STUDENTS’ QUALITATIVE UNDERSTANDING OF THE RELATIONSHIP
BETWEEN THE GRAPH OF A FUNCTION AND THE
GRAPHS OF ITS DERIVATIVES

By

John R. Stahley

B.A. Bowling Green State University, 2001

A THESIS
Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Teaching

The Graduate School
The University of Maine

May, 2011

Advisory Committee:

Robert Franzosa, Professor of Mathematics, Co-Advisor
Natasha Speer, Assistant Professor of Mathematics, Co-Advisor
William Bray, Professor of Mathematics
On behalf of the Graduate Committee for John R. Stahley, I affirm that this manuscript is the final and accepted thesis. Signatures of all committee members are on file with the Graduate School at the University of Maine, 42 Stodder Hall, Orono Maine.

________________________________________________________________________
Committee co-chair’s signature, name, and title (Date)

________________________________________________________________________
Committee co-chair’s signature, name, and title (Date)
LIBRARY RIGHTS STATEMENT

In presenting this thesis in partial fulfillment of the requirements for an advanced degree at The University of Maine, I agree that the Library shall make it freely available for inspection. I further agree that permission for “fair use” copying of this thesis for scholarly purposes may be granted by the Librarian. It is understood that any copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Signature: ____________________________
Date: 2-3-11
The purpose of this research was to understand the thought processes of students as they solve or attempt to solve questions about derivatives. I report on the common and uncommon inaccurate ideas students displayed as they completed conceptually based tasks about graphs of functions and their derivatives.

Findings indicate that first semester calculus students have three main difficulties when attempting to sketch the first and second derivatives of a given graphical function. First, students have the tendency to sketch derivatives that represent opposite behavior of the original function. Second, proceeding left to right, many first derivative sketches begin correctly, but somewhere along the sketching process students become confused and the sketch ends up being incorrect. Lastly, many students express confusion about the second derivative and always sketch a linear function for the second derivative.

Prior research in this area is reviewed and discussed, as well as the ramifications of this study and other research that would be beneficial in understanding students’ qualitative comprehension of derivatives of graphical functions.
ACKNOWLEDGEMENTS

To my wife, Heather, for bringing me closer to God and believing in me every step of the way from Ohio to Maine, and beyond. You never gave up on me even when the stress was overwhelming. You are the hardest worker I know and have been an inspiration to me during times when my motivation was lacking. I am so blessed to be married to such a wonderful human being. Thank you my Love!

To my thesis advisors, Natasha Speer and Bob Franzosa, for their incredible patience and wonderful guidance in helping make this thesis a reality. Thank you for answering all my questions and helping to keep me on task when I thought I would never finish. I am truly grateful!

To Susan McKay for always understanding my situation and for introducing my family and I to Maine. Your advice through the years has been invaluable!

Lastly, I give many thanks to Erik Dasilva. From tea and chess to final printing, Erik’s generosity, wisdom, and friendship has always been there at the right time. My gratitude is never-ending!
TABLE OF CONTENTS

ACKNOWLEDGEMENTS ........................................................................................................... iii

LIST OF TABLES ................................................................................................................... vi

LIST OF FIGURES ................................................................................................................ vii

CHAPTER

1. INTRODUCTION ................................................................................................................ 1

2. LITERATURE REVIEW .................................................................................................... 5
   2.1 Student Understanding of Calculus ........................................................................... 5
      2.1.1 Student understanding of derivative ................................................................. 6

3. METHODOLOGY ............................................................................................................... 13
   3.1 The Research Subjects .............................................................................................. 15
   3.2 The Interviews ........................................................................................................... 16
   3.3 The Second Task (2010 Interviews Only) ............................................................... 23
   3.4 Analyzing the Data ................................................................................................. 25

4. RESULTS .......................................................................................................................... 28
   4.1 Opposite Sketches ..................................................................................................... 30
   4.2 Linear Second Derivative ....................................................................................... 33
   4.3 Proper Start – Improper Finish ............................................................................ 35

5. DISCUSSION OF RESULTS .......................................................................................... 37
   5.1 Interpretation of the Opposite Sketches ................................................................. 37
   5.2 Linear Second Derivative ....................................................................................... 44
   5.3 Proper Start – Improper Finish ............................................................................ 51
6. CONCLUSION ........................................................................................................... 52
REFERENCES ............................................................................................................. 56
BIOGRAPHY OF THE AUTHOR .................................................................................. 61
LIST OF TABLES

Table 1: Numerical breakdown of each pattern.................................................................28

Table 2: Percent Correct/Incorrect.......................................................................................30
LIST OF FIGURES

Figure 1.1: Original Task ................................................................. 3

Figure 3.1: Original Task ................................................................. 14

Figure 3.2: Original function graph used in rubric ................................. 18

Figure 3.3: First derivative graph used in rubric ................................... 19

Figure 3.4: Second derivative graph used in rubric ................................ 20

Figure 3.5: Sample of rubric used to sort data .................................... 20

Figure 3.6: Second Task Graphs ......................................................... 24

Figure 4.1: Original Function f(x) ...................................................... 29

Figure 4.2: First Derivative f'(x), of original function f(x) ....................... 29

Figure 4.3: Second Derivative f''(x) of original function f(x) ................. 29

Figure 4.4: Examples of students’ opposite sketches (original function, 1st derivative) ... 31

Figure 4.5: Examples of students’ opposite sketches (1st derivative, 2nd derivative) ...... 32

Figure 4.6: Examples of Students’ Linear Second Derivative Sketches ............ 34

Figure 4.7: Additional Examples of Students’ Linear Second Derivative Sketches ........ 35

Figure 4.8: Sketching Examples of Students with a Proper Start – Improper Finish ....... 36

Figure 5.1: Opposite Sketches (First Derivative Only) ............................ 38

Figure 5.2: Opposite Sketches: (Second Derivative Only) ....................... 39

Figure 5.3: Carol’s Original Task ....................................................... 40

Figure 5.4: Sharon’s Original Task ...................................................... 42

Figure 5.5: Tod’s Original Function and First Derivative Sketch .............. 43

Figure 5.6: Enlarged section of Figure 5.5 ........................................... 43

Figure 5.7: Example of Student’s First Derivative Opposite Sketch ............ 44
Figure 5.8: Enlarged Section of Figure 5.7 ................................................................. 44
Figure 5.9: Ester’s Original Task .............................................................................. 45
Figure 5.10: Linear Second Derivative Sketches of Constant Value .................... 48
Chapter 1

INTRODUCTION

To explain why I am involved in this research I turn to the article, *Calculus Student’s Graphical Constructions of a Population Growth Model* by Blanton, Hollar, and Coulombe (1996). I believe the first paragraph of this article correlates very well with my own research intentions. They state:

One of the major goals of university mathematics education is the knowledge of the main themes of calculus: limits, differentiation, and integration. Each of these themes has intuitive definitions rooted in the graphs of functions. Understanding the concepts of calculus requires a thorough understanding of the relationship between functions and their graphs (Dunham & Osborne, 1991) (p. 15).

My focus is on how students understand the relationship between derivatives and the graphical functions they are derived from. So why is this research so important? Well, as an educator I want to understand why students think the way they do. By understanding their thought processes on certain content, we, as educators, are better equipped to instruct students in a manner for which they will gain the greatest amount of knowledge and conceptual understanding. This idea is also supported by Blanton et al. (1996) when they say, “We must seek to understand students’ ideas and through the constructive process of learning, help students use their existing knowledge as a bridge to deeper understanding” (p. 23). Zimmermann (1991) also supports this notion when he stated, “With proper preparation, students can develop insight and understanding of functions through the study of the their graphs, but the intuition and knowledge required does not
come automatically. It must be learned” (p. 128). To reiterate, in order to help students learn the knowledge that does not come automatically we need to understand how to best instruct them, which requires insight into how they process information.

