?
20 Liz    Local, um, f
21 Zach  What’s the critical point here?
22 Liz    Zero.
23 Zach When x is zero. And what’s the second derivative when x is zero?
24 Lizy Zero.
25 Nicole Zero.
26 Zach Is that true?
27 Liz No wait, no wait.
28 Zach Suppose this is like the parabola –x^2.
29 Nicole [inaudible]
30 Zach What’s the second derivative?
31 Liz –2.
32 Zach –2. Is that a zero?
33 Liz No.
34 Zach Didn’t work. How could we change so that the second derivative would be zero?
35 Liz We need a 2x.
36 Zach OK, how can you get a 2x as the second derivative?
37 Liz We need a cubic function.
38 Zach OK, so why don’t you try
39 Liz We had that.
40 Zach What happens to a cubic function though?
41 Liz Um.
42 David You don’t get a local max.
43 Zach You don’t get a local max but you do, do you get the second part?
44 David Yeah.
45 Liz Yeah.
46 Zach Yeah. So for x^2 you got the first part, but not the second. For x^3 you got the second part but not the first. So how, how do you think we can get both?
47 Liz x^{3/2}? [laughs]
48 Zach 2, 3, 3/2, that’s a funny sequence.
49 Liz I’m sorry, um.
50 Zach 2, 3, 4, well you could try 4, maybe that’ll work.
51 Liz Ok, let’s try 4.

Discussion

When Zachary asked the students if they were done with the problem, they said they were struggling (line 4). He began to help them think about the desired characteristics of the function and point in line 11. He started by asking them for a function with a local maximum. After Liz suggested –x^2 (line 16), Zachary asked if that satisfied all the conditions from part (a). This provided students with opportunities to connect the question he asked with the original statement of the problem. The questions Zachary asked in this opening section of the
discussion assisted the students in thinking through an example that satisfied half the constraints in the problem.

When the students did not appear to know if all the conditions had been satisfied, he asked about the location of the critical point (line 21). Liz gave the correct critical point (zero). Then, after some slight confusion, they arrived at the correct answer of –2 for the second derivative (line 31). Zachary’s decisions during this interaction lead them to conclude that their example did not satisfy all of the necessary conditions.

Once the fact that their equation failed to meet the problem’s criteria had been aired, Zachary asked them how they might change their answer so the second derivative would be zero (line 34). Again, here and during the subsequent turns of talk, he chose to emphasize the potential connection between what the students had done and the objective of the problem. Liz said they needed it to be 2x and Zachary asked how they could obtain 2x as the second derivative (lines 35–36). She said they needed a cubic function. Zachary confirmed her answer and started to make a suggestion when Liz interrupted (line 39) and said they had already tried a cubic function.

In line 40, Zachary asked what had happened when they tried the cubic function. David said it had not given them a local maximum. Zachary repeated David’s answer and asked if they did get the second condition (f''(c)=0). Both David and Liz said they had. Then Zachary reviewed what they had just discussed and asked how they might get both features in their example. As in the previous exchanges, Zachary was building on the students’ existing work. He summarized the discussion, highlighting key features, and then restated the question the students should think about. Again, Zachary’s decisions were consistent with his beliefs about learning being connections and his desire to help them make the connections for themselves.

Near the end of the episode, Liz, almost jokingly, suggested they try $x^{3/2}$. In response, Zachary listed the exponents they had used, plus the one Liz had just suggested, and said they made a “funny sequence” (line 48). After suggesting that they try 4 next, he left the group. In response to Liz’s suggestion, Zachary modeled how they could think through the examples they had already tried and rely on the pattern in the exponents to generate a plausible next guess.

During this final exchange, Zachary did not focus on the central conceptual issues to the same extent he had at other points. His suggestion to try $x^4$ was tied to the pattern in the sequence of exponents and not to the underlying features of exponential functions. Although this choice might not have provided the students with extensive opportunities to learn about characteristics of exponential functions, he did tie his suggestion to the work that the students had already completed. In addition, he modeled a useful problem-solving technique of noticing and utilizing patterns.
CONCLUSIONS AND IMPLICATIONS

Return to the Original Research Questions

The main motivations for this research stemmed from the disconnect that exists between research on teachers’ beliefs and research on reform-oriented professional development programs. In particular, the general nature of the characterizations of beliefs and practices and the typically correlational findings from such research provide little insight into connections between beliefs and practices at levels of detail where development in practices appears to occur most productively.

