STUDENT UNDERSTANDING IN THE CONCEPT OF LIMIT IN CALCULUS: HOW STUDENT RESPONSES VARY DEPENDING ON QUESTION FORMAT AND TYPE OF REPRESENTATION

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Research indicates that calculus students have difficulties with limit. However, underlying reasons for those difficulties and possible influences of question format have not been examined in detail. Since limit is foundational to calculus it would help the mathematics education community to know not only the difficulties students have, but also how questions used to access knowledge affect responses. Data for this study came from surveys administered to 111 first semester calculus students. Survey questions focused on limit in multiple representations including graphs, mathematical notation and definitions. Questions were multiple choice and free response. Student difficulties documented in previous research were evident in this population. Findings also indicate that difficulties students exhibited in one question were sometimes different than the difficulties those same students exhibited when asked about the same idea in a different representation. Students in general had less difficulty with graphical representations than mathematical notation or definition questions.

Key words: calculus, limits, undergraduate students’ thinking, multiple representations, survey question design

Introduction

This study is a preliminary step into research of teachers’ knowledge of student difficulties with limit. The goal of the survey used was to elicit student difficulties that could be used in subsequent interviews with teachers. The survey used several question formats and representations in order to uncover multiple student difficulties so those difficulties could be used in the design of research into teachers’ knowledge of student thinking. Because the limit concept is foundational in calculus it would help the mathematics education community to know not only the difficulties that students have, but also how the questions and representations used to assess their knowledge affect their responses. This knowledge of how students understand mathematical topics based on the format and representation of questions may help inform and improve instruction.

The concept of limit in calculus is difficult (Oehrtman, 2002; Benzuidenhout, 2001; Williams, 1991). Some research has shown that teachers with many years of experience teaching calculus have difficulty with the limit concept (Simonsen, 1995). Difficulties are observed when different aspects of student thinking are stimulated based on the problem presented (Tall & Vinner, 1981). Some researchers have found that question format can significantly influence student responses. Some of this research has been performed in the context of attitudinal surveys (Tanur, 1992; Schuman & Presser, 1981) and the areas of confirmation bias (Nickerson, 1998), answer confidence (Koriat, Lichtenstein & Fischhoff, 1980) and response elicitation (Garthwaite, Kadane & O’Hagan, 2005). However this issue of links between question format and what data on student thinking is generated has not been examined for student thinking about limits. Knowing whether students perform differently on questions in different formats could be of
importance to researchers examining student thinking of limit as well as instructors who use written tasks to assess student learning.

**Student Thinking About Limit**

Research indicates that calculus students have difficulties with the concept of limit (Oehrtman, 2002; Oehrtman, 2008; Bezuidenhout, 2001; Williams, 1991). Researchers have found that first-year university students’ knowledge and understanding are based on isolated facts and procedures (Bezuidenhout, 2001). Research has shown that students see limits in a variety of ways. Some see it as a boundary that cannot be passed (Williams, 1991; Davis & Vinner, 1986) while others view it as an approximate value obtained through an evaluative process or by imagining points on a graph getting closer to the limit (Williams, 1991). Some students believe the limit is as an infinite process (Williams, 1991; Orton, 1983) and others see limit as a value reached at the end of a process (Orton, 1983; Davis & Vinner, 1986). Some of these conceptions are combined in a dynamic viewpoint where the limit can be deduced by finding function values closer and closer to a given point (Williams, 1991). Students also tend to show conflicting conceptions of limit, continuity, and differentiability (Bezuidenhout, 2001). These conflicting conceptions may be reflective of the informal mental models students have formed from prior experiences, including nonmathematical intuitions of limit (Williams, 1991; Oehrtman, 2002). These informal mental models often result in student difficulties in later calculus topics (Oehrtman, 2002). The goal of teachers and researchers is for students to form a formal understanding of limit that will apply globally. However, students tend to use different informal mental models that often cause student difficulties when applied to various situations. Prior experience appears to play a role in the choice of finding a limit as well. Students’ faith in the use of graphs and formulas may be due to hours of experiences using them (Williams, 1991). However, students often fail to apprehend the concepts involved when using graphs and formula.

Tall and Vinner (1981) defined a concept image in this way:

> We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. (p. 152)

They went on to discuss how certain stimuli can activate different parts of the concept image. They also stated that the concept image as a whole may not be coherent. Thus the stimuli used to activate part of a students’ concept image may affect student response. Oehrtman (2002) notes that while students may learn new ideas, old ideas may not be necessarily ‘obliterated.’ Therefore, the type and representation of a question may activate different aspects of a students’ concept image resulting in inconsistent responses. While the research into student ideas and thinking has flourished, research into how students interact with the format and representation of a question is lacking. This study extends existing work on student thinking about limits by examining how students respond to questions given in different formats and representations. In particular, students were asked questions that involved mathematical notation/symbols, definitions and graphs to investigate whether students demonstrated different levels of success.