This research began as an attempt to find out if the students enrolled in first semester calculus were meeting certain objectives. According to the description of the first semester calculus course provided by the university website, this particular mathematics course is heavily involved in helping students understand the concept of derivative by studying limits. After students become familiar with differentiating common pre-calculus and trigonometric functions they are introduced to integration and finding the area under a curve. My particular objective in this study was to determine how well students could qualitatively determine the derivatives of a given graphical function. For the purposes of this study, to qualitatively determine a derivative means to sketch the derivative of a function for which one does not know the algebraic representation. This process is different from quantitatively determining a derivative due to the fact that it does not directly involve numbers and variables, but the graphical implications of numbers and variables and their related equations. I believe Amit and Vinner (1990) said it well when they talked about the importance of the concept of derivative. They maintain that if it is poorly understood then this, in turn, may lead to inadequate understanding of related concepts that involve derivatives, i.e. the concept of velocity and rate of change in the natural sciences and the concept of marginal value in economics and business administration.
Many students can differentiate an algebraically presented function, but differentiating, or sketching the derivative of a graphically presented function requires dissimilar thought processes and analytical skills (Berry & Nyman, 2003).

In order to assess whether or not students were meeting the particular objective being able to qualitatively determine the derivatives of a given graphical function, they were given the following task (see Figure 1.1 below). In this task they were to let $f$ be the function whose graph is shown. On the axes below the original graph ($f$), students were asked to sketch the first and second derivatives of the original function $f$.

![Figure 1.1: Original Task](image)
Interviews were conducted with students about their thought processes while solving or attempting to solve this task. From those interviews and their related transcriptions, ideas about student thought processes while sketching the derivative of a given graphical function were formulated. This research is motivated by the following questions: What type of thought processes are students using when sketching the derivatives of a given graphically-presented function? How do the ideas of a derivative, formulated from a visual-spatial context, differ from those constructed in a pure numerical or arithmetical context?
Chapter 2

LITERATURE REVIEW

One of the driving forces behind this research is to use the knowledge gained from it to help students better understand derivatives in calculus. This is consistent with Ferrini-Mundy & Graham (1994) when they state:

University mathematics faculty have a responsibility to help students understand central mathematical ideas in a way that will be useful to the students that opens up for them the beauty of the mathematics, and that empowers them to learn more mathematics (p. 44).

This responsibility, in my opinion, should not be taken lightly. So how do we help students understand mathematics better? We conduct research to find out what makes them tick. In fact, there has been a “call to research” so to speak, within the mathematics education community within the last quarter century. This is evident by the increased number of research articles published on this topic during the last 25 years. This push for increased research is encouraged by those who see a disconnect between the procedural (following procedures, rote learning) and conceptual (understanding the behavior of functions, the whole picture) knowledge in mathematics. One area of study, in particular, looks at students’ abilities to understand, explain, and interpret various calculus concepts.

2.1 Student Understanding of Calculus

For most students, the notion of limits, derivatives, and integration is unknown before beginning a calculus course. Many textbooks, such as Jon Rogawski’s (2007) for example, follow this order for instructing new calculus students. The new apprentices are first introduced to the concept of limits and tangent lines and how they relate to rate of
change. This segues into the concept of derivatives, which is the instantaneous rate of change at a given point. Integration, also known as anti-differentiation, comes next and is used to find the area under a curve and other mathematical phenomena.

2.1.1 Student understanding of derivative.

Many students in calculus are primarily taught using rote instruction and textbook design (White & Mitchelmore, 1996). They learn various steps to take in order to differentiate an equation. The same can also be said about graphing derivatives. The majority of students have a pointwise, almost procedural, understanding of what the derivative of a graph might look like (Monk, 1988; Asiala, Cottril, Dubinsky, & Swingendorf, 1997). They know to look for certain points on the original function graph, and at those certain critical points, i.e. relative maximum and minimum points, they follow a set of rules for how to proceed with sketching the derivative of that original function’s graph. While this pointwise view of extrapolating derivative sketches very often yields correct results, it does not mean the student following the procedures truly understands what the graph is saying. Across-time understanding of derivative graphs means that the student is aware of the behavior of the original graph and understands the relationship between the original graph and its derivative sketch (Monk, 1988).

When we think about calculus, for many of us in the mathematical field it brings up notions of derivatives. In fact, some research has looked at students’ capacity to define the concept of derivative (Tiwari, 2007). Since this paper is also looking at how students understand a version of the derivative, I find it fitting to the research of student definitions of derivative. Through these definitions we can see how students use particular language and how they interpret the definition of a derivative and how closely
it relates to the textbook definition. The following are quotes from students regarding
derivatives from Tiwari (2007):

The derivative of a function is the rate of change of the function. It can be
used to find velocities, critical points, marginal profit, and list of other real world
problems. Also, derivative can be used to find the slope of a tangent line, which is
the best approximation of the slope of a graph at a given point…The derivative of
a function is used for many different mathematical operations. The first derivative
is used to show relative extrema, critical points, and open intervals for which f(x)
is increasing or decreasing is also found here. The second derivative is used to
show concavity and inflection points (p. 7).

Differentiation is the process of finding or taking the derivative of a given function or
equation and is one of the key concepts for learning and applying calculus. If we consider
a function as an object, Dreyfus, Artigue, Eisenberg, Tall, & Wheeler, (1990) state very
simply that, “Differentiation generates a new function from a given one” (p. 114). This
definition is not very elaborate, but it is nonetheless correct if the function can be
differentiated.

A copious amount of research has been conducted on students’ abilities to
differentiate analytical and graphical equations within a calculus context (Orton, 1983;
Asiala et al., 1997; Aspinwall, Shaw, & Presmeg, 1997; Burgmeier, 1991; Borgen &
Manu, 2002; Viholainen, 2005; Zandieh, 2000; Ubuz, 2007; Tall, 1985, 1996; Stroup,
2002; A. Selden, J. Seldon, Hauk, & Mason, 1999; Roddick, 2001; Nemirovsky & Rubin,
can be represented graphically, verbally, physically and symbolically (p. 105). This
representation corresponds with, and exceeds, the “Rule of Three”, which says that wherever possible topics should be taught graphically, numerically, and analytically, which is one of the guiding principles for institutions when designing a new calculus course (Hallett, 1991).

As mentioned earlier, the last quarter century has seen an increased demand in mathematical education research. Within this research there has been an encouragement for emphasizing the conceptual and graphical side of calculus and differentiation. According to Dreyfus et al. (1990), “Students cannot understand what a differential equation means unless they have well understood the concepts (rather than the techniques) of differentiation…” (p. 114). This idea is also shared with Orton (1983) as she stated,

It is known that some students are introduced to differentiation as a rule to be applied without much attempt to reveal the reasons for and justifications of the procedure. Many regard this as bad educational practice, and, in fact, it should not be necessary (p. 242).

This encouragement has lead to an increase in research examining students’ graphical abilities with derivatives (Abbey, 2008; Ubuz & Kirkpinar, 2000; Aspinwall & Shaw, 2002; Baker, Cooley, & Trigueros, 2000; Berry et al., 2003; Chappell & Kilpatrick, 2003; Roddick, 2001; Heid, 1988; Asiala et al., 1997; Haciomeroglu, 2010; Mathews, 1991; Nemirovsky, 1997; Tiwari, 2007). Graphical ability, as defined by Kwon (2002) is “an ability to use a graph as a qualitative analysis of a whole picture” (p. 58). So we are pressed with the task of researching students’ graphical abilities, specifically, how they interpret a graph. According to Clement (1985), “The ability to interpret graphs is
important for mathematical literacy and for understanding the concepts of function and variable, as well as for developing basic concepts in calculus” (p. 369). Particularly, we are looking at how students interpret and understand derivatives graphically.

Many researchers have separated this idea of students’ graphical understanding of the derivative into two categories: “pointwise” and “across-time” (Monk, 1988). I select my definition of pointwise, graphical understanding of a derivative as;

…the relationship between the derivative of a function at a point and the slope of the line tangent to the graph of the function at that point. This forms a foundation for understanding the derivative as a function, which among other things, gives for each point in the domain of the derivative the corresponding value of the slope (Asiala et al. 1997).