In this study, methods were used that permitted access to beliefs and practices in fine-grained detail. The work confirmed the hypothesis that a unit of analysis could be chosen that was sufficiently detailed to make it possible to examine the role of beliefs in shaping teachers’ decisions as they relate to specific instructional practices. In particular, the construct “collections of beliefs” was proposed as a unit of analysis for beliefs that captures teachers’ views about particular issues in ways that make it possible to understand, explain, and even predict teachers’ decisions.

The second research question concerned whether particular collections of beliefs appeared to be especially prominent in shaping teachers’ practices. In the case presented here, there were certain practices the teacher used very frequently and two collections of beliefs were especially influential on those practices. In particular, beliefs about what counts as evidence of student understanding were important in the teacher’s decisions about when to ask questions and what kinds of questions to ask, and beliefs about learning shaped how the teacher provided support and guidance to students as they were solving problems collaboratively.

Collections of Beliefs as Useful Unit of Analysis

It might be possible to “understand” Zachary’s teaching practices in a very general way through a top-level characterization of his beliefs and practices. For example, Zachary indicated that he valued questions and he did in fact use questions extensively in his classroom. If one analyzed only whether he used questions, one could potentially conclude various things about the relationship between his beliefs and practices. This description, although accurate, is hardly the whole story. Much more became evident from a closer examination of his beliefs and his teaching practices.

Through the examination of moment-to-moment teaching interactions across a range of modes of interaction, Zachary consistently justified his decisions by invoking certain beliefs. No single belief or typical category of beliefs (e.g., teaching, learning, students, or the nature of mathematics) could be used to accurately
characterize the rationale he gave for his decisions. Instead, his justifications made reference to certain *collections of beliefs* that spanned several of the categories typically found in research on beliefs. For example, when discussing decisions he had made about when and what kinds of questions to ask, Zachary made frequent reference to beliefs about what counted as evidence that students understood the ideas and problems being discussed. This collection of beliefs about *criteria for evidence of student understanding* consists of what he took as sufficient evidence that the students have understood an idea or problem. This collection of beliefs also reflects the explanations he generated when students are unable to produce evidence of understanding. He was not inclined to give his students the benefit of the doubt that they understood—he wanted strong evidence. He consistently asked students to explain what they meant by terms they used or answers they gave and he took students’ inability to answer questions as a sign that they did not understand the ideas. Although more general descriptions of Zachary’s beliefs might have provided some insight into some aspects of his practice, broad characterizations typical of teacher belief research would not provide explanations for the particular moment-to-moment decisions he made.

Analysis also revealed how a different collection of beliefs appeared to be related to how Zachary provided problem-solving support and scaffolded student work. Zachary provided his students with guidance in solving the problems and had students produce most of the solutions on their own instead of presenting information directly to them. In Zachary’s class, discussions frequently centered on conceptual issues related to the problem and how those issues were related to one another. He often pointed out particular approaches to a problem and highlighted relevant aspects of meta-cognition. From analysis of the reasons Zachary gave for decisions, beliefs about student learning were reflected in his responses. In particular, his rationales made frequent reference to *what he believed learning was and how he believed learning occurred*. Zachary believed that learning entailed “making something your own” by connecting new ideas to ideas that were already understood. He believed this could only happen if students worked through problems and made those connections for themselves. Again, while more general descriptions of Zachary’s beliefs could play a role in understanding some aspects of his practice, such characterizations would not provide significant insight into the decisions he made in interaction with students.