A fair amount of what we know about student thinking about limits was generated with data from written surveys (Williams, 1991; Oehrtman, 2002; Bezuidenhout, 2001). This study focuses on the representation of the questions asked to illicit student conceptions. The design focused on student understanding of limit using data generated from tasks from multiple sources (see below for details) and using multiple question formats. The goals were to examine student
responses to differently formatted questions and investigate interactions between question format and the knowledge of limit students displayed in those question formats.

**Research Design**

Because a goal of the research was to gather data on student thinking by combining items that had been used in prior research, the work was done from the theoretical perspective common across those studies, namely a cognitive perspective on knowing and understanding. This perspective, based on an assumption that written artifacts are a reasonable source of data on human thinking, has been explicitly or implicitly used in most studies of student thinking about limit (e.g., Oehrtman, 2008; Bezuidenhout, 2001; Orton, 1993; Williams, 1991).

Survey data was collected in the middle of the spring semester from 111 students in a first semester calculus course at a public university in the northeastern United States. Because of the timing of the course (second semester of the academic year), many of the students had taken calculus in the previous semester and either not passed or had not earned a grade high enough to satisfy the requirements of their majors. Although not representative of the larger population of students who study calculus, this population was a rich source of a range of difficulties and ways of thinking about limit because of their multiple exposures to the ideas from both their current instructor and the ones they had the previous semester. The university offers only one version of calculus and as a result, the students in the course were pursuing majors in physical sciences, engineering, mathematics, biological science as well as social sciences and humanities. During the data collection semester, students were using the textbook *Calculus, 4th Edition*, by Hughes-Hallett et al. During the previous semester, students would have used one of several other commonly used textbooks (there was no single book required of all instructors for calculus during this particular academic year.)

In designing the survey, some of the questions were adopted or adapted from other researchers’ studies on student thinking of limit (Benzuïdenhout, 2001; Oehrtman, 2002; Williams, 1991) and some were created. See Appendix A for the set of questions. Students were asked to explain their definition and meaning of limit in various contexts and representations. Students were asked to describe what limit means at the beginning of the questionnaire and at the end as well. Two similar multiple-choice/multiple-answer mathematical notation questions were given using different limits to see if students would give consistent answers. Two graphical representations of limit that addressed the same concepts as the multiple choice questions were given to see if students would answer consistently across various representations. Lastly, a true/false multiple-answer question was given to see what definition of limit students hold. A final question asked which definition would best describe their definition of limit. Responses were examined using methods from other researchers’ studies where possible (e.g., Benzuïdenhout, 2001; Williams, 1991) to help ensure the validity of the tasks and using a Grounded Theory (Strauss & Corbin, 1990) approach in other cases. Responses were examined for correctness, inconsistencies between answers and between questions asked with different representations. Definition questions were checked for consistency throughout the questionnaire. In the mathematical notation questions the correctness was based on circling or otherwise indicating the appropriate choice on the multiple-choice questions. Graphical questions were considered correct if the answers to the yes/no questions posed were correct. The definition question taken for the Williams (1991) study was considered correct based on the method used in that study.

**Data**
Students showed a much higher correct response rate for graphical tasks than mathematical notation or definition tasks. Refer to Appendix A for the tasks and categories.

<table>
<thead>
<tr>
<th>Mathematical Notation Tasks</th>
<th>Graphical Tasks</th>
<th>Definition Tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>19.8% - correct (Q3)</td>
<td>67.5% - correct (Q5)</td>
<td>21.6% - correct (Q8)</td>
</tr>
<tr>
<td>1.8% - correct (Q4)</td>
<td>65.7% - correct (Q6)</td>
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The responses to question 4 were examined for mutual contradiction similar to the Benzuidenhout (2001) study where the researcher claimed that these contradictory responses indicated that students have an underdeveloped concept of limit. The student response percentages were similar to the Benzuidenhout (2001) study. The responses to question 7 and 8 were examined using the Williams (1991) methods. The student response percentages were similar to that study as well.

The students had difficulty with question 4 item D. Most of the students avoided circling this item. It is possible that students had not seen this type of notation previously. Therefore only question 3 was used in the comparison of mathematical notation and graphical questions. Students answering the mathematical notation task (Q3) correctly were compared for their correct response to the graphical tasks (Q5 & Q6). Similarly the students answering the graphical tasks correctly were compared for their correct response to the mathematical notation task (Q3). The results are given below:

- Q3 notation task correct = 21
- 18 students correctly answered graphical tasks
- 3 students did not correctly answer graphical tasks
- Both (Q5 & Q6) graphical tasks correct = 65
- 18 students correctly answered mathematical notation task Q3
- 47 students did not correctly answer mathematical notation task Q3

The comparison above is striking in that students who answered the graphical tasks correctly were not as successful (27.7%) in answering the mathematical notation question (Q3) correctly. Conversely, students who answered the mathematical notation question correctly were very successful (85.7%) in correctly answering the graphical questions (Q5 & Q6). On the surface it appears that the questions are asking about similar concepts. Did the format or representation of the question stimulate different student mental models based on their personal concept image or were there other factors causing this disparity? In question 7 there are several different true/false statements about limit. Only one of those statements represents a formal definition of limit. Only one student answered question 7 by selecting just the formal definition as being true, yet 78.4% (87) of the students did select the formal definition. Almost all of the students (96.4%) selected more than one of the definitions.