This pointwise viewing and understanding of derivatives within a graphical setting is very common among calculus students (Eisenberg & Dreyfus, 1991). But Monk (1988) states, “…this is not the way the concept of function is actually used in calculus” (p. 1). According to Monk, an across-time understanding of derivatives is actually what students need to comprehend in addition to a pointwise understanding. Monk defines across-time understanding of derivatives as the ability to answer certain questions about a function. These questions are; “How does change in one variable lead to change in others? How is the behavior of the output variables influenced by variation in the input variable?” (Monk, 1998, p. 1). From the evidence the students produced for this research, it appears that most errors when sketching the derivative of a given graph occur when students misinterpret information from a given graph and try to apply it in a pointwise manner.
Other investigations into student graphical understanding of a derivative has lead many researchers, including this one, to look at students’ aptitude for sketching and/or understanding the first and/or second derivative of a given graph (Chappell et al., 2003; Aspinwall et al., 1997; Aspinwall et al., 2002; Ubuz et al., 2000; Roddick, 2001; Hallett, 1991; Zandieh, 2000). Ubuz and Kirkpinar (2000) gave a research question to 59 first year undergraduate students in four sections of a Calculus I course offered at Middle East Technical University during the fall of 1996-1997, which was very similar to the task given for this research, except for the fact that the students were only asked to sketch the first derivative and not the second (p. 248). In addition, tasks given by Aspinwall and Shaw (2002) relate to mathematical processing by having students sketch the first derivative of a given parabolic function (p. 434). Research completed by Chappell and Kilpatrick (2003) had students sketch the first derivative of a more difficult function with a cusp (p. 27) and Lithner (2000) had students complete the first derivative sketch of a given graphical function that would result in a piece-wise derivative function (p. 177). In Chappell and Kilpatrick’s study, the students had to extend previous knowledge to a new situation or task, which was sketching the derivative of a given graphical function.

Overall, students who were taught in a conceptual, rather than procedural, learning environment, completed the derivative sketches very well with little confusion. In Lithner’s study, a particular student talks about using a rule to complete their sketch, but does not understand the implications of this rule or why it works. This student uses algebraic methods in order to understand what to sketch, but his algebraic answer contradicts with his sketch. I also experience similar situations, as mentioned in the Chappell & Kilpatrick study and the Lithner study, with students in my study.
Researchers try to decipher why some students fail to comprehend the graphical view of derivatives. These, and other, researchers discuss the role of visualization, or lack thereof, as hindering this comprehension (Goldenberg, 1987; Eisenberg et al., 1991; Hallett, 1991; Tall, 1991; Vinner, 1989). These studies talk about the benefits of visualization for students and sketching. If a student can visualize what the derivative sketch might look like prior to sketching, they can compare this visualization with what they sketch using procedures. Aspinwall et al. (1997) combines information from Presmeg (1986) article to define “visualization” to mean a mental scheme depicting spatial or visual information (p. 302). By scheme I am using the definition originally given by Piaget and used in the Baker et al. article, which states, “Whatever is repeatable and generalizable in an action is what Piaget refers to as a scheme” (2000, p. 560).

One aspect of visualization that can hinder students’ abilities to sketch derivatives of a given graphical function is known as uncontrollable mental imagery. Aspinwall et al. (1997) talk about students’ uncontrollable mental imagery:

Such images are uncontrollable both in the sense that they appear to rise unbidden in an individual’s thought, and also in their persistence even in the face of contrary evidence. An uncontrollable image, then, is one which is beyond the volition of the cognizing individual (p. 303).

Lithner (2000) also discussed uncontrollable mental imagery, putting it this way, “…students often focus mainly on what they can remember and what is familiar within limited concept images. This focus is so dominating that it prevents other approaches from being initiated and implemented” (p. 93).
To summarize, the research shows that the majority of students taking calculus demonstrate procedural and pointwise understanding of derivatives, and that in addition to this type of understanding, it would be beneficial if students would gain a conceptual and across time understanding of derivatives. It is still unclear why some students lack confidence when differentiating a given graph, and how students fully understand the conceptual relationship between a given graphical function and its derivatives. Through this investigation I hope to expand on the literature of students’ conceptual understanding of derivatives and enhance our understanding of students’ thought processes while differentiating a given graphical function.
Chapter 3

METHODOLOGY

The research methods and goals closely follow those established by Strauss & Corbin (1990). Strauss and Corbin state their interest in the learning behavior of calculus students and how they interpret this behavior from examining the students’ written work and analyzing information via interviews. A similar type of methodology was employed by Ferrini-Mundy and Graham (1994) who state that:

Methodologies employed in studies of this type are often qualitative and descriptive, based on interviews with students as they complete mathematical tasks…The intention is to provide rich and defensible descriptions of student understandings that can serve as springboards for acknowledging the great complexities to be understood in learning about student knowledge (p. 32).

In order to find out what students are thinking and what approaches they use when sketching the first and second derivatives from an original function graph, data consisting of student written work and interviews was collected and analyzed. Only a small portion of data was analyzed quantitatively by computing various percentages of students who sketched correctly. The majority of the data was analyzed qualitatively by transcribing interviews and searching for conversations that related to a particular misconception or discussion of a sketch. The tasks given to students in order to probe their understanding about derivatives are similar to tasks given by Tall and Watson (2001), Tall and Thomas (1989), and Williams (2002).
During the fall of 2007, spring of 2008, and spring of 2010, students enrolled in first semester calculus at a university in the northeastern United States were given the task to complete in Figure 3.1. For this task students were required to sketch the first and second derivatives of the given graphical function (f).

![Figure 3.1: Original Task](image)
The task was created in cooperation with a mathematics professor at the university where the research was conducted. The reason behind giving students this task to complete was to gain insight into students’ thought processes of sketching derivatives of a graphical function and if they followed a more procedural or conceptual method. A procedural method would most likely indicate if students have a pointwise understanding of derivatives and how critical points, inflection points, and concavity relate to the sketch of the first and second derivatives. A conceptual method would most likely indicate if students have an across time understanding of derivatives and provide an opportunity to learn about students’ conceptual understanding of the behavior of graphical derivatives. This type of task and supported reasoning is similar to ones stated in Zimmermann (1991).

3.1 The Research Subjects

A total of 139 students completed the task in the fall of 2007, at the end of the semester, as a mandatory part of a final exam. Ninety-nine students completed the task in the spring of 2008, at the end of the semester, but it was not a mandatory part of the final exam. Fifty-one students completed the task as a mandatory assignment in the spring of 2010, but it was administered before the semester was halfway completed. My data is compiled, primarily, from the approximately 51 students who completed the task in the spring of 2010, but it also includes data from the fall 2007 students. I decided against using the spring 2008 data since some professors included the task as extra credit and not a mandatory assignment. I believe this would affect how the students viewed the task and that they would not give the task their full attention or effort and that this possible lack of effort would negatively affect the validity of the data.
3.2 The Interviews

In March and April of 2008 I conducted interviews of 15 students who completed the sketching task in the fall of 2007, and in March through May of 2010 I conducted interviews of seven students who completed the task in early March, 2010. The reason for interviews coincides with that given by Zandieh (2000) in which she states: “Each of the interviews allowed the student to provide information about her or his understanding of derivative by answering open-ended questions and solving problems involving the concept of derivative” (p. 112). Dexter (1970) put it simply when he referred to an interview as being a conversation with a purpose. My purpose with the interviews was to gain insight into student thinking and understanding of sketching derivatives of a given graph. Lincoln and Guba (1985) talk about interviews being time-intensive and that they offer insight into students’ thinking that written responses do not always capture. These interviews were a planned part of the research before the data was gathered. The reason for the interviews was necessary for gaining explanation, from the students, as to why they sketched certain portions of the derivatives the way they did.