**Implications for Theory and Methods**

Much research on beliefs is based on the premise that there are underlying, basic connections between teachers’ beliefs and their instructional practices, yet these presumed connections are rarely the object of direct, specific investigation. Researchers have found, for instance, both consistencies and inconsistencies between
beliefs and practices, leading some to question the existence of beliefs and/or their role in shaping teaching practices. Findings from the current research suggest, however, that seemingly inconclusive or contradictory findings may be an artifact of the methods used. It is plausible that the presumed or elusive connections are actually visible at fine-grained levels of detail even if they are not evident with general/broad characterizations of beliefs and practices.

Having evidence that these connections do exist at detailed levels does not imply that they do not exist at other levels of detail. The findings suggest, however, that research questions about relationships between beliefs and practices may need to be carried out in ways other than the ones that dominate this area of research if the goal is to understand the connections in ways that can advance theories of teacher cognition. Analysis at a finer grain size, conducted on interview data from discussions of specific instances of practice may be what is needed to develop rigorous explanations for classroom observational data. Using video clip interviews to elicit teachers’ thinking and reasoning about their teaching practices provides one way to access information that it not possible to obtain using typical interviews and/or observations. The resulting data are more likely to reflect the complex, context-sensitive nature of the relationship between beliefs and teaching practices that other researchers (e.g., diSessa et al., 2002; Skott, 2001; Thompson, 1992) have suggested exists.

The findings do not imply that survey and interview research that generates more general information about teachers’ beliefs is not valuable. It may be, however, that it is not productive to approach research in those ways if the goal is to examine and understand connections between beliefs and instructional practices. Those methods may be fine for documenting beliefs held by teachers, but those beliefs, at that level of detail, may carry little explanatory power for teachers’ instructional decisions.

Implications for Practice

It is not possible to draw any generalizations from the current study about what teachers might need to believe in order to implement reform-oriented teaching practice. Contained in the findings and conclusions, however, are some hints about beliefs that may carry special weight in such situations. Findings suggest that some specific collections of beliefs may be especially compatible with reform-oriented practices. For example, if teachers believe they should not give their students the benefit of the doubt that they understand, they may be more inclined to probe students’ responses and request reasons and justifications for answers. If teachers believe that learning is about connecting ideas, they may be more likely to highlight those connections for their students and beliefs about their role in forming those connections may have particular significance for their decisions. If they believe
that they are incapable of making those connections for the students and that their role can only be as a guide, they may create very different opportunities for their students than if they believe their role is to show the connections to the students. In addition, because questioning and scaffolding are often central to reform-oriented approaches to instruction, these might also be fruitful places to look to understand more about how reform is and is not being implemented.

Despite all the efforts associated with “Calculus Reform” at the college level, there is virtually no research on the in-class instructional practices of college mathematics instructors (Smith, Speer, & Horvath, 2007), let alone research that examines beliefs consistent with the practices called for in the reforms. From the findings, it also appears that college mathematics instructors are capable of developing reform-oriented teaching practices. Zachary displayed teaching practices consistent with some of the goals of mathematics reform. He was utilizing the collaborative group work structure, giving students substantial opportunities to work on problems during classes, and refraining from telling answers to their students or showing the steps in the solution to them directly. He was also using questions as his primary means for interacting with students. Given what is known about reform in K–12 education, teaching in these ways is not commonplace or an easily achieved accomplishment. Much more research is called for, however, to determine how teachers such as Zachary came to develop his teaching practices in these ways and how we might support further development as well.

The study described here focused on connections between beliefs and practices at the very level of detail when development and change appear to occur—the moment-to-moment decisions and practices of teachers. As a result, the findings could help inform the design of professional development programs. For example, knowing that particular collections of beliefs appear to shape moment-to-moment decisions about when to ask questions, it may be possible to design activities that help teachers develop beliefs consistent with the particular goals of the reform project. What those activities look like would be a function of the features of the project and the nature of the reform it was aimed at. The identification of particular collections of beliefs, however, may assist professional developers in making choices about where to focus their efforts. This additional understanding of the relationship of beliefs to moment-to-moment teaching decisions may be productively incorporated into the design of professional development and reform programs that can in turn enrich students’ opportunities to learn mathematics.

REFERENCES


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