**Conclusions and Implications**

There were some interesting patterns apparent in the more detailed analysis of student responses. In particular, the student response to mathematical notation and definition tasks were low even though those students had correctly responded to graphical tasks about limit. This difference could be due to student prior experience with graphical questions (e.g., they may have had more opportunities to consider and learn about limits in the presence of this representation than with the others). However, the population was varied and many had been taught with more then one instructor and textbook so there existed at least the potential for many to be familiar with multiple representations. Further research is needed to determine what it is that students who answered the graphical questions correctly understand about limit and why that understanding is not being demonstrated on the notation-type tasks. It could be that students may know how to respond to graphical tasks without having a solid conceptual foundation about
limit or it could be that students are able to demonstrate their solid understanding of the ideas when interpreting a graph but are, for some reason, unable to do so when reading the notation-type questions. This raises the question: Did the format and representation of the question activate different ideas from their concept image? It could be that students are activating different parts of their concept image. Tall and Vinner (1981) noted that different aspects of a students concept image may be invoked based on the problem presented. It may be that the format of the question or the representation used could affect different aspects of student thinking. It is also possible that there is a marked difference in the degree of difficulty between the mathematical notation and graphical questions. In other words, it may be that there are characteristics of questions asked in graphical representations that make them easier no matter what the content is. Further research is needed to determine whether these discrepancies in performance are occurring because of features of the content being assessed (e.g., the challenging concept of limit) or because of features of notation and definition questions (that would lead to lower performance on questions about other content as well). If there are differences in difficulty between question types in general, then the differences in difficulty may have affected student responses to these limit tasks. The degree of difficulty between questions was not considered since the purpose of the survey was to only elicit student difficulties.

There were a high number of students (96.4%) who selected as true more than one of the limit definitions in question 7. Only one of the definitions represents a formal definition. There were many students (78.4%) who included the formal definition among the (multiple) options they selected in question 7 as true. Question 8 was designed by Williams (1991) to make the students commit to one of the definitions as the best definition. Only 21.6% of the students in this study selected the formal definition as the best definition, which is similar to the percentage in the Williams study. It may be that the concept image of the students has conflicting aspects.

Further research could include interviews designed to examine student thinking about their responses. The design of the interviews could include having students explain their answers to the mathematical notation questions. The interviews could also investigate further the students reasoning on the graphical questions. Did the order of the questions influence student response? It would be interesting to investigate the response of students based on question order as well. The response of students to question 7 seems to indicate that they do have a wide range of ideas about limit. Do these ideas only surface based on the format or representation of questions? The format of questions and representations used by researchers and instructors may have a significant impact on what knowledge of limits we ascribe to students.

Appendix A – (Survey)

3) Given an arbitrary function \( f \), if \( \lim_{x \to 3} f(x) = 4 \), what is \( f(3) \)?
   a. 3  
   b. 4  
   c. It must be close to 4.  
   d. \( f(3) \) is not defined.  
   e. Not enough information is given. ANS:_________________

4) In this question circle the number in front of your choice(s). Which statement(s) in A to E below must be true if \( f \) is a function for which \( \lim_{x \to 2} f(x) = 3 \)? Circle letter F if you think that none of them are true.
   a. \( f \) is continuous at the point \( x = 2 \)  
   b. \( f(x) \) is defined at \( x = 2 \)  
   c. \( f(2) = 3 \)
d. \[ \lim_{h \to 0} \{ f(2 + h) - 3 \} = 0 \]

e. \[ f(2) \text{ exists} \]

f. None of the above-mentioned statements.

5) For this question, refer to the following graph:

![Graph 1](image1)

- a) What is the value of the function at \( x = 2 \)?
- b) How did you figure out your answer to (a)?
- c) Does the function have a limit as \( x \) approaches 2?
- d) How did you figure out your answer in (c)?

6) For this question, refer to the following graph:

![Graph 2](image2)

- a) What is the value of the function at \( x = 2 \)?
- b) How did you figure out your answer to (a)?
- c) Does the function have a limit as \( x \) approaches 2?
- d) How did you figure out your answer in (c)?

7) Please mark the following six statements about limits as being true or false.

A. \[ \text{T} \quad \text{F} \]
   A limit describes how a function moves as \( x \) moves toward a certain point.

B. \[ \text{T} \quad \text{F} \]
   A limit is a number or point past which a function cannot go.

C. \[ \text{T} \quad \text{F} \]
   A limit is a number that the \( y \)-values of a function can be made arbitrarily close to by restricting \( x \)-values.

D. \[ \text{T} \quad \text{F} \]
   A limit is a number or point the function gets close to but never reaches.

E. \[ \text{T} \quad \text{F} \]
   A limit is an approximation that can be made as accurate as you wish.

F. \[ \text{T} \quad \text{F} \]
A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

8) Which of the above statements best describes a limit as you understand it? (Circle one)
   A  B  C  D  E  F  None

References