I designed the interviews to be “structured”, based on information from Lincoln and Guba (1985). They stated, “…the structured interview is the mode of choice when the interviewer knows what he or she does not know and can therefore frame appropriate questions to find it out…” (p. 269). (For example, I do not know why students were sketching derivatives that looked to be opposite shapes as their original graphs and why a linear second derivative was so common among incorrect sketches.) I feel that my 2008 interview questions were too structured and that I did not allow adequate room or opportunity for elaboration from the students, nor did I ask students to elaborate or
explain further. For the 2010 interviews I had a better idea of what students might talk about, but I still knew that I did not know for certain what the students were thinking when they completed the task. This is one reason why my spring 2010 questions were much more open-ended and not structured. I gave the students much more opportunity to explain and elaborate on their sketches and thought processes behind them.

In regards to the 2008 interviews, during the interview, I read through the questions in order, exactly as they were written. I then had the student respond with either an answer to a question and/or a sketch. In the spring 2008 interviews I asked the students to explain what they were sketching while they were in the process of sketching. After thorough examination of the fall 2007 task, and spring 2008 interview data I noticed that the majority of students actually sketched both derivatives correctly, but there was a great deal of variety in the different correct sketches. Each graph was then separated into segments I thought were important for determining correctness (see figure 3.15). A rubric was then created and used, following these correct characteristics from the graphs, to determine various percentages of correctness for the student sketches. Graphs from this set of data I denote as “correct” were determined to be 100% correct using the rubric.
To begin, we will look at the graph of the original function.

The original function graph has the following attributes:

- The segment from A to N is decreasing with a negative slope
- The curve NB has a slope that is negative and increasing
- Point B is a relative minimum with a zero slope
- The slope of the curve from minimum point B, to point C, is positive and increasing.
- Point C corresponds to the location of the maximum slope along BD.
- The slope of the curve from point C to D is positive and decreasing.
- D is a relative maximum with a zero slope.
- The slope of the curve from point D to point E is negative and decreasing.
- Point E corresponds to the location of the minimum slope along DF.
- The slope of the curve from point E to point F is negative and increasing.
- The curve FK has a slope that is positive and increasing.
- Segment KG is increasing with a positive slope

Next, we will look at what a correct example of the graph of the first derivative of the original function may look like. We will use the reference points N’ – K’ to help us discern what characteristics of the first derivative graph must be included for it to be marked correct.
There are numerous answers that would be considered correct. A first derivative graph marked “correct” should have the following attributes:

- Graph segment N'B’ is negative and increasing.
- Point B’ has a value of zero.
- Curve B’C’ is positive and increasing.
- Maximum point C’ is positive with a slope of zero.
- Curve C’D’ is positive and decreasing.
- Point D’ has a value of zero and a negative slope.
- Graph segment D’E’ is negative and decreasing.
- Point E’ is a minimum with a zero slope.
- Curve E’F’ is negative and increasing.
- Point F’ has a value of zero.
- Segment F’K’ is positive and increasing.

Next, we will look at what a correct example of the graph of the second derivative of the original function may look like. We will use the reference points N’’, C’’, H’’, E’’, and K’’ to help us discern what characteristics of the second derivative graph must be included for it to be marked correct.
There are numerous answers that would be considered correct. A second derivative graph marked “correct” should have the following attributes:

- Graph segment $N'C'$ is positive.
- $C'$ is zero.
- Curve $C'E'$ is negative.
- Point $H'$ is a minimum with zero slope.
- $E'$ is zero.
- Segment $E'K'$ is positive.

Below is a copy of a portion of one sample of the rubric used to help sort the data.

<table>
<thead>
<tr>
<th>First Derivative Graph</th>
<th>#</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph segment $N'B'$ is negative and increasing</td>
<td>32</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This example shows how I sorted and evaluated correctness for a first derivative sketch. The student work was given a number: in this case 32. Each segment from the student’s first derivative sketch was compared with the graph I created to determine correctness (see Figure 3.3). If the student’s sketch matched the description of the corresponding segment, then the column was given a “1,” meaning correct. A “0” (zero) for the column
would indicate that the segment of the student sketch did not match the pattern, or
description, of the segment of the graph used to determine correctness. In the example
above, student 32 correctly sketched the first three segments of the first derivative sketch.
There were eleven different segments that comprised the first derivative sketch. I took the
number of correctly sketched segments (each indicated with a number 1), added them
together and divided by eleven, the total. This ratio gave me a “percent correct” for each
student. I only considered students’ sketches to be truly correct if they scored 100%
correct for the particular category ie, first and second derivative sketches.

In addition to the columns that described the various segments, I also had a
“comments” column that I could use to write down any strange or unusual patterns or
symbols the student left on their task. After both the first and second derivatives were
categorized I created additional coding for each student’s answer that compared the
overall percent correct for each student along with comparisons of which derivative, first
or second, had the majority of correct sketches.

In regards to the incorrect sketches, I noticed some unusual patterns, but I was
unable to corroborate the patterns with transcripts of the interviews. I had data, but I did
not have student explanations as to what they were thinking when they created the
sketches. I decided that more qualitative data was needed to get information about the
students’ actual thinking.

The interview questions stated below are from the spring 2008 interviews:

1. Please sketch the first derivative of the original function on the axis provided.
2. Please sketch the second derivative of the original function on the next axis
   provided.
3. Please describe how you figured out where to cross the x-axis while sketching the
   first derivative.
4. Please describe how you figured out where to cross the x-axis while sketching the second derivative.
5. Do the min’s and max’s of the original function have any relevance on your sketches?
6. Do the inflection points, (where the graph changes concavity), of the original function have any relevance on your sketches?
7. Do the min’s and max’s of the first derivative have any relevance on your sketch of the second derivative?
8. Do the inflection points of the first derivative have any relevance on your sketch of the second derivative?
9. What do you look for when sketching the second derivative? i.e. What information do you use from the graph of the original function and your sketch of the first derivative when sketching the second derivative?

Below is the set of interview questions from the spring of 2010 interviews. These questions were more open ended in order to allow the students greater freedom in their explanations and descriptions.

1. Please sketch the first and second derivatives of the original function on the axis provided.
2. Tell me what you were thinking when you sketched the first derivative. How did you know what to graph?
3. How did you know what to sketch when creating the second derivative?
4. Did concavity, or the notion of concavity, arise at all in your thinking, where and why and how?
5. To what extent were you thinking about increasing or deceasing behavior/direction and/or the slope/steepestness of your sketch?

I took a different method to interviewing for my spring 2010 students. I followed all the same procedures in note taking, recording, and observation as I had with the first set of interviews, along with some new procedures inspired by Jorgensen (1989). Jorgensen discusses the importance of interviewing, along with some good techniques, such as restating what interviewees say for clarification. By doing this we, the interviewers, help show the interviewee that we are interested in what they are talking about. That leads to greater rapport and comfort with the interview and also gives the interviewee the opportunity to correct any mistakes in the interpretation. For the 2010
interviews I asked the students to complete their sketches first, then I had them talk about their approaches and thought processes. The latter method made for more of a reflective atmosphere during the interview process. For both sets of interviews I observed what students were sketching and made notes about their sketches. Borgen & Manu (2002) summarized work done by Pirie & Kieren (1992b, 1994) stating that

Understanding a student’s understanding requires not only a close examination of his/her written work, but also involves a careful analysis of the student’s thoughts and ideas while performing the computations (p. 152).

Along with observing the sketches I also observed body language that I thought was relevant to the student’s explanation of their sketches and would often include spoken notes within the interview to describe what the student was doing. For example, one particular student angled their pencil in the air in order to visualize the slope or pattern of the sketch. Observing and recording body language coincides with interviewing methods discussed by Borgen & Manu (2002). In addition, Borgen & Manu (2002) recapitulate previous work by Borgen (1998) by emphasizing,

Body language may be as important if not more important, than the actual words spoken in determining and individual’s thoughts since the verbal expressions of a student may not sufficiently explain what he/she is trying to relay (p. 152).

3.3 The Second Task (2010 Interviews Only)

In addition to the original task (figure 3.1), six of the seven interviewed students from the spring of 2010 were given a second task to complete during their interview. This
task involved four separate pieces of paper. On each paper was an identical original graph, but the first derivative graph was different on all four papers. The student was asked to choose the correct first derivative graph, and then sketch the second derivative on that same piece of paper in the space provided. In the following figure show the original graph on top, and the four versions of the first derivative graph below.

![Second Task Graphs](image)

The purpose of this second task was to gain further insight on the students’ qualitative understanding of derivatives of a given graphical function. I was also curious to see if any students reified, remembered, or changed their thinking about sketching the derivatives after going through this multiple-choice exercise. Students think in various ways; this task being multiple choice may have allowed the students logically deduce what the first derivative is. Many of the students who completed this task eliminated certain choices before choosing their final answer and sketching the second derivative. This task allowed students’ pointwise understanding of derivatives to be more prominent since they could align certain points on the original function with those correlating points on the first
derivative. It helped to fill in some gaps of knowledge about derivatives that may have been vague to the students and gave me an inside look as to whether this type of thinking inhibited or helped the student better understand the relationship between the graph of a function and it’s derivative.

3.4 Analyzing the Data

I took a quantitative approach to analyzing the 2007 sketches. A rubric was created (see figure 3.3) that broke down the first and second derivative sketches into various segments. For example, one segment would be from a relative minimum to an inflection point, etc. (see figure 3.4) I examined all 139 sketches from 2007 using this rubric and took detailed notes about the direction or behavior of the sketch. In the end, however, the quantitative data from this rubric did not offer the kind of insight into student thinking about sketching derivatives that I was expecting. One of the reasons was that the majority of students actually sketched the derivatives correctly. I was able to determine, from some of the sketches and using the rubric, that many of the correct sketches were the result of pointwise understanding of derivatives. This was evident by what I call “rain” on the paper. Students would sketch vertical dashes, which looked like rain, from a particular critical point on the original function to the first derivative sketch and sometimes to the second derivative sketch. This type of student understanding was already well documented in research (Monk, 1988; Asiala et al., 1997; Eisenberg et al., 1991). I wanted the focus of my research to include more of the across time understanding of students, and the rubric did not allow me to gain data regarding this.

My review of the 2010 sketches was much less structured. I examined the overall behavior of the derivative sketches to determine correctness. The places where the
derivatives crossed the x-axis should be in the approximate area where the original function had tangent lines with zero slopes. I also checked to see if the value of the derivative graph was positive or negative according to the correlating slope of the tangent in the original graph.

Although the interviewing tactics between 2008 and 2010 were quite different, the analyses of the interviews were identical. After audio-recording the interviews I typed up transcripts and then went back to reviewing the sketches. I started noticing patterns within the incorrect sketches; namely resemblance of the original function on the first derivative, opposites, proper start – improper finish, and linear second derivatives. (There were also other incorrect sketches that I was unable to categorize because the pattern of those derivative sketches did not fit into any of the previously named categories.) These patterns were initially noticed in the 2007 data, but not with the consistency of the 2010 data. Since the only incorrect sketching pattern that had been established and discussed in research was resemblance (Nemirovsky et al., 1992), I decided to focus my research on the other patterns. Once these patterns were established, for the 2007 and 2010 data, I went back and studied the transcripts again, looking for students discussing these patterns in our conversations. Since the majority of the interviewees from 2008 correctly sketched both derivatives, I focused more attention on the 2010 data. While searching through the transcripts I looked for words such as, “opposite” and “confusion” within interviews of students I saw patterns. (I present numerous student quotes, mostly from the 2010 group, that describe student thinking about opposites and linear second derivatives. I chose these quotes because they are easy to follow and are fairly clear in their descriptions.) I also re-read the transcripts paying close attention to conversations where the students talked
about the behavior of the graph or their sketch or how they went about sketching a derivative. I provide examples of student quotes later in Chapters 4 and 5.
Chapter 4

RESULTS

This chapter begins with a discussion of what correct sketches should resemble. Then I reveal the percentage of correct and incorrect sketches generated by participants in the research. Continuing on I disclose the three patterns I focused on from incorrect sketches. From this statement, one can presume that there are more than three patterns, and that presumption would be correct. One of the most re-occurring patterns found in the incorrect sketches, especially with the first derivative, was where the first derivative sketch resembles the shape of the original graph. These findings correlate with those by Nemirovsky et al. (1992). The other three patterns; opposite sketches, linear second derivatives, and proper start – improper finish, are a bit more complex than resemblance since very little research, if any at all, exists regarding these specific patterns. Table 4.01 gives a numerical breakdown of these three patterns, indicating from which set of data they occur. (Some data may have more than one pattern associated with it.)

<table>
<thead>
<tr>
<th>Pattern</th>
<th>2007 Final Exam</th>
<th>2010 Mid-semester task</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opposites-1&lt;sup&gt;st&lt;/sup&gt; Derivative</td>
<td>4/139 (2.88%)</td>
<td>7/51 (13.73%)</td>
<td>11/190 (5.79%)</td>
</tr>
<tr>
<td>Opposites-2&lt;sup&gt;nd&lt;/sup&gt; Derivative</td>
<td>6/139 (4.32%)</td>
<td>11/51 (21.57%)</td>
<td>17/190 (8.94%)</td>
</tr>
<tr>
<td>Linear Second Derivative</td>
<td>5/139 (3.60%)</td>
<td>15/51 (29.41%)</td>
<td>20/190 (10.52%)</td>
</tr>
<tr>
<td>Proper Start – Improper Finish</td>
<td>14/139 (10.07%)</td>
<td>16/51 (31.37%)</td>
<td>30/190 (15.79%)</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>29/139 (20.86%)</td>
<td>49/51 (96.08%)</td>
<td>78/190 (41.05%)</td>
</tr>
</tbody>
</table>

Table 1: Numerical breakdown of each pattern

There are numerous answers that would be considered correct, but taking into account the fact that the task does not have any type of scaling, examples of the most accurate graphs of the first and second derivatives would look similar to these:
Figure 4.1: Original Function $f(x)$

Figure 4.2: First Derivative $f'(x)$, of original function $f(x)$

Figure 4.3: Second Derivative $f''(x)$ of original function $f(x)$
From the 2007 and 2010 data there is compelling evidence of student confusion while sketching the first and second derivatives of a given graphical function. Out of the 139 students who attempted the task in 2007, and 51 students who attempted the task in 2010, a table of results indicates the percent correct and incorrect for each category. This table was created following the pattern categorization I mentioned in the methods sections regarding how I determined correct sketches. To reiterate, I examined the overall behavior of the derivative sketches to determine correctness. The places where the derivatives crossed the x-axis should be in the approximate area where the original function had tangent lines with zero slopes. I also checked to see if the value of the derivative graph was positive or negative according to the correlating slope of the tangent in the original graph.

<table>
<thead>
<tr>
<th></th>
<th># correct</th>
<th>% correct</th>
<th># incorrect</th>
<th>% incorrect</th>
</tr>
</thead>
<tbody>
<tr>
<td>2007</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>First Derivative</td>
<td>98</td>
<td>70.50%</td>
<td>41</td>
<td>29.50%</td>
</tr>
<tr>
<td>Second Derivative</td>
<td>89</td>
<td>64.03%</td>
<td>50</td>
<td>35.97%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th># correct</th>
<th>% correct</th>
<th># incorrect</th>
<th>% incorrect</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>First Derivative</td>
<td>16</td>
<td>31.37%</td>
<td>35</td>
<td>68.63%</td>
</tr>
<tr>
<td>Second Derivative</td>
<td>9</td>
<td>17.65%</td>
<td>42</td>
<td>82.35%</td>
</tr>
</tbody>
</table>

Table 2: Percentage Correct/Incorrect

4.1 Opposite Sketches

According to the 2007 data, four out of 41 students, or 9.76% of the incorrect first derivative sketches and six out of 50 students, or 12% of the incorrect second derivative sketches appeared to be “opposites.” Compare this with the 2010 data where seven out of 35 students, or 20% of the incorrect first derivative sketches and 11 out of 42 students, or 26.19% of the second derivative sketches appear to be “opposites.” I am defining
“opposite” as the derivative being a sketch of the original graph approximately reflected over a horizontal axis. In regards to the second derivative sketch, the “original graph” in my interpretation, would be the first derivative sketch. In Figure 4.4, one can see that the student begins their sketch in a direction that corresponds to the original function flipped over a horizontal axis. As the initial slope of the original function decreases to a relative minimum, the initial slope of the first derivative increases to a relative maximum.

![Figure 4.4: Examples of students’ opposite sketches (original function, first derivative)](image)

Below are a couple of examples of students’ work that show opposite sketching. You will notice that for both sketches, the first derivative appears to be the opposite of the original function and the second derivative appears to be the opposite of the first derivative sketch.
In addition to the sketches, numerous students talked about opposites during their interviews. For example, during an interview with a student named Sharon\textsuperscript{1}, I asked her what her reasoning was for starting a sketch in a particular quadrant. Her response:

Um, because it was increasing, uh, well technically decreasing in the second quadrant or first quadrant I had it going increasing, I had it doing the opposite of what its initial function was doing...To kind of indicate it (first derivative sketch) was doing the opp, like the opposite, like what I had in mind it was doing the opposite behavior as its initial function.

In an interview with a student named Lucinda, this is what she had to say when I asked her to sketch the first derivative of the graphical function, “Um I don’t really remember exactly how to do it. Um, I don’t know, I don’t remember if it is supposed to be the opposite shape or...I’m not too sure.” These two quotes, along with others I will discuss

\textsuperscript{1} All names are pseudonyms
later, reinforce the notion that some students’ derivative sketches seem to take on the opposite shape of its original function. There does, however, seem to be some underlying confidence issues with this form of reasoning. I will elaborate on this later.

4.2 Linear Second Derivative

Another type of incorrect sketch that arose very often from the student work and the interviews was a second derivative sketch that was linear; in other words, a straight line. In the majority of cases this line was either horizontal and above the x-axis, increasing from quadrant III to quadrant I, or decreasing from quadrant II to quadrant IV (see figure 4.6). From 2007, five out of 50, or 10.00% of the incorrect second derivative sketches were of this manner and 15 out of 42, or 35.71% of incorrect sketches from the 2010 data were in the form of a linear second derivative.

During an interview with Ester, she made this comment when I inquired about her second derivative sketch;

“Yeah. On the first task for the second derivative I had no idea, and I still really don’t. So I just kind of drew a line for the second derivative, just out of some crazy assumption…”

Another example of a student explaining their sketch of the second derivative is found in my interview with Carol. Her explanation is different from Ester’s, but the result is the same. In the course of the interview I asked Carol why she drew a horizontal line above the x-axis for her second derivative on the original task. Her response was:

“I have the idea that if your first derivative is positive, then your second derivative is going to be positive. And if your first derivative is negative, the second is positive.”
(In other words, no matter what value the first derivative had, be it positive or negative, Carol interpreted the related portion of the second derivative as some kind of constant positive value.) After Carol stated this I asked her if she could elaborate as to why she sketched a line instead of a curve. She was unable to give an explanation. I chose these responses from Carol and Ester for their uniqueness in describing their viewpoint about the second derivative.

Here are some examples of students sketching a linear second derivative.

![Figure 4.6: Examples of Students’ Linear Second Derivative Sketches](image-url)
4.3 Proper Start – Improper Finish

On 14 out of 41 incorrect cases (34.15%) for 2007 and 16 out of 35 incorrect cases (45.71%) for 2010, the first derivative sketch begins in the correct direction (when reading left to right), but the final result is not considered correct. Below are some examples of incorrect sketches that begin in the correct direction.
Figure 4.8: Sketching Examples of Students with a Proper Start – Improper Finish

It appears, from the evidence, that the majority of students who began sketching the first derivative correctly had problems when they reached the first critical point, which correlates to the first relative minimum of the original graph. Many of the “opposite” sketches fall in this category.
Chapter 5

DISCUSSION OF RESULTS

First, the correlation between the data and student performance over time is discussed. The 2010 data was taken mid-semester and the 2007 data was taken at the end of the semester, as shown in Table 4.1. A possible reason for the significant difference in the percentage of correct and incorrect derivative sketches may be due to when the task was given during the semester. A possible interpretation of the data is that students’ abilities to more correctly sketch the derivatives of a given graphical function improve the longer they are in the course. But the data also emphasizes something else, the importance of students learning how to extrapolate data from the original graph and sketch the derivatives with that data. The reason this is so important is shown in the data. Mid-semester, nearly 70% (35 out of 51) of participating students sketched the first derivative incorrectly and approximately 80% (42 out of 51) sketched the second derivative incorrectly. If students have misconceptions or incomplete knowledge about the derivative of a given function and/or lack the expertise for how to process the information from the graph, they may mis-interpret the relationship between the graph and its derivative. This tells us, as educators, that we need to continue to address any misconceptions, pre-conceptions, or gaps in knowledge that calculus students have about the derivative of a given function.

5.1 Interpretation of the Opposite Sketches

Much of the information I have about why students sketch the first and second derivatives of a given graphical function incorrectly comes from the interviews I conducted with them in 2008 and 2010 and from various research articles on the subject.
Using student quotes and notes from the interviews I am able to infer some of the logic and thought processes behind students’ sketches.

The first set of sketches presented previously in Figures 4.4 and 4.5, show this opposite sketching phenomena very well. As you can see, in both pieces of student work, the first derivative seems to be a representation of the original function reflected across some horizontal axis. When sketching the second derivative, it appears as though both students have an “opposite” second derivative sketch in relation to the first derivative sketch. This, in turn, makes the second derivative a similar shape as the original function. Below are some more examples of opposite sketching, but in these sketches, only the first or second derivative is an opposite sketch.

![Figure 5.1: Opposite Sketches (First Derivative Only)](image-url)
There is very little research done previously that discusses the topic of students sketching derivatives as opposite. In an interview that Zandieh (2000) conducted with a high school calculus student about their graphical interpretation of derivative, the following conversation occurred:

MZ: *Do* you remember what a derivative is?

Brad: [laughs] Isn’t it like the opposite of a – no, what’s the opposite of a limit or something. I don’t know. I have no idea how to say it. It’s been a while. [pause] Isn’t it like all those formulas for velocity and acceleration and something like that? (p. 114)

Although this conversation does not mimic the definition of “opposite” sketches I have presented, in my opinion it does state something rather interesting and relevant. The student does not specifically state confusion with derivative sketches deriving from “opposites”, but he does allude to the idea that opposites might have something to do

Figure 5.2: Opposite Sketches: (Second Derivative Only)
with velocity and acceleration. Since acceleration is the derivative of velocity, it is the opinion of this researcher, that this student’s confusion is similar the type expressed by interviewed students in my research. In fact, one of the students I interviewed, Carol, went into further detail about her thinking of the derivative being opposite. She stated that she looks at the direction (increase or decrease) of the original function when sketching the first derivative. When I asked her to explain further she said, "I think that when you sketch the derivative it’s the opposite of the function.” Originally, Carol lightly sketched a large, concave down, parabola as her first derivative. She then erased this drawing and sketched the function labeled 1st, in Figure 5.3. When I asked Carol why she did this, the following conversation occurred:

**Figure 5.3: Carol’s Original Task**

As we can see from this conversation, Carol created this pattern of opposite sketching. It is the opinion of this researcher that Carol created this pattern as a method of coping with confusion when she encountered sketches such as this one.
Another student, Sharon, whom I quoted previously in the Results section, had this thought about sketching derivatives of a given graphical function. When asking Sharon if concavity had any impact on her sketches, she replied, “Yeah I usually kind of, I don’t know if it’s correct, but based on what I was getting from my tutor it’s almost like the opposite, so as you’re watching the function decreasing, the derivative is increasing up to that point…” My opinion of Sharon’s quote is that she interpreted examples given by her tutor as derivatives of graphs being the opposite shape. This idea is corroborated with another quote from Sharon:

“Like I was saying before, I used to watch that as the function was decreasing you had it increased, and then as you started seeing it increase, you start having it decrease again. I was just following kind of like almost like the opposite of what the actual function is doing, was what I always got out of my tutor to do.”

Below is an excerpt of the conversation I had with Sharon, along with a picture of her work (Figure 5.4). As you can derive from the picture, Sharon, besides having ideas about opposites, also had mass confusion and uncertainty about how to create the first and second derivative sketches. Her sketches for each derivative are labeled in consecutive order as she created them.
RS: And you learned to do the opposites from your tutor?
S: Kind of. She was trying to help, help me look at graphs. And I, she would show me pictures as well and ask which one would be the first, the second, and so forth. And I would use that (making opposite = derivative) as a way to pick out the picture. I might be looking like straight into it. It shouldn’t be just that, that you’re considering that it is the opposite. But I just kind of used it as a starting point for me. You know I can’t remember which ones I’ve seen in class, but for the most part I think it’s been OK in helping me make the graph.

Figure 5.4: Sharon’s Original Task

Recapping the quote from Sharon in the Results section, I asked her about why she started sketching in a particular quadrant. Here is a portion of her previous quote that I want to emphasize:

“…To kind of indicate it (first derivative sketch) was doing the opp, like the opposite, like what I had in mind it was doing the opposite behavior as its initial function.”

Prior to the discussion of Sharon’s interview and sketch, Carol talked about creating a pattern for sketching the derivative of a given graphical function, and I mentioned earlier that it was the opinion of this researcher that she created this pattern as a coping method for when she was confused about sketching the derivative of a given graphical function. I believe Carol’s created pattern and Sharon’s learned behavior about sketching the
opposite behavior are very much related. I will elaborate further. Below is an example of work done by another student named Tod.

![Figure 5.5: Tod’s Original Function and First Derivative Sketch](image)

![Figure 5.6: Enlarged section of Figure 5.5](image)

This is a sketch of Tod’s first derivative in Figure 5.5. As one can see, it is correct, and Tod seems to understand the relationship between the slope of the original function and the sketch of the derivative, especially at the maximum and minimum values of the original function. The sketch on the right, Figure 5.6 is an enlarged view of the left side of Figure 5.5. I want to emphasize the initial relationship between the direction of the original function compared to the initial direction of the first derivative sketch. When read left to right, the graph of the original function and sketch of the first derivative appear to be going in “opposite” directions, but as I mentioned before, Tod’s first derivative sketch is correct, not incorrect. Maybe there is something to this opposite sketching after all.

The following is an example of a student whose first derivative sketch, Figure 5.7, is incorrect and an enlarged view of the left side is in Figure 5.8. One can definitely see
the similarity in the initial direction of this student’s graph when compared to Tod’s. This is to further emphasize the idea that both correct and incorrect sketches initially begin the same way.

5.2 Linear Second Derivative

Many students were confused about how to sketch the second derivative. In a study conducted by Baker et al. (2000) they also talked about students struggling with the concept of the second derivative. In their study, students either ignored the second derivative, worked from memorized notions of what they thought the second derivative should look like, or were unable to coordinate the first and second derivatives across intervals. In an interview with Sharon, when I asked her about the second derivative, her response was: “Once I go to the second derivative I pretty much have to guess a little bit, based on what I’m trying to remember from my tutor last year.”
The second derivative, at times, exemplifies a mystery to calculus students. When asked to graphically determine the second derivative from a given graphical function, some students are left baffled and unsure of how to complete this task. The following conversation with Ester, and her associated sketch in Figure 5.9, demonstrate this idea. (Note: Ester’s initial second derivative sketch, from her interview, was the horizontal line above the x-axis.)

RS: Let’s talk about your second derivative sketch. What were you thinking when you sketched that?
E: I really had no idea so I just went with a line that was parallel to the x-axis because I know a lot of the times the second derivative is just a number.
RS: Second derivative is just a number?
E: Like when you get down to that point all of your variables are gone. I don’t know I can’t even like explain what I was thinking.

Figure 5.9: Ester’s Original Task

Even when students have a correct idea or procedure they want to initiate, the confidence in their decision is waning. Some students say they understand the concept that the second derivative sketch can also be the derivative of the first derivative sketch. This idea is correct, but some students seem unable to demonstrate this procedure. Following are some quotes by Ester that support this:

“Yeah. On the first task for the second derivative I had no idea, and I still really don’t. So I just kind of drew a line for the second derivative, just out
of some crazy assumption…I think as we were talking about the first task I was thinking about it out loud to you and how it (the second derivative) could relate to the first derivative in the same way that the first derivative relates to the original function. And that hadn’t been something (I thought of) like, oh that is actually an option.”

Here is an additional quote from Ester:

“OK Well I decided since I really don’t still understand how the second derivative relates to the first derivative or the original function. I just kind of went with the idea that the second derivative has the same relationship to the first derivative that the first derivative has to the original function, in that the graph of the second derivative would be the values of the slope of the tangent to the first derivative graph. So I kind of went with that.”

This latter quote from Ester is sort of ironic since she states that she still does not understand how the second derivative relates to the first, but then she goes on to explain the very concept she said she did not understand. In her statement, Ester seems to confirm and contradict the idea made by Baker et al. (2000), which is, “Notably, students could not interpret the relationship of the second derivative to the first” (p. 576).

One possible reason for why students have this misconception may be due to a lack of conception about the second derivative (Baker et al. 2000). It seems obvious from the quantitative and qualitative data that this confusion revolving around second derivative linear sketch, if not confronted, may continue to skew the conceptual understanding of students’ visual interpretation of the second derivative.
I have mentioned research and presented evidence that students lack conceptual understanding about the second derivative and how it relates to the first derivative, but this does not fully explain the existence of numerous examples of students sketching the second derivative as a linear function. One possibility is that this phenomenon is happening due to the use of certain examples. The example of taking the first and second derivative of a cubic function is not used often in the classroom, nor does this type of example relate very well to real-world problems. Here is a quote from Ester that alludes to this theory,

“I just went with how I’d seen a lot of second derivatives before. I really had no idea. I just kind of picked. It has nothing to do with the other two graphs at all.”

One major use of this type of example in mathematics and physics classes, which supports the theory that the second derivative is always a horizontal line, is taking the first and second derivatives of a position versus time graph. In the majority of cases where this researcher has seen these examples used, the position versus time graph is in the shape of a parabola. This is used to model the trajectory of a rocket being launched or a ball being thrown in the air. The first derivative is then the velocity of the object, which would be a linear function with a positive or negative slope. The second derivative would be the constant acceleration of that object, presented as a linear function with a constant value (i.e. a horizontal line with zero slope). See Figure 5.10:
In Zandieh’s (2000) article, one of the students she interviewed had a difficult time explaining what a derivative is. He said, “Isn’t it like all those formulas for velocity and acceleration and something like that?” (p. 114). This quote, along with others from
Juter (2009), emphasize the fact that students see this type of example used quite a bit. This type of example becomes a routine one for differentiation in mathematics and physics classes. “The problem is that such routines very soon become just that—routine, so that student(s) begin to find it difficult to answer questions that are conceptually challenging” (Tall 1996, p. 17). Since the original graph used in this research is not as simple as a parabolic function representing a projectile trajectory, this researcher would consider it a conceptually challenging task for students to complete. There is a limited amount of real-world situations that involve the use of a graphically-defined function and its derivatives.

In light of this research, it appears that students may come to the misunderstanding that all second derivatives are linear. Once again, this may be due to the student trying to rationalize the answer without truly understanding the concept of differentiation. A linear second derivative sketch is another coping method used by various students when they are at an impasse and do not know how to continue the problem with clarity. This is supported by Ester’s quote in section 4.2.

During one interview in particular, I received an answer regarding the second derivative that seemed to have nothing to do with routine or the overuse of examples. During the interview with Carol, I asked her why she drew a horizontal line above the x-axis for her second derivative on the original task. Her response: “I have the idea that if your first derivative is positive, then your second derivative is going to be positive. And if your first derivative is negative, the second is positive.”

The student could not elaborate as to why she sketched a line instead of a nonlinear curve. One theory behind this response that may also be linked to other
examples of students’ linear second derivatives is the idea of uncontrollable mental imagery. According to Aspinwall et al. (1997):

“Such images are uncontrollable both in sense that they appear to rise unbidden in an individual’s thought and also in their persistence even in the face of contrary evidence. An uncontrollable image, then, is one which beyond the volition of the cognizing individual.” (p. 303)

In a study by Lithner (2000), he asked four undergraduate students to sketch the derivative of a given graphical function. In his summary he states:

“The four students often focus mainly on what they can remember and what is familiar within limited concept images. This focus is so dominating that it prevents other approaches from being initiated and implemented.” (p. 93-94)

What this means in this situation is that the students who do sketch a linear second derivative might not have any other direction to follow in their mind; so they decide to follow the path their mind produces or the image in their mind is so prominent that they do not even think about other visual options. According to Eisenberg and Dreyfus (1991) students are not able to understand the context of the calculus problem in a visual sense. They state:

“Many believe this lack in understanding is to a large extent due to the failure to establish explicit and detailed connections between the visual and analytic aspects of mathematical concepts and procedures.” (p. 27)
5.3 Proper Start – Improper Finish

So how does this information affect students who properly started sketching in the correct direction, but in the end their graphs were incorrect? Well, it is apparent from the interview transcripts that Carol and Sharon learned or created a pattern of initially sketching the derivative of a given graphical function in the opposite direction of that graphical function. It is the hypothesis of this researcher that if or when the students are confused about what to do next, especially with the first derivative, they resorted back to this understanding, or learned pattern, of sketching the opposite direction of the original function through its entirety. The evidence shows that students who sketch the opposite of the first derivative initially begin sketching in the correct direction when the graph is viewed left to right. When they became confused along the way, these students continued this pattern of sketching the opposite behavior due to the fact that they did not have thorough understandings of the relationship between the derivative of a given graphical function and the function itself. This process of students properly starting and improperly finishing a sketch includes, but is not limited to, those students who sketched opposites. Many students appear to repeat the process of sketching in the opposite direction when they get confused. This process of coping by repeating a procedural action is discussed in Tall (1996). In this case, Sharon’s and Carol’s coping method is to sketch the derivative as an opposite shape of the original function.
Although students did show differences in performance between student populations, which lead me to believe that students may have a better grasp of sketching derivatives of a given graphical function towards the end of the semester, there were still some vivid misconceptions that lingered. The notion of students sketching the derivative of a given graphical function as the opposite shape of the original graph is apparent in the data and supported by students’ descriptions from interviews. After examining numerous student derivative sketches, both correct and incorrect, especially those sketches dealing with the first derivative, it is apparent that many students begin their sketches the same way, correctly with a graph that goes in the opposite direction of the original function. Overall correctness, past this initial point, seems to rely on the depth of the student’s understanding of the relationship between the graph of the original function and, primarily, the sketch of its first derivative. If this understanding is unclear, then I believe students simply continue to follow their initial procedure of sketching in the opposite direction of the original function. Another theory of why students get confused along the process of sketching is that they may have only pieces of knowledge about the derivative that they are relying on while sketching. According to Scherr (2006), “If we hold a pieces model of student reasoning, we will not necessarily expect coherence among a student’s responses to various questions” (p. 3). In fact, the knowledge in pieces model may very well describe why student responses during interviews are widely varied and fluctuating and why some students change their minds frequently (Scherr 2006). Unlike the misconceptions model, the knowledge in pieces model of student thinking says that
students have different knowledge pieces that are “…small-scale knowledge elements that can be applied productively in many different settings” (Wittmann 2006, p. 1); one of those settings being tasks preformed for this research. For example, if students begin sketching in the correct direction for a derivative sketch, then come to a point where they become confused about what to do next, they may pull from a different piece of knowledge in order to continue sketching. The piece of knowledge, which is different from the one they used when they began the sketch, may or may not result in a correct sketch. It was not until late in the research and multiple examinations of the interviews, that the knowledge-in-pieces model came to light. Up to that point, my analysis of the data from an educational psychology perspective, was limited to the broad, and notably overused term, misconception. I used the term “misconception” to encompass anything that seemed to be confusing or any type of misunderstanding the students seemed to encounter while completing the task or stated during the interviews. Further investigation into this data would probably reveal that a knowledge-in-pieces model may help to better explain the thought processes of the confused students.

Another prominent misunderstanding apparent in the data was when students sketched the second derivative as a linear function. One possible method to confront this difficulty is to break outside the stereotypical examples that so often dominate the calculus and physics classroom. I am not saying these examples are bad, just that it might be worth exploring the potential impact of adding some more variation to these examples when teaching students about derivatives; in particular sketching derivatives of a given graphical function. I believe students may also see very common examples of taking the derivative of a quadratic function, which usually associates with a position versus time
The derivative of this quadratic function would then be a linear function with a positive or negative slope, which would relate to the velocity of the object. Lastly, the second derivative would then become a linear function with zero slope representing a constant acceleration. Because the example given to students in this research was not in a quadratic form, but more in the shape of a quartic function that is not used in examples frequently, I would consider it more conceptually challenging. Also, the second derivative is not linear but actually more like a parabola in shape. Many students probably do not have much experience with differentiating a fourth degree function in a graphical setting.

I believe both of these types of incorrect sketches are coping methods for students and calculus. According to Tall (1996):

“When faced with conceptual difficulties, the student must learn to cope. In previous elementary mathematics, this coping involves learning computational and manipulative skills to pass exams. If the fundamental concepts of calculus (such as the limit concept underpinning differentiation and integration) prove difficult to master, one solution is to focus on the symbolic routines of differentiation and integration. At least this resonates with earlier experiences in arithmetic and algebra in which a sequence of manipulations are performed to get an answer” (p. 17).

For the students who sketched the derivative, the symbolic routines would be always sketching the opposite behavior as a derivative. For the students who sketched a linear second derivative, the routine would be for them sketch a second derivative similar to the ones they have seem dominate the examples in the classroom and textbooks.
If we continue to stress conceptual understanding of differentiation in calculus, I believe many students will be able to overcome confusion and uncontrollable mental imagery to create sketches of derivatives that truly represent understanding of the relationship between an original function and the graphs of its derivatives.

Some directions for future research include, but are not limited to: research that specifically targets the notion of opposites when sketching derivatives, research trying to identify why students start with a correct graph but then lead into a sketch of opposite behavior, and per another observation from the research, why some students have the ability to sketch the first derivative correctly, but not the second. Where is the disconnect between sketching the first and second derivative and how can we help students to bridge that gap of confusion?
REFERENCES


BIOGRAPHY OF THE AUTHOR

John (Randy) Stahley was born in Kettering, Ohio on December 24, 1978. He was raised in Centerville, Ohio and graduated from Centerville High School in 1997. He attended Bowling Green State University and graduated in 2001 with a Bachelor’s degree in General Business. He worked for a small reprographics company and became a Hewlett Packard Designjet Service Technician. In September of 2006 he relocated to Orono, Maine and entered an advanced degree program for teachers at The University of Maine in September of 2007. In August of 2010 he relocated, once more, to Enfield, New Hampshire where he is currently a Mathematics Teacher at Mascoma Valley Regional High School in Canaan, New Hampshire.

After receiving his degree, John will continue to teach at Mascoma Valley Regional High School. John is a candidate for the Master of Science in Teaching degree from the University of Maine in May, 2011.